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DISCRIMINETING BETWEN GREEN AND INFLUENCE FUNCTIONS

У статиці немає жодної відмінності між функцією впливу і функцією Гріна, проте аналізуючи динамічні процеси, можливо вказати на їхню істотну різницю. Основою для розрізнення між ними слугує значення функції щільності вантажу. Показано два приклади функції Гріна для простої балки Bernoulli з в'язкопружними властивостями матеріалу (моделі Войта і Максвела).

Ключові слова: функція Гріна, функції щільності вантажу.

In statics there is no distinction between influence function and Green function, while analysing dynamical processes it is possible to indicate a significant difference. The basis for the discriminating between them comes from the meaning of load function density. Two examples of Green function for simply supported Bernoulli beam with viscoelastic properties of its material (Voigt and Maxwell models) are shown here.

Keywords: Green function, load function density.

Introduction. Green function, thanks to it superpositional properties, seems to be the crown of most engineering linear problems. Even now, when numerical methods domination is unquestioned, this analytical tool still plays an important role.

The meaning of Green function is strongly connected with Laplace potential and was first formulated by George Green [1] (1793 - 1841). Volterra and Lauricella in several papers (for example [2]) used Green function in integral form as a tool for boundary problem investigations. The basis for Green function theory is to be found in Love [3], Trefftz [4], Kellog [5], Gurtin [6], Wladimirov [7]. the work by Zielinski [8], it is worth recalling here where significantly complicated boundary problems were solved by means of boundary Green functions.

The aim of this consideration is to indicate and define the criterion for discriminating between Green function and influence function. Two simple visco-elastic problems in case of Bernoulli beam are enclosed as examples.

1. Definition of dynamical influence surface. In statics, for plate bridge caring decks for example, the influence surface converges with the Green function for any assumed boundary problem. The opposite situation takes place when the dynamical or quasi-statical process is analysed, in cases of which we can precisely define influence function as the limitation of Green one.

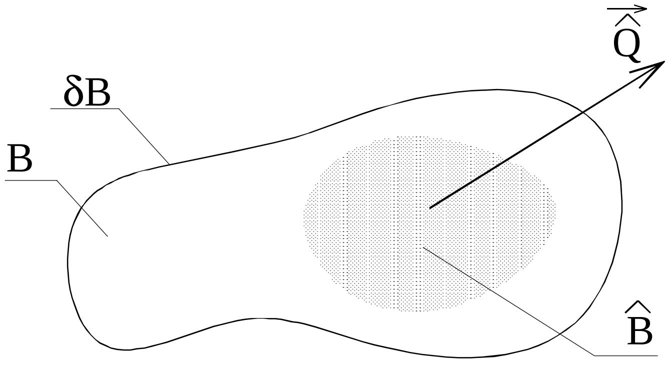

Fig. 1. Resultant force taken over open subregion $\hat{B}$

The understanding of load density is basic for this consideration. The following definition of load density is assumed -

$$\vec{q} = \lim_{\hat{V} \to 0} \frac{\vec{\hat{Q}}}{\hat{V}} \qquad , \tag{1}$$

where $\hat{Q}$ is the resultant force taken as integral over open subregion $\hat{B}$ (Fig. 1)

$$\vec{\hat{Q}} = \int_{\hat{B}} \vec{q} \sqrt{g}\, dq_1\, dq_2 \ldots dq_k \,, \tag{2}$$

and the adequate measure of subregion $\hat{B}$,volume for example, fulfils formula

$$\hat{V} = \int_{\hat{B}} \sqrt{g}\, dq_1\, dq_2 \ldots dq_k \qquad ; \tag{3}$$

and where $\sqrt{g}$ - stands for Jacobi determinant of arbitrary coordinate system transformation to θ_j (j=1,2,3, ... ,k) system.

For three dimensional elastokinetic problems, load density depends on fourth arguments (k=4), i.e. geometrical coordinates $\theta_1, \theta_2, \theta_3$ and $\theta_k = \theta_4 = t$ which stands for time axis.

In general case of three dimensional geometrical space and for an arbitrary curvilinear coordinate system characterised by basic vectors $\vec{g}_j$ and for an arbitrary point of the body load density could be presented as follows

$$\vec{q}(t) = \sum_{m=1}^{3} \vec{q}_m(t) = \sum_{m=1}^{3} q_m(t) \frac{\vec{g}_m}{|\vec{g}_m|} \;, \tag{4}$$

where q_m - are physical components.

When density depends on some of possible arguments, i.e. only on geometrical ones for example, then we get partial densities, so it could be: linear, surface or volume density. Considering dependence on all arguments the following definition of influence function can be stated

The complete influence surface of analyzed mechanics' magnitude is its geometrical image of the structure answer to the load in the form of Dirac's impulses obeying all arguments

$$\vec{q}_m = \vec{1}_m \; d\left(q_1 - q_1^*\right) d\left(q_2 - q_2^*\right) \ldots .\; d\left(q_j - q_j^*\right) \ldots .\; d\left(q_k - q_k^*\right), \tag{5}$$

where $m = 1,2,3, \ldots ., (k-1)$

Considering the load function applied to plate structure in Cartesian coordinate system

$$\vec{q}_3 = \vec{i}_3 \; q\left(x_1, x_2, t\right) \tag{6}$$

where $\vec{i}_3$ is normal to the plate plane, $q\left(x_1, x_2, t\right)$ stands for partial density, in detail it is surface density depending on time parameter. The correspondence between surface density q and density q_ρ could be expressed by

$$q = \int_0^t q_r\left(x_1, x_2, q_3\right) dq_3 \,, \tag{7}$$

$$q_r = h(t) \frac{\partial}{\partial t} q \,; \tag{8}$$

where $\eta(t)$ is the Heavisidea's step function.

Using the definition formulated above, we search for dynamical influence flexure function as an effect of q action in the form

$$q = \int_0^t d\left(x_1 - x_1^*\right)\, d\left(x_2 - x_2^*\right)\, d\left(q_3 - q_3^*\right) dq = q_x\, q_t\,; \tag{9}$$

where

$$q_x = d\left(x_1 - x_1^*\right)\, d\left(x_2 - x_2^*\right) \tag{10}$$

and

$$q_t = h\left(t - t^*\right). \tag{11}$$

Expression (9) shows the connection between influence surface and Green function for which

$$q_t = d\left(t - t^*\right). \tag{9.1}$$

Per analog we can find such relations for linear densities in beam problems, too.

2. Examples of dynamical Green functions

Assumptions:

- the simple Bernoulli beam model is undertaken,
- the material is homogenous and isotropic, is characterized by visco-elastic Voigt and Maxwell models,
- by virtue of [10], [11] is taken that $$\frac{l}{m} = \frac{\hat{l}}{\hat{m}}, \tag{12}$$
- the initial conditions are uniform and for each function at t=0, its values are zeros,
- load density function is of the form

$$q = d\left(x - x^*\right)\, d\left(t - t^*\right). \tag{13}$$

Notation:

$\mu, \lambda, \hat{\lambda}, \hat{\mu}$ - generalized Lamé elastic and viscous material constancies,

E, ν - Young and Poisson modules,

w, l, A, J - adequately are: flexure, span, area of cross-section, principal second moment of beam cross-section area,

ρ - material volume mass density,

x, t – Cartesian coordinate along beam span, time,

$\xi = \frac{x}{l}$, $\tau = \frac{t}{t_o}$, $\omega = \frac{w}{l}$ - dimensionless coordinates and flexure,

$\varepsilon_{ij}, \sigma_{mn}, J_1, I_1$ - strain, stress tensors and its linear invariants,

$C_{ij}^{\ kl}, B_{ij}^{\ kl}, D_{ij}^{\ kl}$ - elastic and viscous stiffness tensors,

g_{ij} - metric tensor,

$\dot{()} = \frac{\partial ()}{\partial t}$ - time derivative,

$\tilde{f}(t)$ - Carson-Laplace transformation of original function $f(\tau)$

$$\tilde{f}(p) = p\int_0^\infty f(t)\exp(-pt)\,dt \ , \tag{14}$$

p - is the transformation complex parameter.

2.1. Voigt model

Voigt constitutive relation is assumed in form

$$s_{ij} = C_{ij}^{\ kl} e_{kl} + D_{ij}^{\ kl} \dot{e}_{kl}\,, \tag{15}$$

which for isotropy and according to (12) could be written as

$$s_{ij} = 2\left(m e_{ij} + \hat{m}\dot{e}_{ij}\right) + \left(l\, J_1 + \hat{l}\, \dot{J}_1\right) g_{ij} = 2\left(m e_{ij} + \hat{m}\dot{e}_{ij}\right) + \frac{n}{1-2n}\left(J_1 + \dot{J}_1\right) g_{ij}\,, \qquad (16)$$

and after applying (14) we arrive at

$$\tilde{s}_{ij} = 2(m + p\hat{m})\left(\tilde{e}_{ij} + \frac{n}{1-2n}\tilde{J}_1 g_{ij}\right). \qquad (17)$$

In case of pure bending the normal stresses are given by

$$\tilde{s}_{33} = 2(m + p\hat{m})(1+n)\tilde{e}_{33} = \left(E + p\hat{E}\right)\tilde{e}_{33}\,, \qquad (18)$$

this implies beam equation of motion as below

$$E\,J\,(1+py)\frac{\partial^4 \tilde{w}}{\partial x^4} + r\,Ap^2\tilde{w} = \tilde{q}\,; \qquad y = \frac{\hat{E}}{E}\,, \qquad (19)$$

which in dimensionless coordinates becomes to

$$\tilde{w}^{IV}(1+py) + p^2\tilde{w} = \tilde{q}\,a\,, \qquad t_o = l^2\sqrt{\frac{r\,A}{EJ}}\,, \qquad a = \frac{l^3}{EJ}\,. \qquad (20)$$

The solution is searched by means of Fourier series presentation of transformed unknowns as well as transformed load function (13)

$$\tilde{w}(x,p) = \sum_{j=1,2}^{\infty} \tilde{w}_j(p)\sin(jpx)\,, \qquad (21.1)$$

$$\tilde{q}(x,p) = \sum_{j=1,2}^{\infty} \tilde{q}_j(p)\sin(jpx)\,, \qquad (21.2)$$

when

$$\tilde{q}_j(p) = p\,\sin\left(jpx^*\right)\exp\left(-pt^*\right). \qquad (22)$$

Substituting (21) into (19) we arrive at

$$\tilde{w}_j = \frac{\tilde{q}_j\;a}{p^2 + (jp)^4(1+py)} = \frac{\tilde{q}_j\;a}{(p-p_1)(p-p_2)} \qquad (23)$$

where

$$p_{1,2} = -\frac{(jp)^4}{2}y\left(1 \pm \sqrt{1 - \left(\frac{2}{(jp)^2 y}\right)^2}\right) \qquad (24)$$

are real and negative singularities.

By virtue of the residuum theorem the originals are found in the form

$$w_j = \sum_{m=1}^{2} \operatorname{Res}\left(\frac{\exp(pt)}{p}\frac{\hat{q}_j\,a}{(p-p_1)(p-p_2)}\right)\Bigg|_{p=p_m} = \frac{a}{2}\sin(jpx_o)\left(\frac{\exp(p_1(t-t_o))}{p_1} + \frac{\exp(p_2(t-t_o))}{p_2}\right). \qquad (25)$$

Substituting (25) into (21.1) the dynamical Green function of beam flexure is obtained.

2.2. Maxwell body

The constitutive law in the Maxwell body case has the form

$$s_{ij} + B_{ij}{}^{kl}\dot{s}_{kl} = C_{ij}{}^{kl}e_{kl}\,, \qquad (26)$$

which in isotropy and (12) becomes

$$e_{ij} = \frac{1}{2}\left(\left(\frac{s_{ij}}{m} + \frac{\dot{s}_{ij}}{\hat{m}}\right) - \frac{n}{1+n}\left(\frac{J_1}{m} + \frac{\dot{J}_1}{\hat{m}}\right)g_{ij}\right). \qquad (27)$$

For beam transformed normal stresses we have

$$\tilde{s}_{33} = 2m(1+n)\frac{\tilde{e}_{33}}{1+pc} = E\frac{\tilde{e}_{33}}{1+pc}, \qquad c = \frac{m}{\hat{m}}. \tag{28}$$

Using (28) the beam equation could be written down as

$$\tilde{w}^{IV}(1+pc)^{-1} + p^2\tilde{w} = \tilde{q}\,a. \tag{29}$$

Again, the expansion into series (21.1-2), (22) take place and leads us to Fourier coefficients as below

$$\tilde{w}_j = \tilde{q}_j\ a\frac{1+pc}{p^3c + p^2 + (jp)^4} = \tilde{q}_j\ a\frac{1+pc}{(p-p_1)(p-p_2)(p-p_3)}. \tag{30}$$

The magnitude $-\dfrac{1}{\chi}$ is not a root of the dominator of (30) expression. Assuming that the roots are single we have two variants of the solution:

a) all three roots are real, then using the residuum theorem we obtain

$$w_j = \frac{a}{2}\sin(jpx_o)\sum_{m=1,2}^{3}\exp(p_m(t-t_o))\frac{1+p_m c}{(p_m-p_n)(p_m-p_k)} \tag{31}$$

when $k \neq m \neq n$ end k, m, n = 1, 2, 3;

b) one root is real and negative, p_1 for example, and two others are complex conjugated, i.e. $p_2 = \overline{p}_3 = \alpha + i\beta$, moreover $\alpha < 0$; then we get

$$\tilde{w}_j = \tilde{q}_j\ a\frac{1+pc}{(p-p_1)\ \left(p^2 - 2pa + a^2 + b^2\right)}. \tag{32}$$

Retransforming (32) we arrive at

$$w_j = a\sin(jpx_o)\left(\exp(p_1(t-t_o))\frac{1+p_1c}{(p_1)^2 - 2a\ p_1 + a^2 + b^2} + \right.$$

$$\left. + \frac{\exp(a(t-t_o))}{b\left(b^2 + (a-p_1)^2\right)}\left(\left((1+ac)(a-p_1) + b^2c\right)\sin(b(t-t_o)) - b(1+ac)\cos(b(t-t_o))\right)\right) \tag{33}$$

Substituting (31) or (33) into (20.1), the dynamical Green function of the beam characterized by Maxwell model becomes derived.

Conclusions. The criterion for discriminating between Green and influence functions has been formulated on the basis of load density form. In both cases the load density is presented by Dirac deltas product, but in the case of influence function the deltas product is related to partial density proper to the particular mechanical problem. In the case of Green function the load density function is ‘pure’ Dirac deltas product according to all considered arguments.

Dynamical Green function or alternatively dynamical influence functions are serviceable only when the really existing damping is taken into consideration. This is available when along with elastics viscosity is taken into account.

In general, in practice, the lack of appropriate material constant values is observed. Many experimental results were derived for different sample dimensions and shapes, also by different and inconsistent each to other laboratory tests [12]. Probably this is the reason of the absence of visco-elastics in codes and engineering design generally and especially in everyday road pavement material and problem treats.

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