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BOUNDARY PROBLEMS FOR ELLIPTIC SYSTEMS WITH ANISOTROPIC NONLINEARITY

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It is proved the well-posedness of boundary problems for some class of elliptic systems of equations with polynomial nonlinearities given in unbounded domains without conditions on the solution's behaviour and increasing of initial data at infinity.

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Introduction

A great many mathematical papers are devoted to boundary problems for elliptic equations and their systems in unbounded domains (see, for instance, [1]—[7] and the literature cited therein). These researches are mainly directed to prove the existence and uniqueness of the solutions of corresponding problems under some growth conditions on the solution and initial data or without them at infinity.

In this paper we investigate the well-posedness of nonlinear elliptic systems of equations generalizing the model equation

$$-\sum_{i=1}^{n}\Big(|u_{x_i}(x)|^{p_i-2}u_{x_i}(x)\Big)_{x_i}+|u(x)|^{p_0-2}u(x)=f(x) \qquad (1)$$

given in unbounded domain Ω with corresponding indices of nonlinearity $p_i>1$ $(i=\overline{0,n})$.

Here we consider the case when boundary problems for such equations and their systems have unique solution without restrictions on it's behaviour and increasing of initial data at infinity. At first such result was obtained in 1984 by H. Brezis in [1] for the elliptic equation

$$-\Delta u+|u|^{p-2}u=f(x), \quad p>2,$$

given in the whole space $\mathbb{R}^n$. Later on new types of equations with constant indices of nonlinearity keeping such property were found in works [2]—[6]. In particular, in work [6] it is established the uniquely solvability of boundary problems for equations generalizing (1) with $1<p_i\le 2$, $p_0>2$.

In the recent years the boundary problems for elliptic equations and their systems with nonstandard growth conditions have aroused increasing interest. Such problems were recently obtained in the hydromechanics of quasi-Newton fluids ([8]). In paper [7] authors considered equations of this type extending (1) with p_i depending on $x\in\Omega$. This work is continuation of investigations made in [7] for the systems of elliptic equations having polynomial nonlinearities that vary at x and are different with respect to different derivatives. We consider mixed boundary conditions: the Dirichlet boundary condition on one part of the boundary and the Neumann boundary condition on the other part. In addition to one-valued solvability of the problem in class of functions with arbitrary behaviour at infinity we consider the question of continuous dependence of the solution on initial data.

I. Preliminaries

We denote by $\mathbb{R}^k$, where $k\in\mathbb{N}$, a linear space consisted of elements $x=(x_1,\dots,x_k)$, where $x_i\in\mathbb{R}$ $(i=\overline{1,k})$, with norm $|x|=\sqrt{x_1^2+\ldots+x_k^2}$. For function $v(z)$, $z\in\widetilde{D}\subset\mathbb{R}^k$, we denote by $v|_D$ its restriction to a set $D\subset\widetilde{D}$.

Let $n\ge 2$ be a natural number, Ω be an unbounded domain in $\mathbb{R}^n$ with piece regular boundary $\Gamma\overset{\mathrm{def}}{=}\partial\Omega$; $\Gamma=\overline{\Gamma}_1\bigcup\overline{\Gamma}_2$, where Γ_1, Γ_2 are open sets on $\partial\Omega$ (one of them can be empty), $\Gamma_1\bigcap\Gamma_2=\varnothing$; ν is an unit exterior normal vector to $\partial\Omega$. We will suppose without loss of generality that $0\in\Omega$. For all $R>0$ we denote by Ω_R the connected component of the set $\Omega\cap\{x:|x|<R\}$ such that $0\in\Omega_R$. Let $S_R=\partial\Omega_R\cap\Omega$, $\Gamma_{k,R}\overset{\mathrm{def}}{=}\Gamma_k\bigcap\partial\Omega_R$, $k\in\{1,2\}$, $R>0$.

Let $C^1_{\mathrm{c}}(\overline{\Omega})$ be the subspace of the space $C^1(\overline{\Omega})$ consisted of functions with bounded support in $\overline{\Omega}$. Put $C^{1,+}_{\mathrm{c}}(\overline{\Omega})\overset{\mathrm{def}}{=}\{v\in C^1_{\mathrm{c}}(\overline{\Omega}) : v\ge 0 \text{ on } \Omega\}$. Denote by $C^1_0(\overline{\Omega},\Gamma_1)$ the subspace of $C^1_{\mathrm{c}}(\overline{\Omega})$ consisting of elements having zero value in the neighbourhood of Γ_1.

Let $r\in L_\infty(\Omega)$ be such function that $r(x)\ge 1$ for a.e. $x\in\Omega$. On the space $C(\overline{\Omega_R})$, where $R>0$ is an

arbitrary number, we introduce the norm
$$\|v\|_{L_{r(\,\cdot\,)}(\Omega_R)} \overset{\text{def}}{=} \inf\{\lambda > 0 : \rho_{r,R}(v/\lambda) \le 1\},$$
where $\rho_{r,R}(v) \overset{\text{def}}{=} \int\limits_{\Omega_R} |v(x)|^{r(x)}\, d\,x$.

The completion of $C(\overline{\Omega_R})$ by this norm is denoted by $L_{r(\,\cdot\,)}(\Omega_R)$ and is called the *general Lebesgue space*. It is obvious that the set $L_{r(\,\cdot\,)}(\Omega_R)$ is a linear subspace of the space $L_1(\Omega_R)$. We define $L_{r(\,\cdot\,),\,\mathrm{loc}}(\overline{\Omega})$ be the closure of the space of continuous functions on $\overline{\Omega}$ in the topology generated by the system of seminorms: $\|\cdot\|_{L_{r(\,\cdot\,)}(\Omega_R)}$, $R > 0$.

Let N be a given natural number. When X is a Banach (topological) space, $(X)^N$ denotes the Cartesian product of X with the corresponding topology and its elements can be written as the column vectors.

Define $\mathbb{P}$ be a set of matrix functions $p = (p_0, p_1, \ldots, p_n)$, $p_k = \mathrm{colon}(p_{k1}, \ldots, p_{kN})$ $(k = \overline{0,n})$, such that $p_{ij} \in L_\infty(\overline{\Omega})$ and $p_{ij}(x) > 1$ $(i = \overline{0,n},\ j = \overline{1,N})$ for a.e. $x \in \Omega$. For a function $p \in \mathbb{P}$ by $p^* = (p_0^*, p_1^*, \ldots, p_n^*)$, $p_k^* = \mathrm{colon}(p_{k1}^*, \ldots, p_{kN}^*)$ $(k = \overline{0,n})$, denote the matrix function such that $1/p_{ij}(x) + 1/p_{ij}^*(x) = 1$ $(i = \overline{0,n},\ j = \overline{1,N})$ for a.e. $x \in \Omega$ (it is obvious that $p^* \in \mathbb{P}$).

For all $R > 0$ define $W^1_{p(\,\cdot\,)}(\Omega_R)$ be the Banach space obtained as the completion of the space $\big(C^1(\overline{\Omega_R})\big)^N$ by the norm
$$\|v\|_{W^1_{p(\,\cdot\,)}(\Omega_R)} \overset{\text{def}}{=} \sum_{j=1}^{N}\sum_{i=0}^{n} \|\partial_i v_j\|_{L_{p\,ij(\,\cdot\,)}(\Omega_R)},$$
where $\partial_i \overset{\text{def}}{=} \partial/\partial x_i$ $(i = \overline{1,n})$, $\partial_0 v \overset{\text{def}}{=} v$. It is obvious that $W^1_{p(\,\cdot\,)}(\Omega_R)$ is a subspace of the space $\big\{v(x),\ x \in \Omega_R :\ \partial_i v_j \in L_{p_{ij}(\,\cdot\,)}(\Omega_R)\ (i = \overline{0,n},\ j = \overline{1,N})\big\}$.

On the space $\big(C_0^1(\overline{\Omega}, \Gamma_1))\big)^N$ consider a locally convex linear topology generated by the system of seminorms: $\|\cdot\|_{W^1_{p(\,\cdot\,)}(\Omega_R)}$, $R > 0$ (see [9]), and let $\overset{\circ}{W}{}^1_{p(\,\cdot\,),\,\mathrm{loc}}(\overline{\Omega}, \Gamma_1)$, $W^1_{p(\,\cdot\,),\,\mathrm{loc}}(\overline{\Omega})$ be the completion of $\big(C_0^1(\overline{\Omega}, \Gamma_1)\big)^N$, $\big(C^1(\overline{\Omega})\big)^N$ respectively in this topology. A sequence $\{v_k\}_{k=1}^\infty$ is convergent to v in $W^1_{p(\,\cdot\,),\,\mathrm{loc}}(\overline{\Omega})$ if $\|v_k - v\|_{W^1_{p(\,\cdot\,)}(\Omega_R)} \underset{k\to\infty}{\longrightarrow} 0$ for all $R > 0$. Note that $v\big|_{\Omega_R} \in W^1_{p(\,\cdot\,)}(\Omega_R)$ for all $R > 0$ provided $v \in W^1_{p(\,\cdot\,),\,\mathrm{loc}}(\overline{\Omega})$.

II. The statement of the problem and main results

Let $p \in \mathbb{P}$. We denote by $\mathbb{A}_p$ the set of function arrays $\{A_{ij} : i = \overline{0,n},\ j = \overline{1,N}\} \equiv \{A_{ij}\}$ such that for every $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, N\}$ function A_{ij} is defined on $\Omega \times \mathbb{R}$, for every $j \in \{1, \ldots, N\}$ function A_{0j} is defined on $\Omega \times \mathbb{R}^N$, and the following conditions hold

1) for a.e. $x \in \Omega$ functions $A_{ij}(x, \cdot) : \mathbb{R} \to \mathbb{R}$, $A_{0j}(x, \cdot) : \mathbb{R}^N \to \mathbb{R}$ $(i = \overline{1,n},\ j = \overline{1,N})$ are continuous and for every $\xi \in \mathbb{R}$ and $\eta \in \mathbb{R}^N$ functions $A_{ij}(\cdot, \xi) : \Omega \to \mathbb{R}$, $A_{0j}(\cdot, \eta) : \Omega \to \mathbb{R}$ $(i = \overline{1,n},\ j = \overline{1,N})$ are measurable (the Caratheodory condition);

1′) $A_{ij}(x, 0) = 0$ $(i = \overline{0,n},\ j = \overline{1,N})$ for a.e. $x \in \Omega$;

2) for a.e. $x \in \Omega$, $\forall \eta, \widetilde{\eta} \in \mathbb{R}^N$ the inequalities hold
$$|A_{0j}(x, \eta)| \le \sum_{k=1}^{N} h_{j\,k}(x) |\eta_k|^{p_{0k}(x)/p^*_{0j}(x)} + h_j(x),\ j = \overline{1,N},$$
$$\sum_{j=1}^{N} (A_{0j}(x, \eta) - A_{0j}(x, \widetilde{\eta}))\,(\eta_j - \widetilde{\eta}_j) \ge K_0 \sum_{j=1}^{N} |\eta_j - \widetilde{\eta}_j|^{p_{0j}(x)},$$
where K_0 is some positive constant, $h_{jk} \in L_{\infty,\,\mathrm{loc}}(\overline{\Omega})$, $h_j \in L_{p^*_{0j}(\,\cdot\,),\,\mathrm{loc}}(\overline{\Omega})$ $(j = \overline{1,N}, k = \overline{1,N})$;

3) for all $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, N\}$ and for a.e. $x \in \Omega$ there exists $\partial A_{ij}(x, \xi)/\partial \xi$, $\xi \ne 0$, and $\forall \xi \in \mathbb{R}, \xi \ne 0$ they satisfy the following inequalities
$$K_{ij} |\xi|^{p_{ij}(x)-2} \le \frac{\partial A_{ij}(x, \xi)}{\partial \xi} \le \widetilde{K}_{ij} (1 + |x|)^{\sigma_{ij}} |\xi|^{p_{ij}(x)-2},$$
where $K_{ij} > 0$, $\widetilde{K}_{ij} > 0$, $\sigma_{ij} \ge 0$ are some positive constants.

Let $\mathbb{F}_p$ for all $p \in \mathbb{P}$ denote the set of matrix functions $(F_{ij}) = (F_0, F_1, \ldots, F_n)$, $F_k = \mathrm{colon}(F_{k1}, \ldots, F_{kN})$ $(k = \overline{0,n})$, such that $F_{ij} \in L_{p^*_{ij}(\,\cdot\,),\,\mathrm{loc}}(\overline{\Omega})$ $(i = \overline{0,n}, j = \overline{1,N})$. On $\mathbb{F}_p$ introduce the topology of Cartesian product of locally convex spaces $L_{p^*_{ij}(\,\cdot\,),\,\mathrm{loc}}(\overline{\Omega})$ $(i = \overline{0,n},$ $j = \overline{1,N})$.

Note that real numbers r, r_k $(k \in \mathbb{N})$ and $q > 1$ satisfy the inequality $(|r_k|^{q-2} r_k - |r|^{q-2} r)(r_k - r) \ge 0$ $(k \in \mathbb{N})$ and we have $(|r_k|^{q-2} r_k - |r|^{q-2} r)(r_k - r) \underset{k\to\infty}{\longrightarrow} 0$ if and only if $r_k \underset{k\to\infty}{\longrightarrow} r$. Thus on the linear space $\overset{\circ}{W}{}^1_{p(\,\cdot\,),\,\mathrm{loc}}(\overline{\Omega}, \Gamma_1)$ we can introduce such concept of convergence that the sequence of elements $\{v_k\}_{k=1}^\infty$ is convergent to v if
$$\int\limits_{\Omega_R} \Big\{ \sum_{j=1}^{N}\sum_{i=1}^{n} \big(|\partial_i v_j^k|^{p_{ij}(x)-2} \partial_i v_j^k - |\partial_i v_j|^{p_{ij}(x)-2} \partial_i v_j \big) \times$$
$$\times \big(\partial_i v_j^k - \partial_i v_j\big) + \sum_{j=1}^{N} |v_j^k - v_j|^{p_{0j}(x)} \Big\}\, d\,x \underset{k\to\infty}{\longrightarrow} 0$$
for all $R > 0$. This space with such concept of convergence we denote by $\mathbb{U}_p$.

Now formulate the investigated problem. Let $\widetilde{\mathbb{P}} \subset \mathbb{P}$ and $\widetilde{\mathbb{A}}_p \subset \mathbb{A}_p$, $\widetilde{\mathbb{F}}_p \subset \mathbb{F}_p$, $\widetilde{\mathbb{U}}_p \subset \mathbb{U}_p$ for $p \in \widetilde{\mathbb{P}}$. The main problem $\mathbf{SA}\big(\widetilde{\mathbb{A}}_p, \widetilde{\mathbb{F}}_p, \widetilde{\mathbb{U}}_p : p \in \widetilde{\mathbb{P}}\big)$ (**S**ystem of equations in **A**nisotropic space) is to find for every $p \in \widetilde{\mathbb{P}}$ and $\{A_{ij}\} \in \widetilde{\mathbb{A}}_p$, $(F_{ij}) \in \widetilde{\mathbb{F}}_p$ the set $\mathbf{SSA}\big(\{A_{ij}\}, (F_{ij})\big)$ (**S**olutions of **S**ystem of equations in **A**nisotropic space) of functions $u \in \widetilde{\mathbb{U}}_p$ such that the equality
$$\int\limits_{\Omega} \sum_{j=1}^{N} \Big\{ \sum_{i=1}^{n} A_{ij}(x, \partial_i u_j)\, \partial_i v_j + A_{0j}(x, u)\, v_j \Big\}\, d\,x =$$
$$= \int\limits_{\Omega} \sum_{j=1}^{N}\sum_{i=0}^{n} F_{ij}(x)\, \partial_i v_j\, d\,x \qquad (2)$$
holds for all $v \in \overset{\circ}{W}{}^1_{p(\,\cdot\,),\,\mathrm{loc}}(\overline{\Omega}, \Gamma_1)$, supp v is a bounded set.

Remark 1. It is seen from the statement of the investigated problem that the restricting condition $\mathbf{1'}$) is not essential. Otherwise we can introduce new functions

$$\widetilde{A}_i(x,\xi)\overset{\text{def}}{=}A_i(x,\xi)-A_i(x,0),\quad \widetilde{F}_i(x)\overset{\text{def}}{=}F_i(x)-A_i(x,0)$$

for a.e. $x\in\Omega$ $(i=\overline{0,n})$, and rewrite the identity (2) with $\widetilde{A}_i$, $\widetilde{F}_i$ instead of A_i, F_i respectively, where functions $\widetilde{A}_i(x,0)=0$ $(i=\overline{0,n})$ for a.e. $x\in\Omega$.

Hereinafter we will use the following concepts. We'll say that $\mathbf{SA}\big(\widetilde{\mathbb{A}}_p,\,\widetilde{\mathbb{F}}_p,\,\widetilde{\mathbb{U}}_p : p\in\widetilde{\mathbb{P}}\big)$ is a *solvable* (*unique*, *uniquely solvable*) problem, if for every $p\in\widetilde{\mathbb{P}}$, arbitrary $\{A_{ij}\}\in\widetilde{\mathbb{A}}_p$ and $(F_{ij})\in\widetilde{\mathbb{F}}_p$ the set $\mathbf{SSA}\big(\{A_{ij}\},(F_{ij})\big)$ is *non-empty* (*contains at most one element*, *has exactly one element*).

We'll say that $\mathbf{SA}\big(\widetilde{\mathbb{A}}_p,\widetilde{\mathbb{F}}_p,\widetilde{\mathbb{U}}_p : p\in\widetilde{\mathbb{P}}\big)$ *is a weakly well-posed* problem, if it is uniquely solvable and for all $p\in\widetilde{\mathbb{P}}$, arbitrary elements $\{A_{ij}\}\in\widetilde{\mathbb{A}}_p$, $(F_{ij})\in\widetilde{\mathbb{F}}_p$ and sequence $\{(F^k_{ij})\}_{k=1}^{\infty}\subset\widetilde{\mathbb{F}}_p$ such that $(F^k_{ij})\underset{k\to\infty}{\longrightarrow}(F_{ij})$ in $\mathbb{F}_p$ we have $u^k\underset{k\to\infty}{\longrightarrow}u$ in $\mathbb{U}_p$, where $u^k\in\mathbf{SSA}\big(\{A_{ij}\},(F^k_{ij})\big)$, $k\in\mathbb{N}$, $u\in\mathbf{SSA}\big(\{A_{ij}\},(F_{ij})\big)$.

It is obvious that problem $\mathbf{SA}\big(\widetilde{\mathbb{A}}_p,\,\widetilde{\mathbb{F}}_p,\,\widetilde{\mathbb{U}}_p : p\in\widetilde{\mathbb{P}}\big)$ can be formally interpreted as the boundary value problem

$$-\sum_{i=1}^{n}\frac{d}{dx_i}A_{ij}(x,\partial_i u_j)+A_{0j}(x,u)=$$

$$=-\sum_{i=0}^{n}\partial_i F_{ij}(x),\quad x\in\Omega,\ j=\overline{1,N},$$

$$u_j(x)=0,\ x\in\Gamma_1,\qquad \frac{\partial u_j(x)}{\partial\nu_A}=0,\ x\in\Gamma_2,\ j=\overline{1,N},$$

where $\{A_{ij}\}\in\widetilde{\mathbb{A}}_p$, $(F_{ij})\in\widetilde{\mathbb{F}}_p$; $\partial u_j(x)/\partial\nu_A\overset{\text{def}}{=}$ $\overset{\text{def}}{=}\sum_{i=1}^{n}A_{ij}(x,\partial_i u_j)\,\nu_i(x)$, $x\in\Gamma_2$.

We search for the sets $\widetilde{\mathbb{P}}$ and $\big\{\widetilde{\mathbb{A}}_p,\,\widetilde{\mathbb{F}}_p,\,\widetilde{\mathbb{U}}_p : p\in\widetilde{\mathbb{P}}\big\}$ such that problem $\mathbf{SA}\big(\widetilde{\mathbb{A}}_p,\,\widetilde{\mathbb{F}}_p,\,\widetilde{\mathbb{U}}_p : p\in\widetilde{\mathbb{P}}\big)$ is uniquely solvable or weakly well-posed. Note that we don't want to put any restrictions on increasing of the elements of the sets $\widetilde{\mathbb{F}}_p$, $\widetilde{\mathbb{U}}_p$ $(p\in\widetilde{\mathbb{P}})$ at infinity.

Here we make the following choice of the requisite sets. Let $\mathbb{P}^*$ be a set of elements $p\in\mathbb{P}$ such that

$$p^-_{0j}\overset{\text{def}}{=}\operatorname*{ess\,inf}_{x\in\Omega}p_{0j}(x)\ge 2,\qquad p^+_{0j}\overset{\text{def}}{=}\operatorname*{ess\,sup}_{x\in\Omega}p_{0j}(x)<\infty,$$

$$p^-_{ij}\overset{\text{def}}{=}\operatorname*{ess\,inf}_{x\in\Omega}p_{ij}(x)>1,\qquad p^+_{ij}\overset{\text{def}}{=}\operatorname*{ess\,sup}_{x\in\Omega}p_{ij}(x)\le 2,$$

$$q^-_{ij}\overset{\text{def}}{=}\operatorname*{ess\,inf}_{x\in\Omega}q_{ij}(x)>n,\qquad q^+_{ij}\overset{\text{def}}{=}\operatorname*{ess\,sup}_{x\in\Omega}q_{ij}(x)<+\infty,$$

where $q_{ij}(x)\overset{\text{def}}{=}\frac{p_{0j}(x)p_{ij}(x)}{p_{0j}(x)-p_{ij}(x)}$, $x\in\Omega$ $(i=\overline{1,n}, j=\overline{1,N})$.

For all $p\in\mathbb{P}^*$ define $\mathbb{A}^*_p$ as the set of function arrays $\{A_{ij}\}\in\mathbb{A}_p$ satisfying the additional condition

4) constants $\sigma_{1j},\ldots,\sigma_{nj}$ $(j=\overline{1,N})$ in condition **3**) are such that $n-q^-_{ij}+\sigma_{ij}\frac{q^+_{ij}}{p^-_{ij}}<0$ $(i=\overline{1,n},\ j=\overline{1,N})$.

Let $\mathbb{F}^*_p\overset{\text{def}}{=}L_{p^*_{01}(\,\cdot\,),\,\mathrm{loc}}(\overline{\Omega})\times\cdots\times L_{p^*_{0N}(\,\cdot\,),\,\mathrm{loc}}(\overline{\Omega})$.

Theorem 1. *The following statements are valid. 1) The problem* $\mathbf{SA}\big(\mathbb{A}^*_p,\,\mathbb{F}_p,\,\mathbb{U}_p : p\in\mathbb{P}^*\big)$ *is uniquely solvable and for all* $p\in\mathbb{P}^*$, $\{A_{ij}\}\in\mathbb{A}^*_p$, $(F_{ij})\in\mathbb{F}_p$ *the (unique) function* $u\in\mathbf{SSA}(\{A_{ij}\},(F_{ij}))$ *for every* $R_0>0$, $R\ge 1$, $R_0<R$, *satisfies the estimate*

$$\int\limits_{\Omega_{R_0}}\sum_{j=1}^{N}\Big\{\sum_{i=1}^{n}|\partial_i u_j(x)|^{p_{ij}(x)}+|u_j(x)|^{p_{0j}(x)}\Big\}d\,x\le$$

$$\le\frac{R^s}{(R-R_0)^s}\Big[C_1\,R^{n-\gamma}+C_2\int\limits_{\Omega_R}\sum_{j=1}^{N}\sum_{i=0}^{n}|F_{ij}(x)|^{p^*_{ij}(x)}d\,x+$$

$$+C_3\int\limits_{\Omega_R}\sum_{j=1}^{N}|h_j(x)|^{p^*_{0j}(x)}\,d\,x\Big],\tag{3}$$

where $\gamma=\min\big\{q^-_{ij}-\sigma_{ij}\frac{q^+_{ij}}{p^-_{ij}} : 1\le i\le n, 1\le j\le N\big\}$; $s>\max\big\{q^+_{ij}:1\le i\le n,1\le j\le N\big\}$; C_1,C_2,C_3 *are some positive constants depending only on* n, s, p^-_{ij}, p^+_{ij} $(i=\overline{0,n},\ j=\overline{1,N})$, q^-_{ij}, q^+_{ij} $(i=\overline{1,n},\ j=\overline{1,N})$.

2) The problem $\mathbf{SA}\big(\mathbb{A}^*_p,\,\mathbb{F}^*_p,\,\mathbb{U}_p : p\in\mathbb{P}^*\big)$ *is weakly well-posed and its solution satisfies the estimate (3) with corresponding simplification.*

III. Auxiliary statements

It is easy to establish that following Lemma is valid (see [10], p. 312).

Lemma 1. *Let* $r\in L_\infty(\Omega)$, $r^-\overset{\text{def}}{=}\operatorname*{ess\,inf}_{x\in\Omega}r(x)>1$, $r^+\overset{\text{def}}{=}\operatorname*{ess\,sup}_{x\in\Omega}r(x)<+\infty$. *Then for every function* $v\in L_{r(\,\cdot\,),\,\mathrm{loc}}(\Omega)$ *and number* $R>0$ *the following inequalities hold*

$$\min\Big\{(\rho_{r,R}(v))^{\frac{1}{r^-}},(\rho_{r,R}(v))^{\frac{1}{r^+}}\Big\}\le\|v\|_{L_{r(\,\cdot\,)}(\Omega_R)}\le$$

$$\le\max\Big\{(\rho_{r,R}(v))^{\frac{1}{r^-}},(\rho_{r,R}(v))^{\frac{1}{r^+}}\Big\},$$

$$\min\Big\{\|v\|^{r^-}_{L_{r(\,\cdot\,)}(\Omega_R)},\|v\|^{r^+}_{L_{r(\,\cdot\,)}(\Omega_R)}\Big\}\le\rho_{r,R}(v)\le$$

$$\le\max\Big\{\|v\|^{r^-}_{L_{r(\,\cdot\,)}(\Omega_R)},\|v\|^{r^+}_{L_{r(\,\cdot\,)}(\Omega_R)}\Big\}.$$

Remark 2. For all $a\ge 0, b\ge 0,\varepsilon>0,\nu>1$ Young's inequality ([9]): $a\,b\le\frac{a^\nu}{\nu}+\frac{b^{\nu^*}}{\nu^*}$, $\nu^*=\frac{\nu}{\nu-1}$, implies the inequality

$$a\,b\le\varepsilon a^\nu+\varepsilon^{1-\nu^*}\,b^{\nu^*}.\tag{4}$$

Remark 3. Young's inequality ([9]): $a\,b\,c\le\frac{a^{\nu_1}}{\nu_1}+$ $+\frac{b^{\nu_2}}{\nu_2}+\frac{c^{\nu_3}}{\nu_3}$, $a\ge 0,b\ge 0,c\ge 0,\nu_1>1,\nu_2>1,\nu_3>1$, $\frac{1}{\nu_1}+\frac{1}{\nu_2}+\frac{1}{\nu_3}=1$, simply implies the inequality

$$a\,b\,c\le\varepsilon a^{\nu_1}+\varepsilon b^{\nu_2}+\varepsilon^{1-\nu_3}\,c^{\nu_3},\quad\varepsilon>0.\tag{5}$$

Lemma 2. *Let $\{A_{ij}\} \in \mathbb{A}_p$ for $p \in \mathbb{P}^*$. For a.e. $x \in \Omega$ and arbitrary $\xi_1, \xi_2 \in \mathbb{R}$ the following inequalities are valid*

$$(A_{ij}(x,\xi_1) - A_{ij}(x,\xi_2))(\xi_1 - \xi_2) \geq$$
$$\geq K_{ij}^{-}\big(|\xi_1|^{p_{ij}(x)-2}\xi_1 - |\xi_2|^{p_{ij}(x)-2}\xi_2\big)\big(\xi_1 - \xi_2\big), \quad (6)$$
$$(A_{ij}(x,\xi_1) - A_{ij}(x,\xi_2))(\xi_1 - \xi_2) \leq$$
$$\leq K_{ij}^{+}(1+|x|)^{\sigma_{ij}}|\xi_1 - \xi_2|^{p_{ij}(x)}, \quad (7)$$

where K_{ij}^{-}, K_{ij}^{+} $(i = \overline{1,n},\ j = \overline{1,N})$ are some positive constants.

□ *Proof.* The inequalities (6), (7) are proved in [7]. ■

Lemma 3. *Let $R_* \geq 1$, $\{A_{ij}\} \in \mathbb{A}_p^*$, $F_{ij} \in L_{p_{ij}^*(\,\cdot\,)}(\Omega_{R_*})$ $(i = \overline{1,n},\ j = \overline{1,N})$ for some $p \in \mathbb{P}^*$ and for every $l \in \{1,2\}$ functions $F_{0j}^l \in L_{p_{0j}^*(\,\cdot\,)}(\Omega_{R_*})$ $(j = \overline{1,N})$, $u^l \in W^1_{p(\,\cdot\,)}(\Omega_{R_*})$ are such that $u^l = 0$ on Γ_{1,R_*} and*

$$\int\limits_{\Omega_{R_*}} \sum_{j=1}^{N}\Big\{\sum_{i=1}^{n} A_{ij}(x,\partial_i u_j^l)\,\partial_i v_j + A_{0j}(x,u^l)\,v_j\Big\}\,dx =$$
$$= \int\limits_{\Omega_{R_*}} \sum_{j=1}^{N}\Big\{\sum_{i=1}^{n} F_{ij}(x)\,\partial_i v_j + F_{0j}^l(x)\,v_j\Big\}\,dx \quad (8)$$

for arbitrary $v \in W^1_{p(\,\cdot\,)}(\Omega_{R_*})$, $v|_{\Gamma_{1,R_*}\cup S_{R_*}} = 0$.
Then for every $R_0 > 0, R \geq 1,\ R_0 < R \leq R_*$, the inequality

$$\int\limits_{\Omega_{R_0}} \sum_{j=1}^{N}\Big\{\sum_{i=1}^{n}(|\partial_i u_j^1|^{p_{ij}(x)-2}\partial_i u_j^1 - |\partial_i u_j^2|^{p_{ij}(x)-2}\partial_i u_j^2)\times$$
$$\times(\partial_i u_j^1 - \partial_i u_j^2) + |u_j^1(x) - u_j^2(x)|^{p_{0j}(x)}\Big\}dx \leq \left(\frac{R}{R-R_0}\right)^s \times$$
$$\times\Big[C_4 R^{n-\gamma} + C_5 \int\limits_{\Omega_R}\sum_{j=1}^{N}|F_{0j}^1(x) - F_{0j}^2(x)|^{p^*_{0j}(x)}\,dx\Big] \quad (9)$$

holds, where s and γ are the same as in the Theorem 1; C_4, C_5 are positive constants which do not depend on u^l, F_{0j}^l $(l = \overline{1,2}, j = \overline{1,N})$, F_{ij} $(i = \overline{1,n}, j = \overline{1,N})$.

□ *Proof.* Pick somehow and fix numbers R_0, R provided that $0 < R_0 < R \leq R_*$, $R \geq 1$. Subtracting integral equalities derived from (8) for $l = 1$ and $l = 2$ and putting in the obtained equality (see [2], p. 220) $v = w\zeta^s$, where $w \overset{\text{def}}{=} u^1 - u^2$,

$$\zeta(x) = \begin{cases} \frac{1}{R}(R^2 - |x|^2), & |x| < R, \\ 0, & |x| \geq R, \end{cases} \quad (10)$$

$s > 1$ is sufficiently large number (value of s will be defined more precisely later), we derive the equality

$$\int\limits_{\Omega_R}\sum_{j=1}^{N}\Big\{\sum_{i=1}^{n}(A_{ij}(x,\partial_i u_j^1) - A_{ij}(x,\partial_i u_j^2))\,\partial_i w_j\,\zeta^s +$$
$$+(A_{0j}(x,u^1) - A_{0j}(x,u^2))\,w_j\,\zeta^s\Big\}dx =$$
$$= \int\limits_{\Omega_R}\sum_{j=1}^{N}(F_{0j}^1 - F_{0j}^2)\,w_j\,\zeta^s\,dx -$$
$$-s\int\limits_{\Omega_R}\sum_{j=1}^{N}\sum_{i=1}^{n}(A_{ij}(x,\partial_i u_j^1) - A_{ij}(x,\partial_i u_j^2))\,w_j\,\zeta^{s-1}\,\partial_i\zeta\,dx. \quad (11)$$

Let's estimate each term of (11). Using inequality (4), we deduce

$$\int\limits_{\Omega_R}\sum_{j=1}^{N}(F_{0j}^1 - F_{0j}^2)\,w_j\,\zeta^s\,dx \leq \eta_1\int\limits_{\Omega_R}\sum_{j=1}^{N}|w_j(x)|^{p_{0j}(x)}\,\zeta^s\,dx +$$
$$+C_6(\eta_1)\int\limits_{\Omega_R}\sum_{j=1}^{N}|F_{0j}^1(x) - F_{0j}^2(x)|^{p^*_{0j}(x)}\,\zeta^s\,dx, \quad (12)$$

where η_1 is an arbitrary number out of $(0;1)$; $C_6(\eta_1) = \max\limits_{1\leq j\leq N} \eta_1^{1-p_{0j}^+}$.

Using the same arguments as in [7] (see p.77), for $s > \max\{q_{ij}^+ : 1 \leq i \leq n, 1 \leq j \leq N\}$ we obtain

$$-s\int\limits_{\Omega_R}\sum_{j=1}^{N}\sum_{i=1}^{n}(A_{ij}(x,\partial_i u_j^1) - A_{ij}(x,\partial_i u_j^2))\,w_j\,\zeta^{s-1}\,\partial_i\zeta\,dx \leq$$
$$\leq sn\eta_2\int\limits_{\Omega_R}\sum_{j=1}^{N}\Big\{\sum_{i=1}^{n}(A_{ij}(x,\partial_i u_j^1) - A_{ij}(x,\partial_i u_j^2))\,\partial_i w_j +$$
$$+|w_j(x)|^{p_{0j}(x)}\Big\}\,\zeta^s\,dx + sC_7(\eta_2)\sum_{j=1}^{N}\sum_{i=1}^{n} R^{n+s-q_{ij}^- + \sigma_{ij}\frac{q_{ij}^+}{p_{ij}^-}}, \quad (13)$$

where $\eta_2 \in (0;1)$ is an arbitrary number; $C_7(\eta_2)$ is a positive constant.

From (11), relying on condition **2**), (12) and (13), with sufficiently small values η_1 and η_2 we get

$$\int\limits_{\Omega_R}\sum_{j=1}^{N}\Big\{\sum_{i=1}^{n}(A_{ij}(x,\partial_i u_j^1) - A_{ij}(x,\partial_i u_j^2))\partial_i w_j +$$
$$+|w_j(x)|^{p_{0j}(x)}\Big\}\,\zeta^s\,dx \leq C_8\sum_{j=1}^{N}\sum_{i=1}^{n} R^{n+s-q_{ij}^- + \sigma_{ij}q_{ij}^+/p_{ij}^-} +$$
$$+C_9\int\limits_{\Omega_R}\sum_{j=1}^{N}|F_{0j}^1(x) - F_{0j}^2(x)|^{p^*_{0j}(x)}\,\zeta^s\,dx, \quad (14)$$

where C_8, C_9 are some positive constants.

Note that $0 \le \zeta(x) \le R$ when $x \in \mathbb{R}^n$ and $\zeta(x) \ge R - R_0$ when $|x| \le R_0$, where $R_0 \in (0,R)$ is any number. Taking into account stated above and, in particular, that $R \ge 1$, in virtue of inequality (6) from (14) we conclude

$$\int\limits_{\Omega_{R_0}} \sum_{j=1}^{N}\Big\{\sum_{i=1}^{n}(|\partial_i u_j^1|^{p_{ij}(x)-2}\partial_i u_j^1 - |\partial_i u_j^2|^{p_{ij}(x)-2}\partial_i u_j^2)\times$$

$$\times(\partial_i u_j^1 - \partial_i u_j^2) + |u_j^1(x) - u_j^2(x)|^{p_{0j}(x)}\Big\} d\,x \le$$

$$\le \left(\frac{R}{R-R_0}\right)^s \Big[C_{10}\sum_{j=1}^{N}\sum_{i=1}^{n} R^{n-q_{ij}^- + \sigma_{ij} q_{ij}^+/p_{ij}^-} +$$

$$+C_{11}\int\limits_{\Omega_R}\sum_{j=1}^{N}|F_j^{0,1}(x) - F_j^{0,2}(x)|^{p^*_{0j}(x)}\,d\,x\Big], \qquad (15)$$

where C_{10}, C_{11} are positive constants depending only on n, s, p_{ij}^-, p_{ij}^+ $(i=\overline{0,n},\ j=\overline{1,N})$, q_{ij}^-, q_{ij}^+ $(i=\overline{1,n},\ j=\overline{1,N})$. Observing in (15) that $n - q_{ij}^- + \sigma_{ij}q_{ij}^+/p_{ij}^- \le n - \gamma$ $(i=\overline{1,n},\ j=\overline{1,N})$, where $\gamma = \min\{q_{ij}^- - \sigma_{ij}q_{ij}^+/p_{ij}^- : 1 \le i \le n,\ 1 \le j \le N\}$, we obtain inequality (9).■

Lemma 4. *Let $p\in\mathbb{P}^*$ and $\{A_{ij}\}\in\mathbb{A}_p^*$, $(F_{ij}) \in \mathbb{F}_p$, $u \in W^1_{p(\,\cdot\,)}(\Omega_{R_*})$ are such that the integral identity (2) holds for arbitrary $v \in W^1_{p(\,\cdot\,)}(\Omega_{R_*})$, $v|_{\Gamma_{1,R_*}\cup S_{R_*}}=0$, where $R_* > 1$ is some number.*

Then for every numbers $R_0 > 0, R \ge 1$, $R_0 < R \le R_*$, the inequality (3) is fulfilled.

□ *Proof.* Let R be any number in the interval $[1; R_*]$. Put in (2) $v = u\zeta^s$, where ζ is defined in (10). After simple transformations we get

$$\int\limits_{\Omega_R}\sum_{j=1}^{N}\Big\{\sum_{i=1}^{n}A_{ij}(x,\partial_i u_j)\,\partial_i u_j + A_{0j}(x,u)\,u_j\Big\}\,\zeta^s\,d\,x =$$

$$= \int\limits_{\Omega_R}\sum_{j=1}^{N}\sum_{i=0}^{n}F_{ij}\partial_i u_j\zeta^s dx + s\int\limits_{\Omega_R}\sum_{j=1}^{N}\sum_{i=1}^{n}F_{ij}u_j\zeta^{s-1}\partial_i\zeta\, d\,x -$$

$$-s\int\limits_{\Omega_R}\sum_{j=1}^{N}\sum_{i=1}^{n}A_{ij}(x,\partial_i u_j)\,u_j\,\zeta^{s-1}\partial_i\zeta\,d\,x. \qquad (16)$$

Arguing the same way as in proof of Lemma 3 (see (12),(13)), we derive

$$\int\limits_{\Omega_R}\sum_{j=1}^{N}\Big\{\sum_{i=1}^{n}|\partial_i u_j(x)|^{p_{ij}(x)} + |u_j(x)|^{p_{0j}(x)}\Big\}\,\zeta^s(x)\,d\,x \le$$

$$\le \widehat{C}_1\sum_{j=1}^{N}\sum_{i=1}^{n}R^{n+s-q^-_{ij}+\sigma_{ij}\frac{q^+_{ij}}{p^-_{ij}}} +$$

$$+\widehat{C}_2\int\limits_{\Omega_R}\sum_{j=1}^{N}\sum_{i=0}^{n}|F_{ij}(x)|^{p^*_{ij}(x)}\,\zeta^s(x)d\,x +$$

$$+\widehat{C}_3\int\limits_{\Omega_R}\sum_{j=1}^{N}|h_j(x)|^{p^*_{0j}(x)}\zeta^s(x)\,d\,x, \qquad (17)$$

where $s > \max\{q_{ij}^+ : 1 \le i \le n, 1 \le j \le N\}$ is an arbitrary number; $\widehat{C}_1$, $\widehat{C}_2$, $\widehat{C}_3$ are some positive constants depending only on n, s, p_{ij}^-, p_{ij}^+ $(i = \overline{0,n}, j = \overline{1,N})$, q_{ij}^-, q_{ij}^+, $(i = \overline{1,n}, j = \overline{1,N})$.

Proceeding just as in proof of Lemma 3 (see (15)), we obtain the estimate (3). ■

IV. Proof of the Theorem 1

Solvability of the problem SA$(\mathbb{A}_p^*$, $\mathbb{F}_p$, $\mathbb{U}_p$: $p\in\mathbb{P}^*)$. Let $\{A_{ij}\} \in \mathbb{A}_p^*$, $(F_{ij}) \in \mathbb{F}_p$ for some $p \in \mathbb{P}^*$ and k is an arbitrary natural number. Put $F_{ij}^k \overset{\text{def}}{=} F_{ij}\chi_k$, $i = \overline{0,n}$, $j = \overline{1,N}$, where $\chi_k \in C^\infty(\overline{\Omega})$, $0 \le \chi_k \le 1$ on $\overline{\Omega}$, $\chi_k \equiv 1$ on $\Omega_{k-3/4}$, $\chi_k \equiv 0$ on $\Omega\backslash\Omega_{k-1/2}$.

Define U_k as the subspace of the space $W^1_{p(\,\cdot\,)}(\Omega_k)$, consisted of functions, satisfying the condition $v|_{\Gamma_{1,k}\cup S_k} = 0$ in a sense of trace. Let U_k' denote the adjoint to U_k space and $\langle\cdot,\cdot\rangle_k$ denote the canonical bilinear form on $U_k' \times U_k$.

Define the operator $L_k : U_k \longrightarrow U_k'$ as follows

$$\langle L_k u, v\rangle_k \overset{\text{def}}{=} \int\limits_{\Omega_k}\sum_{j=1}^{N}\Big\{\sum_{i=1}^{n}A_{ij}(x,\partial_i u_j)\,\partial_i v_j +$$

$$+A_{0j}(x,u)\,v_j\Big\}\,d\,x \quad \forall u, v \in U_k.$$

It is easy to verify that the operator $L_k : U_k \longrightarrow U_k'$ is strictly monotone, bounded, coercive and hemicontinuous. This fact can be proved by analogy to the case of constant exponent of nonlinearity with the aid of inequalities in Lemma 1.

We search for a function $u^k \in U_k$ satisfying for all $v \in U_k$ the equality

$$\langle L_k u^k, v\rangle_k = \int\limits_{\Omega_k}\sum_{j=1}^{N}\sum_{i=0}^{n}F_{ij}^k\,\partial_i v_j\,d\,x. \qquad (18)$$

The existence of a function u^k can be proved by Galerkin's method (see, for instance, [11, p. 22]). The uniqueness of function u^k follows from strictly monotonicity of operator L_k.

Given functions u^k for all $k\in\mathbb{N}$. Extend them by zero in Ω. Keep the notation u^k of these extensions. We claim that the sequence $\{u^k\}_{k=1}^\infty$ contains the subsequence converging to $u\in$**SSA**$(\{A_{ij}\},(F_{ij}))$ in some sense.

Indeed, let k and l be arbitrary natural numbers and $1 < k < l$; R_0, R are arbitrary real numbers such that $0 < R_0 < R \le k-1$, $R \ge 1$. Observe that $F_{ij}^k = F_{ij}^l$ $(i = \overline{0,n},\ j = \overline{1,N})$ on Ω_{k-1}. From Lemma 3, taking $R_* = k-1$, we get

$$\int\limits_{\Omega_{R_0}}\sum_{j=1}^{N}\Big\{\sum_{i=1}^{n}(|\partial_i u_j^k|^{p_{ij}(x)-2}\partial_i u_j^k - |\partial_i u_j^l|^{p_{ij}(x)-2}\partial_i u_j^l)\times$$

$$\times(\partial_i u_j^k - \partial_i u_j^l) + |u_j^k - u_j^l|^{p_{0j}(x)}\Big\}dx \le C_4\frac{R^s}{(R-R_0)^s}R^{n-\gamma}, \qquad (19)$$

where $C_4>0$, $s>0$ are constants not depending on k, l, R_0 and R; γ is such that $n-\gamma<0$ (it can be assigned in such a way on the basis of Theorem's 1 assertion).

Let $\varepsilon>0$ be an arbitrary number. Fix any value of $R_0>0$ and take $R>\max\{1;R_0\}$ sufficiently large to make right-hand side of inequality (19) be less than ε. Then for every $k\geq R+1$ and $l>k$ the left-hand side of inequality (19) is less than ε. It means that the sequence $\{u_j^k|_{\Omega_{R_0}}\}_{k=1}^{\infty}$ is fundamental in $L_{p\,0j(\,\cdot\,)}(\Omega_{R_0})$ $(j=\overline{1,N})$. Since R_0 is an arbitrary positive number, there exist functions $u_j\in L_{p\,0j(\,\cdot\,),\mathrm{loc}}(\overline{\Omega})$ $(j=\overline{1,N})$ such that

$$u_j^k\underset{k\to\infty}{\longrightarrow}u_j\quad\text{strongly in}\quad L_{p\,0j(\,\cdot\,),\mathrm{loc}}(\overline{\Omega}),\quad j=\overline{1,N}. \tag{20}$$

Show that the sequences $\{u^k\}_{k=1}^{\infty}$, $\{A_{0j}(\cdot,u^k(\,\cdot\,))\}_{k=1}^{\infty}$, $\{A_{ij}(\cdot,\partial_i u_j^k(\,\cdot\,))\}_{k=1}^{\infty}$ are bounded in $W^1_{p(\,\cdot\,),\mathrm{loc}}(\overline{\Omega})$, $L_{p^*_{0j}(\,\cdot\,),\mathrm{loc}}(\overline{\Omega})$, $L_{p^*_{ij}(\,\cdot\,),\mathrm{loc}}(\overline{\Omega})$ $(i=\overline{1,n}, j=\overline{1,N})$ respectively.

Indeed, let R_0, R be some real numbers such that $0<R_0<R$, $R\geq 1$. According to Lemma 4 for every natural number $k>R+1$ we conclude

$$\int\limits_{\Omega_{R_0}}\sum_{j=1}^{N}\Big\{\sum_{i=1}^{n}|\partial_i u_j^k(x)|^{p_{ij}(x)}+|u_j^k(x)|^{p_{0j}(x)}\Big\}d\,x\leq C_{12}(R_0), \tag{21}$$

where $C_{12}(R_0)>0$ is constant not depending on k but possibly depending on R_0.

Combining condition **2**), inequality (7) and estimate (21), we obtain

$$\int\limits_{\Omega_{R_0}}|A_{0j}\big(x,u^k(x)\big)|^{p^*_{0j}(x)}\,d\,x\leq$$

$$\leq\int\limits_{\Omega_{R_0}}\Big(\sum_{m=1}^{N}h_{jm}|u_m^k(x)|^{\frac{p\,_{0m}(x)}{p^*_{0j}(x)}}+h_j\Big)^{p^*_{0j}(x)}\,d\,x\leq C_{13}(R_0), \tag{22}$$

$$\int\limits_{\Omega_{R_0}}|A_{ij}\big(x,\partial_i u_j^k(x)\big)|^{p^*_{ij}(x)}d\,x\leq$$

$$\leq\int\limits_{\Omega_{R_0}}\big(K^+_{ij}(1+|R_0|)^{\sigma_{ij}}\big)\big)^{p^*_{ij}(x)}\,|\partial_i u_j^k|^{p_{ij}(x)}\,d\,x\leq C_{14}(R_0), \tag{23}$$

where $i=\{1,\ldots,n\}$, $j=\{1,\ldots,N\}$, $C_{13}(R_0)>0$, $C_{14}(R_0)>0$ are constants not depending on k but probably depending on R_0.

Condition **1**) and (20)—(23) yield the existence of a subsequence $\{u^{k_m}\}_{m=1}^{\infty}$ of the sequence $\{u^k\}_{k=1}^{\infty}$ and functions $v\in W^1_{p(\,\cdot\,),\mathrm{loc}}(\overline{\Omega})$, $\chi_{ij}\in L_{p^*_{ij}(\,\cdot\,),\mathrm{loc}}(\overline{\Omega})$ $(i=\overline{0,n}, j=\overline{1,N})$ such that

$$u^{k_m}\underset{m\to\infty}{\longrightarrow}v\quad\text{weakly in}\quad W^1_{p(\,\cdot\,),\mathrm{loc}}(\overline{\Omega}), \tag{24}$$

$$u^{k_m}\underset{m\to\infty}{\longrightarrow}u\quad\text{a.e. on}\quad\Omega, \tag{25}$$

$$A_{0j}\big(\cdot,u^{k_m}(\,\cdot\,)\big)\underset{j\to\infty}{\longrightarrow}\chi_{0j}(\,\cdot\,)\quad\text{weakly in}\quad L_{p^*_{0j}(\,\cdot\,),\mathrm{loc}}(\overline{\Omega}), \tag{26}$$

$$A_{0j}\big(x,u^{k_m}(x)\big)\underset{m\to\infty}{\longrightarrow}A_{0j}\big(x,u(x)\big)\quad\text{for a.e.}\quad x\in\Omega, \tag{27}$$

$$A_{ij}\big(\cdot,\partial_i u_j^{k_m}(\,\cdot\,)\big)\underset{m\to\infty}{\longrightarrow}\chi_{ij}(\,\cdot\,)\quad\text{weakly in}\quad L_{p^*_{ij}(\,\cdot\,),\mathrm{loc}}(\overline{\Omega}). \tag{28}$$

In view of (20), (24)—(27) and Lemma 1.3 in [11, p.25] we deduce that

$$v=u,\qquad\chi_{0j}(\,\cdot\,)=A_{0j}\big(\cdot,u(\,\cdot\,)\big),\quad j=\overline{1,N}. \tag{29}$$

Show that

$$\chi_{ij}(\,\cdot\,)=A_{ij}(\cdot,\partial_i u_j(\,\cdot\,)),\qquad i=\overline{1,n},\ j=\overline{1,N}. \tag{30}$$

By virtue of inequality (6) we have

$$\int\limits_{\Omega}\sum_{j=1}^{N}\sum_{i=1}^{n}\big(A_{ij}(x,\partial_i u_j^{k_m})-A_{ij}(x,\partial_i w_j)\big)\times$$

$$\times\big(\partial_i u_j^{k_m}-\partial_i w_j\big)\,\psi\,d\,x\geq 0 \tag{31}$$

for all $m\in\mathbb{N}$, $w\in W^1_{p(\,\cdot\,),\mathrm{loc}}(\overline{\Omega})$, $\psi\in C^{1,+}_{\mathrm{c}}(\overline{\Omega})$.

Observe that for every $m\in\mathbb{N}$ the equality

$$\int\limits_{\Omega_{k_m}}\sum_{j=1}^{N}\Big\{\sum_{i=1}^{n}A_{ij}(x,\partial_i u_j^{k_m})\partial_i v_j+$$

$$+A_{0j}(x,u^{k_m})v_j-\sum_{i=0}^{n}F_{ij}^{k_m}\partial_i v_j\Big\}\,d\,x=0 \tag{32}$$

holds for all $v\in W^1_{p(\,\cdot\,),\mathrm{c}}(\overline{\Omega})$, $v|_{\Gamma_{1,k_m}}=0$, $\operatorname{supp}v\subset\overline{\Omega_{k_m}}$. Let us take $v=u^{k_m}\,\psi$, where $\psi\in C^{1,+}_{\mathrm{c}}(\overline{\Omega})$. Combining the obtained equality and (32), we conclude

$$-\int\limits_{\Omega}\sum_{j=1}^{N}\sum_{i=1}^{n}A_{ij}(x,\partial_i u_j^{k_m})\partial_i u_j^{k_m}\,\psi\,d\,x=$$

$$=\int\limits_{\Omega}\sum_{j=1}^{N}\Big\{A_{0j}(x,u^{k_m})u_j^{k_m}\,\psi-\sum_{i=0}^{n}F_{ij}^{k_m}\,\partial_i u_j^{k_m}\,\psi+$$

$$+\sum_{i=1}^{n}A_{ij}(x,\partial_i u_j^{k_m})\,u_j^{k_m}\,\partial_i\psi-\sum_{i=1}^{n}F_{ij}^{k_m}\,u_j^{k_m}\,\partial_i\psi\Big\}\,d\,x. \tag{33}$$

From (31) and (33) we have

$$\int\limits_{\Omega}\sum_{j=1}^{N}\Big\{A_{0j}(x,u^{k_m})u_j^{k_m}\,\psi-$$

$$-\sum_{i=0}^{n}F_{ij}^{k_m}\,\partial_i u_j^{k_m}\,\psi\;+\sum_{i=1}^{n}A_{ij}(x,\partial_i u_j^{k_m})\,u_j^{k_m}\,\partial_i\psi-$$

$$-\sum_{i=1}^{n}F_{ij}^{k_m}\,u_j^{k_m}\,\partial_i\psi\Big\}\,d\,x+\int\limits_{\Omega}\sum_{j=1}^{N}\sum_{i=1}^{n}\Big(A_{ij}(x,\partial_i u_j^{k_m})\,\partial_i w_j+$$

$$+A_{ij}(x,\partial_i w_j)\,(\partial_i u_j^{k_m}-\partial_i w_j)\Big)\;\psi\,d\,x\leq 0 \tag{34}$$

for every $w\in W^1_{p(\,\cdot\,),\mathrm{loc}}(\overline{\Omega})$, $\psi\in C^{1,+}_{\mathrm{c}}(\overline{\Omega})$.

Passing to the limit in (34) and keeping in mind the definition of $F_{ij}^{k_m}$, (20), (24), (26), (28) and (29), we derive

$$\int\limits_{\Omega}\sum_{j=1}^{N}\Big\{A_{0j}(x,u)u_j\,\psi-\sum_{i=0}^{n}F_{ij}\,\partial_i u_j\,\psi\;+\sum_{i=1}^{n}\chi_{ij}\,u_j\,\partial_i\psi-$$

$$-\sum_{i=1}^{n}F_{ij}\,u_j\,\partial_i\psi\Big\}\,d\,x+\int\limits_{\Omega}\sum_{j=1}^{N}\sum_{i=1}^{n}(\chi_{ij}\partial_i w_j+$$

$$+A_{ij}(x,\partial_i w_j)(\partial_i u_j-\partial_i w_j))\;\psi\,d\,x\le 0 \qquad (35)$$

for every $w\in W^1_{p(\,\cdot\,),\,\mathrm{loc}}(\overline{\Omega})$, $\psi\in C^{1,+}_{\mathrm{c}}(\overline{\Omega})$.

Let $\psi\in C^{1,+}_{\mathrm{c}}(\overline{\Omega})$ be an arbitrary function and $l\in\mathbb{N}$ be such that $\operatorname{supp}\psi\subset\Omega_{k_l}$. Put in equality (32) $v=u\,\psi$ for $m>l$ and pass to the limit as $m\to\infty$. We conclude

$$-\int\limits_{\Omega}\sum_{j=1}^{N}\sum_{i=1}^{n}\chi_{ij}\partial_i u_j\psi d\,x=\int\limits_{\Omega}\sum_{j=1}^{N}\Big\{A_{0j}(x,u)u_j\psi-$$

$$-\sum_{i=0}^{n}F_{ij}\,\partial_i u_j\,\psi\;+\sum_{i=1}^{n}\chi_{ij}\,u_j\,\partial_i\psi-\sum_{i=1}^{n}F_{ij}\,u_j\,\partial_i\psi\Big\}\,d\,x. \qquad (36)$$

From (35) and (36) it follows that

$$\int\limits_{\Omega}\sum_{j=1}^{N}\sum_{i=1}^{n}(\chi_{ij}-A_{ij}(x,\partial_i w_j))\,(\partial_i u_j-\partial_i w_j)\;\psi\,d\,x\ge 0 \qquad (37)$$

for every $w\in W^1_{p(\,\cdot\,),\,\mathrm{loc}}(\overline{\Omega})$, $\psi\in C^{1,+}_{\mathrm{c}}(\overline{\Omega})$.

Taking in (37) $w=u-\lambda g$, $\lambda>0$, $g\in W^1_{p(\,\cdot\,),\,\mathrm{loc}}(\overline{\Omega})$ and dividing by λ, we deduce

$$\int\limits_{\Omega}\sum_{j=1}^{N}\sum_{i=1}^{n}\Big(\chi_{ij}-A_{ij}\big(x,\partial_i(u_j-\lambda g_j)\big)\Big)\partial_i g\,\psi\,d\,x\ge 0$$

for every $g\in W^1_{p(\,\cdot\,),\,\mathrm{loc}}(\overline{\Omega})$. Let us tend here λ to 0. Combining Lebesgue Theorem on passage to the limit under the integral, conditions **1**), **1′**) and inequality (7), we get

$$\int\limits_{\Omega}\sum_{j=1}^{N}\sum_{i=1}^{n}(\chi_{ij}-A_{ij}(x,\partial_i u_j))\,\partial_i g\,\psi\,d\,x\ge 0. \qquad (38)$$

for every $g\in W^1_{p(\,\cdot\,),\,\mathrm{loc}}(\overline{\Omega})$.

Since (38) is fulfilled for any $g\in W^1_{p(\,\cdot\,),\,\mathrm{loc}}(\overline{\Omega})$, assigning first $g(x)=x_l$, $l=\overline{1,n}$, and then $g(x)=-x_l$, $l=\overline{1,n}$, we obtain (30).

Let $v\in\overset{\circ}{W}{}^1_{p(\,\cdot\,),\,\mathrm{loc}}(\overline{\Omega},\Gamma_1)$, supp v is a compact in $\overline{\Omega}$. In view of the definition of u^{k_m} for every $m\ge l$, where $l\in\mathbb{N}$ is such that $\operatorname{supp}v\subset\overline{\Omega_{k_l}}$, we have

$$\int\limits_{\Omega_{k_m}}\sum_{j=1}^{N}\Big\{\sum_{i=1}^{n}A_{ij}(x,\partial_i u_j^{k_m})\partial_i v_j+A_{0j}(x,u^{k_m})v_j-$$

$$-\sum_{i=0}^{n}F_{ij}^{k_m}\partial_i v_j\Big\}\,d\,x=0. \qquad (39)$$

Let us pass to the limit in (39) as $m\to+\infty$ with regard to (26) and (29), (28) and (30). As a result we obtain (2) for the given function v. As v is an arbitrary function and $0=u^{k_m}\to u$ on Γ_1, we have proved that $u\in\mathbf{SSA}\big(\{A_{ij}\},(F_{ij})\big)$.

Uniqueness of the problem $\mathbf{SA}\big(\mathbb{A}^*_p,\mathbb{F}_p,\mathbb{U}_p:p\in\mathbb{P}^*\big)$.

Let $\{A_{ij}\}\in\mathbb{A}^*_p$, $(F_{ij})\in\mathbb{F}_p$ for some $p\in\mathbb{P}^*$. We claim that the set $\mathbf{SSA}\big(\{A_{ij}\},(F_{ij})\big)$ contains at most one element. Arguing by contradiction, we assume that there are two (different) elements u^1, u^2 from $\mathbf{SSA}\big(\{A_{ij}\},(F_{ij})\big)$. From Lemma 3 ($R_*$ is an arbitrary number) we conclude

$$\int\limits_{\Omega_{R_0}}\sum_{j=1}^{N}\Big\{\sum_{i=1}^{n}(|\partial_i u_j^1|^{p_{ij}(x)-2}\partial_i u_j^1-|\partial_i u_j^2|^{p_{ij}(x)-2}\partial_i u_j^2)\times$$

$$\times(\partial_i u_j^1-\partial_i u_j^2)+|u_j^1-u_j^2|^{p_{0j}(x)}\Big\}d\,x\le C_4\frac{R^s}{(R-R_0)^s}R^{n-\gamma}, \qquad (40)$$

where R_0, R are some constants, $0<R_0<R$, $R\ge 1$; $\gamma>0$ is such that $n-\gamma<0$; $C_4>0$, s are the constants not depending on R_0 and R.

Fix $R_0>0$ and pass to the limit in (40) as $R\to+\infty$. As a result we obtain $u^1=u^2$ on Ω_{R_0}. Since $R_0>0$ is an arbitrary number, $u_1=u_2$ a.e. on Ω.

Weakly well-posedness of the problem $\mathbf{SA}\big(\mathbb{A}^*_p,\mathbb{F}^*_p,\mathbb{U}_p:p\in\mathbb{P}^*\big)$.

Problem $\mathbf{SA}\big(\mathbb{A}^*_p,\mathbb{F}^*_p,\mathbb{U}_p:p\in\mathbb{P}^*\big)$ is a partial case of problem $\mathbf{SA}\big(\mathbb{A}^*_p,\mathbb{F}_p,\mathbb{U}_p:p\in\mathbb{P}^*\big)$, therefore its uniquely solvability follows from uniquely solvability of problem $\mathbf{SA}\big(\mathbb{A}^*_p,\mathbb{F}_p,\mathbb{U}_p:p\in\mathbb{P}^*\big)$.

Let $\{A_{ij}\}\in\mathbb{A}^*_p$, $(F^k_{0j})\underset{k\to\infty}{\longrightarrow}(F_{0j})$ in $\mathbb{F}^*_p$ and $u\in\mathbf{SSA}\big(\{A_{ij}\},(F_{ij})\big)$, $u_k\in\mathbf{SSA}\big(\{A_{ij}\},(F^k_{ij})\big)$, $k\in\mathbb{N}$. In view of the definition of the functions u and u^k, $k\in\mathbb{N}$, we have

$$\int\limits_{\Omega}\sum_{j=1}^{N}\Big\{\sum_{i=1}^{n}A_{ij}(x,\partial_i u_j)\partial_i v_j+A_{0j}(x,u)v_j-F_{0j}v_j\Big\}dx=0, \qquad (41)$$

$$\int\limits_{\Omega}\sum_{j=1}^{N}\Big\{\sum_{i=1}^{n}A_{ij}(x,\partial_i u_j^k)\partial_i v_j+A_{0j}(x,u^k)v_j-F^k_{0j}v_j\Big\}dx=0, \qquad (42)$$

where $v\in\overset{\circ}{W}{}^1_{p(\,\cdot\,),\,\mathrm{loc}}(\overline{\Omega},\Gamma_1)$, $\operatorname{supp}v$ is a compact in $\overline{\Omega}$. Let R_0 and R be arbitrary constants such that $0<R_0<R$, $R\ge 1$. From (41) and (42) by virtue of Lemma 3 we deduce for arbitrary k

$$\int\limits_{\Omega_{R_0}}\sum_{j=1}^{N}\Big\{\sum_{i=1}^{n}(|\partial_i u_j^k|^{p_{ij}(x)-2}\partial_i u_j^k-|\partial_i u_j|^{p_{ij}(x)-2}\partial_i u_j)\times$$

$$\times(\partial_i u_j^k-\partial_i u_j)+|u_j^k-u_j|^{p_{0j}(x)}\Big\}\,d\,x\le\left(\frac{R}{R-R_0}\right)^s\times$$

$$\times\Big[C_4 R^{n-\gamma} + C_5 \int\limits_{\Omega_R} \sum_{j=1}^{N} |F_{0j}^k - F_{0j}|^{p^*_{0j}(x)}\, d\,x\Big]. \qquad (43)$$

Let $\varepsilon > 0$ be any however small number. Fix arbitrary selected $R_0 > 0$ and pick $R \geq \max\{1; 2R_0\}$ so large that

$$C_5 \left(\frac{R}{R-R_0}\right)^s R^{n-\gamma} < \frac{\varepsilon}{2}. \qquad (44)$$

Observing $\|F_{ij}^k - F_{0j}\|_{L_{p^*_{0j}(\cdot)}(\Omega_R)} \underset{k\to\infty}{\longrightarrow} 0$, $j = \overline{1,N}$, we derive that the left-hand side of (43) tends to zero when $k \to \infty$. Because of $\frac{R}{R-R_0} \leq 1 + \frac{R_0}{R-R_0} \leq 2$, all said above yields the existence of a natural number k_0 such that for every $k \geq k_0$

$$C_5 \left(\frac{R}{R-R_0}\right)^s \int\limits_{\Omega_R} \sum_{j=1}^{N} |F_{0j}^k(x) - F_{0j}(x)|^{p^*_{0j}(x)}\, d\,x < \frac{\varepsilon}{2}. \qquad (45)$$

Taking into account (44) and (45), from (43) we deduce for all $k \geq k_0$

$$\int\limits_{\Omega_{R_0}} \sum_{j=1}^{N} \Big\{ \sum_{i=1}^{n} (|\partial_i u_j^k|^{p_{ij}(x)-2} \partial_i u_j^k - |\partial_i u_j|^{p_{ij}(x)-2} \partial_i u_j) \times$$

$$\times (\partial_i u_j^k - \partial_i u_j) + |u_j^k(x) - u_j(x)|^{p_{0j}(x)} \Big\}\, d\,x \leq \varepsilon.$$

Hence it follows $u^k \underset{k\to\infty}{\longrightarrow} u$ in $\mathbb{U}_\mathrm{p}$. Thus we have proved the well-posedness of the problem $\mathbf{SA}(\mathbb{A}_p^*, \mathbb{F}_p^*, \mathbb{U}_p : p \in \mathbb{P}^*)$. ■

Thus we've considered the boundary problems for the system of elliptic equations in general anisotropic Lebesgue-Sobolev spaces. We've proved the weakly well-posedness of such problems in class of functions without conditions at infinity. The boundary conditions are mixed. The coefficients in the main part of the equations allow the polynomial growth with respect to space variable.

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КРАЙОВІ ЗАДАЧІ ДЛЯ СИСТЕМ ЕЛІПТИЧНИХ РІВНЯНЬ З АНІЗОТРОПНОЮ НЕЛІНІЙНІСТЮ

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Доведено коректність крайових задач для деякого класу систем еліптичних рівнянь зі степеневими нелінійностями, заданих у необмежених областях, без умов на поведінку розв'язку та зростання вихідних даних не нескінченності.

Ключові слова: нелінійна еліптична система рівнянь, узагальнені анізотропні простори Лебега-Соболєва

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