A SPECIAL CASE OF THE TRIANGLE SOLUTION WITH THE LAW OF SINES IN GEODETIC APPLICATION

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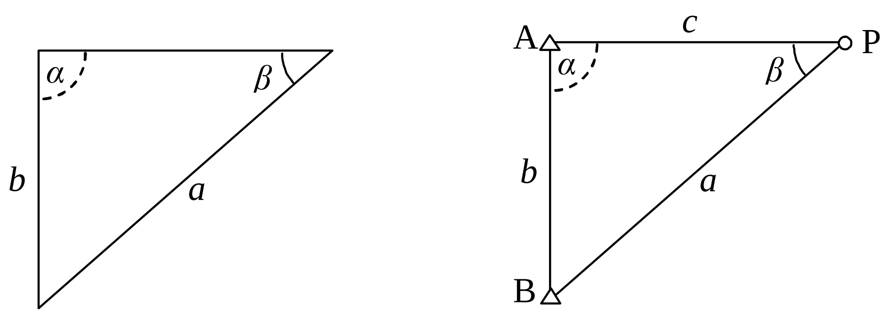
1. Introduction and formulation of the issue

The law of sines is often applied in solving various geodetic problems which are based on a triangle structure (elements of geodetic networks, intersections, transfer of coordinates, etc.) – cf. (Lazzarini et al., 1990)). Solving a triangle without excess elements (e.g. 2 lengths and 1 angle – cf. Fig 1a) is necessary, e.g. in calculating the coordinates in so called modular networks (Gargula, 2004), where the situation is similar to that in Fig. 1a.



a *б*

Fig. 1. An example solution of a triangle (a) in coordinates transfer (b)

Calculating the desired angle α from the sines formula

$$\sin\alpha = \frac{a \cdot \sin\beta}{b} \tag{1}$$

should be preceded by considering the following cases (Bronsztejn and Siemiendiajew, 1990):

1) if $b \geq a$ $\rightarrow$ $\alpha < 90^{\circ}$,

2) if $b < a$ and:

a) $a \cdot \sin\beta < b \rightarrow \alpha_1 < 90°$; $\alpha_2 = 180° - \alpha_1$ (two solutions),

b) $a \cdot \sin\beta = b \rightarrow \alpha = 90°$,

c) $a \cdot \sin\beta > b \rightarrow$ the triangle does not exist (no solution).

The problem may emerge in case 2b) of formula (1) or in a similar situation, i.e. if angle α is close to the right angle. The sine function graph (Fig. 2) shows that for an angle $\alpha \approx 90^{\circ}$ (100^{g}), even a slight change in the function value ($\Delta sin\alpha$) is accompanied by a large change in the angle value ($\Delta\alpha$). A conclusion that can be drawn from the fact is that in such cases, an angle calculated from the sines formula (1) may have an error which is unacceptable in terms of the accuracy required in geodetic applications.

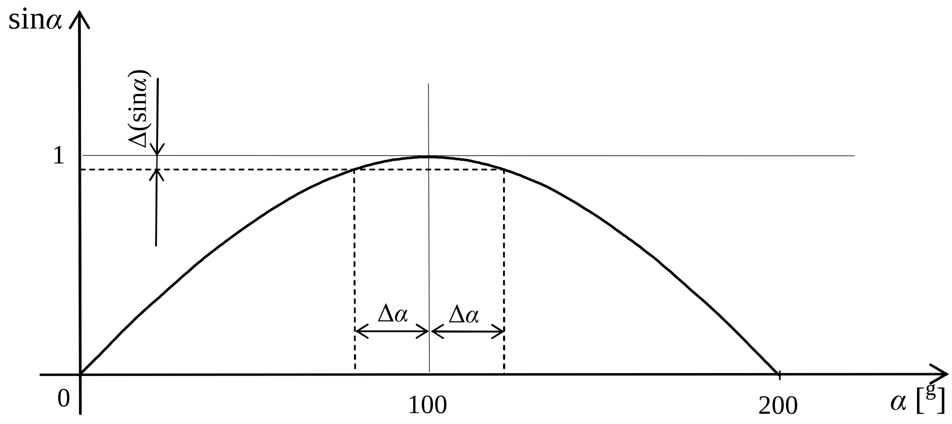


Fig. 2. The relationship between the angle increase (value close to 100g)
and the increase in the function value

The Snellius law (Bronsztejn and Siemiendiajew, 1990) also results in the following:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = 2R \qquad (2)$$

where: R – the radius of the circle circumscribed on the triangle.

If α is the right angle, side a goes through the centre of the circle, i.e.

$$\alpha = 90^{\circ} \rightarrow a = 2R \qquad (3)$$

This is the general case when all the three vertices of a right-angled triangle (in coordinates transfer those will be both the known points *A*, *B* and the determined point *P* – cf. Fig. 1b) lie on one circle (Fig. 3). As is seen from the diagram in Fig. 2, angle α has the maximum error. The situation is also unfavourable (from the point of view of geodetic applications) if the vortex of angle α (point *A*) is situated close to the circle whose diameter is *a*. The circle is an "unsafe circle" in determination with the law of sines of an angle which is close to the right angle.

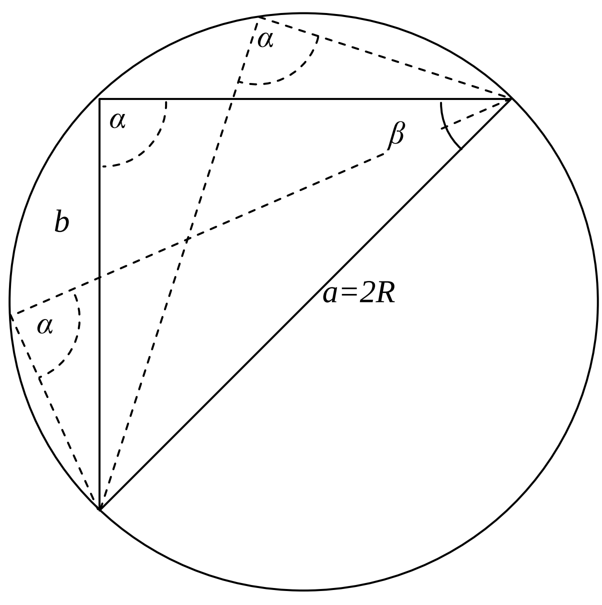


Fig. 3. An "unsafe circle" when a general sines law is applied in solving a triangle

It will be possible to assess the significance of errors (inaccuracies) occurring when the sines law is applied only after the relevant numerical tests have been performed.

2. Numerical example no. 1.

The example is based on Fig. 1b. Our aim is to determine the coordinates of point *P* from the known coordinates of points *A* and *B* (x_A = 1100.00; y_A = 1000.00; x_B = 1000.00; y_B = 1000.00) and observations of *a* and *β* (table 1).

First, the values of angle *α* (column 6) were calculated for a given length of *a* (column 2 in table 1), which was followed by the length of *c* (column 8). The determined values of *α* and *c* were in turn used for the calculation of the coordinates of point *P* (column 10, 11). In the first variant (line 1) the length of $a = 100\sqrt{2}$ was adopted (ideal triangle) within an accuracy of $1{\cdot}10^{-6}$ m. As it turns out, it is case 2c) of the sines law (1). The problem does not have any solutions, as *a* slightly exceeds (by about $8{\cdot}10^{-7}$) the value of $100\sqrt{2}$. If the actual error of the length measurement of *a* is adopted as $-2{\cdot}10^{-7}$ (line 2), a solution will be achieved, but if the error of angle *α* is $-0{,}0037^{g}$, which in turn generates an error of length of *c* equal to 0.006 m and a linear deviation of the determined point *P*: f_L = 0,008 m. Further decreasing of the length of *a* by minimum values (starting with fractions of a millimetre) results in rapid increase in the errors of the calculated values (lines 3-8). Making a substantial error of measurement of *a* of about 1 cm (line 7) results in the angle error of over 1^{g} and the linear deviation of point P of over 1.5 m. Such errors for a geodesist are absolutely unacceptable.

Table 1

The accuracy of determination of angle *α* and the coordinates of point *P* depending on the accuracy of the length of *a*.

No.	*a* [m]	f_a [m]	*b* [m]	*β* [g]	*α* [g]	f_α [g]	*c* [m]	f_c [m]	x_P [m]	y_P [m]	f_x [m]	f_y [m]	f_L [m]
1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*
1	141,421357	$-8{\cdot}10^{-7}$	100,00	50,0000	*	*	*	*	*	*	*	*	*
2	141,421356	$-2{\cdot}10^{-7}$	100,00	50,0000	99,9963	-0,0037	100,006	0,006	1099,994	1100,006	-0,006	0,006	0,008
3	141,42135	$-6{\cdot}10^{-6}$	100,00	50,0000	99,9811	-0,0189	100,030	0,030	1099,970	1100,030	-0,030	0,030	0,042
4	141,4213	$-6{\cdot}10^{-5}$	100,00	50,0000	99,9432	-0,0568	100,089	0,089	1099,911	1100,089	-0,089	0,089	0,126
5	141,421	$-4{\cdot}10^{-4}$	100,00	50,0000	99,8571	-0,1429	100,224	0,224	1099,775	1100,224	-0,225	0,224	0,317
6	141,420	-0,001	100,00	50,0000	99,7212	-0,2788	100,437	0,437	1099,560	1100,436	-0,440	0,436	0,619
7	141,410	-0,011	100,00	50,0000	99,1932	-0,8068	101,259	1,259	1098,717	1101,251	-1,283	1,251	1,792
8	141,400	-0,021	100,00	50,0000	98,8936	-1,1064	101,723	1,723	1098,232	1101,707	-1,768	1,707	2,458

Notation:
a, b, β – given elements of a triangle (according to Fig. 1b); *a, c* – calculated elements of the triangle (law of sines);
f_a, f_α, f_c – deviations of the respective values from their theoretical values (error-free);
x_P, y_P – calculated coordinates of point *P*; f_a, f_α, f_c – deviations of the respective values from their theoretical values;
f_L – linear deviation of point *P*; * – no solution (sin*α* > 1).

3. Numerical example no. 2.

The aim of this example is to establish the relationship between the error of the determined angle *α* and the value of the angle. In order to do this, formulae for the mean error of angle *α* and formulae for the mean errors of the coordinates of point *P* as functions of observations α and β will be derived for our example from Fig. 1a.

Applying the law of propagation the mean errors of independent values (e.g. Baran, 1999), it can be established that the mean error of angle α calculated from formula (1) is equal to:

$$m_\alpha = \sqrt{\frac{\sin^2\beta\cdot(m_a\cdot\rho)^2 + a^2\cdot\cos^2\beta\cdot m_\beta^2}{b^2 - a^2\cdot\sin^2\beta}} \tag{4}$$

where m_a, m_β – the mean errors of determination of the length of a and angle β (respectively), ρ – calculation coefficient between the arc measure and the grade (or degree) measure of the angle.

In order to determine the coordinates *xy* of point *P* (Fig. 1b), angle α previously calculated from (1) is used to calculate the length of *c*:

$$c = b\cdot\frac{\sin(\alpha+\beta)}{\sin\beta} \tag{5}$$

Therefore, the formulae needed to calculate the coordinates as functions of parameters α and β will have the following form:

$$\begin{cases} x_P = x_A + b\cdot\dfrac{\sin(\alpha+\beta)\cdot\cos(A_{AB}-\alpha)}{\sin\beta} \\ y_P = y_A + b\cdot\dfrac{\sin(\alpha+\beta)\cdot\sin(A_{AB}-\alpha)}{\sin\beta} \end{cases} \tag{6}$$

where A_{AB} – azimuth of side *AB*.

For thus calculated coordinates (6), based on the law of error propagation, the values of their mean errors will be determined:

$$\begin{cases} m_x = \dfrac{b}{\sin^2\beta}\cdot\sqrt{\sin^2\beta\cdot\cos^2(A_{AB}-2\alpha-\beta)\cdot\left(\dfrac{m_\alpha}{\rho}\right)^2 + \sin^2\alpha\cdot\cos^2(A_{AB}-\alpha)\cdot\left(\dfrac{m_\beta}{\rho}\right)^2} \\ m_y = \dfrac{b}{\sin^2\beta}\cdot\sqrt{\sin^2\beta\cdot\sin^2(A_{AB}-2\alpha-\beta)\cdot\left(\dfrac{m_\alpha}{\rho}\right)^2 + \sin^2\alpha\cdot\sin^2(A_{AB}-\alpha)\cdot\left(\dfrac{m_\beta}{\rho}\right)^2} \end{cases} \tag{7}$$

Taking into account equation (4), the final formulae for the mean errors of coordinates as functions of the measured values – the length of *a* and angle β, – are obtained:

$$\begin{cases} m_x = \sqrt{\dfrac{b^2\cos^2(A_{AB}-2\alpha-\beta)}{b^2-a^2\sin^2\beta}\cdot m_a^2 + \left(\dfrac{a^2b^2\cos^2(A_{AB}-2\alpha-\beta)\cos^2\beta}{b^2\sin^2\beta-a^2\sin^4\beta} + \dfrac{b^2\sin^2\alpha\cdot\cos^2(A_{AB}-\alpha)}{\sin^4\beta}\right)\cdot\left(\dfrac{m_\beta}{\rho}\right)^2} \\ m_y = \sqrt{\dfrac{b^2\sin^2(A_{AB}-2\alpha-\beta)}{b^2-a^2\sin^2\beta}\cdot m_a^2 + \left(\dfrac{a^2b^2\sin^2(A_{AB}-2\alpha-\beta)\cos^2\beta}{b^2\sin^2\beta-a^2\sin^4\beta} + \dfrac{b^2\sin^2\alpha\cdot\sin^2(A_{AB}-\alpha)}{\sin^4\beta}\right)\cdot\left(\dfrac{m_\beta}{\rho}\right)^2} \end{cases} \tag{8}$$

The starting coordinates of points *A* and *B* in this example are adopted as identical as in example 1. The error-free nature of the coordinates is also assumed. The variable quality for various calculation variants is angle α (table 2 column 2).

Table 2

Mean errors of angle α and the errors of position of point P depending on the angle value

No.	α [g]	β [g]	b [m]	a [m]	m_α [g]	c [m]	m_c [m]	Δx_{AP} [m]	Δy_{AP} [m]	m_x [m]	m_y [m]	m_P [m]
1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*
1	85,0000	50,0000	100,00	137,514	0,0105	120,582	0,018	-28,149	117,250	0,020	0,018	0,027
2	90,0000	50,0000	100,00	139,680	0,0157	114,412	0,029	-17,898	113,004	0,028	0,029	0,041
3	95,0000	50,0000	100,00	140,985	0,0314	107,538	0,064	-8,437	107,206	0,053	0,064	0,083
4	96,0000	50,0000	100,00	141,142	0,0392	106,082	0,082	-6,661	105,872	0,065	0,081	0,104
5	97,0000	50,0000	100,00	141,264	0,0523	104,600	0,111	-4,927	104,484	0,086	0,111	0,140
6	98,0000	50,0000	100,00	141,352	0,0784	103,092	0,169	-3,238	103,041	0,127	0,169	0,211
7	99,0000	50,0000	100,00	141,404	0,1568	101,558	0,343	-1,595	101,546	0,250	0,343	0,424
8	99,5000	50,0000	100,00	141,4170	0,3136	100,782	0,691	-0,792	100,779	0,496	0,691	0,851
9	99,6000	50,0000	100,00	141,4186	0,3920	100,626	0,865	-0,632	100,624	0,620	0,865	1,064
10	99,7000	50,0000	100,00	141,4198	0,5227	100,470	1,156	-0,473	100,469	0,825	1,156	1,420
11	99,8000	50,0000	100,00	141,4207	0,7840	100,314	1,736	-0,315	100,313	1,235	1,736	2,131
12	99,9000	50,0000	100,00	141,4212	1,5680	100,157	3,478	-0,157	100,157	2,467	3,478	4,264
13	99,9100	50,0000	100,00	141,42121	1,7422	100,141	3,865	-0,142	100,141	2,740	3,865	4,738
14	99,9200	50,0000	100,00	141,42124	1,9599	100,126	4,348	-0,126	100,126	3,083	4,348	5,330
15	99,9300	50,0000	100,00	141,42127	2,2399	100,110	4,970	-0,110	100,110	3,522	4,970	6,092
16	99,9400	50,0000	100,00	141,42129	2,6133	100,094	5,800	-0,094	100,094	4,109	5,800	7,108
17	99,9500	50,0000	100,00	141,42131	3,1359	100,079	6,961	-0,079	100,078	4,930	6,961	8,530
18	99,9600	50,0000	100,00	141,4213283	3,9199	100,063	8,702	-0,063	100,063	6,161	8,702	10,663
19	99,9700	50,0000	100,00	141,4213405	5,2265	100,047	11,605	-0,047	100,047	8,214	11,605	14,218
20	99,9800	50,0000	100,00	141,4213493	7,8398	100,031	17,410	-0,031	100,031	12,319	17,410	21,327
21	99,9900	50,0000	100,00	141,4213545	15,6795	100,016	34,826	-0,016	100,016	24,633	34,826	42,657
22	**100,0000**	**50,0000**	**100,00**	**141,4213562**	**$1{,}8\cdot10^5$**	**100,000**	**$4{,}1\cdot10^5$**	**0,000**	**100,000**	**$2{,}7\cdot10^5$**	**$4{,}1\cdot10^5$**	**$5{,}0\cdot10^5$**
23	100,0100	50,0000	100,00	141,4213545	15,6795	99,984	34,837	0,016	99,984	24,626	34,837	42,662
24	100,0200	50,0000	100,00	141,4213493	7,8398	99,969	17,421	0,031	99,969	12,311	17,421	21,332
25	100,0300	50,0000	100,00	141,4213405	5,2265	99,953	11,616	0,047	99,953	8,206	11,616	14,222
26	100,0400	50,0000	100,00	141,4213283	3,9199	99,937	8,713	0,063	99,937	6,153	8,713	10,667
27	100,0500	50,0000	100,00	141,4213126	3,1359	99,921	6,972	0,078	99,921	4,922	6,972	8,534
…	…	…	…	…	…	…	…	…	…	…	…	…
-	-	m_β=0,001	m_b= 0	m_a = 0,005	-	-	-	-	-	-	-	-

Notation:
a, b, c, α, β – according to Fig. 1b ; $m_a, m_b, m_c, m_\alpha, m_\beta$ – mean errors of the respective values;
Δx_{AP}, Δy_{AP} – calculated increments of the coordinates on side AP;
m_x, m_y – mean errors of the coordinates of point P; m_P – position error of point P.

Let us start the calculations (line 1) with such an angle value (85^g), for which the error of the determined point P (0.027 m) is still acceptable. When the value of angle α approximates 100^g (lines 2÷22), the mean errors of the following determined values are increasing: angle α (column 6), length of c (column 8) and position of point P (column 13). For angle α = 100.0000^g (line 22) the values of the errors become incredibly high. This stems from formula (4), in which the denominator (under square root) for the right angle α is equal to zero.

$$\left(\alpha = 100^g,\ \beta = 50^g\right) \rightarrow b^2 - a^2 \cdot \sin^2 \beta = 100^2 - \left(100\sqrt{2}\right)^2 \cdot \left(\frac{\sqrt{2}}{2}\right)^2 = 0$$

In this case the error m_α cannot be determined (infinity). The value of the error in table 2 (line 22) has been calculated for a finite value of the length of *a:*

$$a = \frac{\sqrt{2}}{2} \cong 141{,}4213562 \quad m$$

Therefore, a general conclusion can be formulated: if the angle value approximates 90° (100^g), the mean error of the angle goes to infinity:

$$\lim_{\alpha \to 90^\circ} m_\alpha = \infty \tag{9}$$

The principle (9) is clearly visible in the diagram (Fig. 4), which shows the relationship between the respective values from columns 6 and 2 in table 2. A symmetric diagram shape (in relation to the straight angle line) indicates that a change of length of side *a* affects the mean error of the angle value to a very slight extent (column 5, table 2).

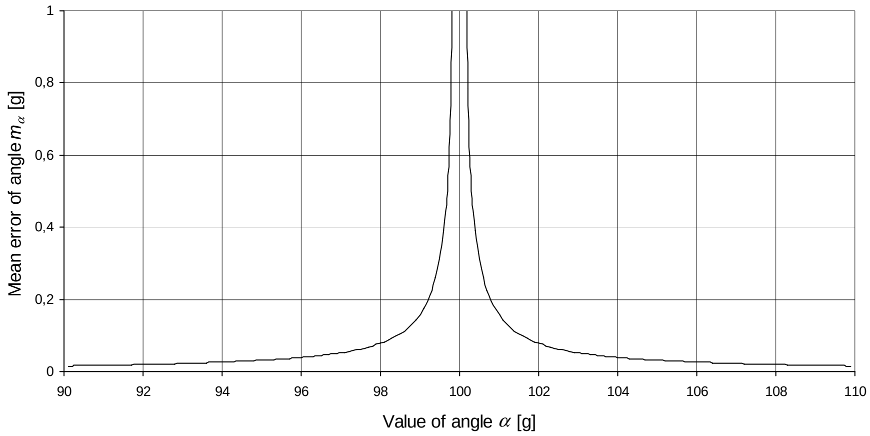


Fig. 4. The relationship between the mean error of angle α and the value of the angle

Fig. 5 shows the diagram of the relationship between the error of position of point *P* and the value of angle α, which was used in calculation of the coordinates of the point (columns 9, 10, table 2). A similar shape of both diagrams (Fig. 4, Fig. 5) suggests that the error of determination of a point (m_P) is mainly affected by the error of the angle determination (m_α), whereas the length of side *c* is of little importance (column 7, table 2).

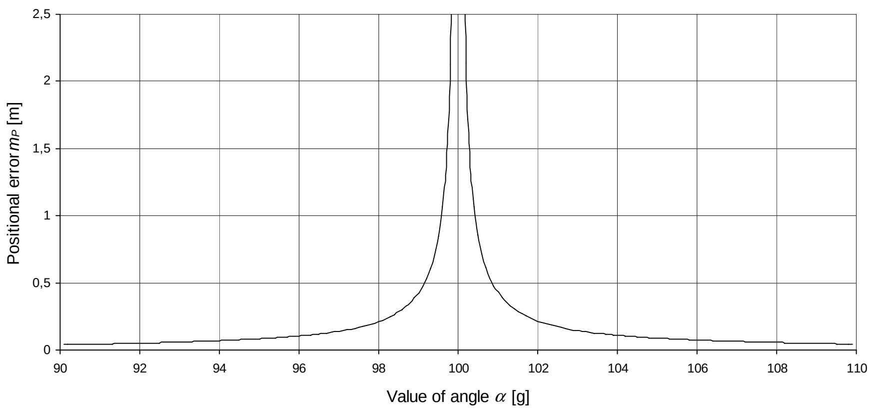


Fig. 5. The relationship between the error of position of point P and the value of angle α

4. Summary and conclusions

The subject of this study is the solution of a triangle with the use of the law of sines (Snellius law) in geodetic applications. A specific case of the problem has been dealt with, when the determined angle is equal to or close to the right angle. The theoretical considerations and numerical tests aimed at finding a relationship between the value of the determined angle and the error of such determination. The tests (conducted in many variants) consisted in solving a triangle, followed by calculating the coordinates of a point as a function of a previously determined triangle elements. For each variant the accuracy of the determined values has been assessed. The results have been presented both in a tabular and graphic form. This provides foundations for the following conclusions:

– for the angle of 100^g the error of its determination has the maximum value, whereas close to that number ($\pm 1^g$) the error of the angle determination is usually equal to several grades,

– the error of the position of a point determined based on such an angle can be as great as several dozen metres (depending on the transfer side),

– in geodetic applications (intersections, coordinate transfer, etc.) such triangle shapes should be avoided in which the triangle calculated from the trigonometric relationships (law of sines or other relationships depended) is close to the right angle,

– a “safe” triangle structure is one in which the determined angle is smaller than 85^g (or larger than 115^g).

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Особливий випадок розв’язання трикутника на підставі теореми синусів та його застосування в геодезії

Т. Гаргула

Подано теоретичний та емпіричний аналіз точності визначення кута у трикутнику на підставі теореми синусів для випадку, коли кут близький до прямого. Дослідження виконано для застосування в геодезії (наприклад, для визначення координат), де вимогою є певна точність визначень. Для різних варіантів трикутника досліджено низку моделей, на підставі яких визначено “ безпечну” величину кута.

Особый случай решения треугольника по теореме синусов и его использование в геодезии

Т. Гаргула

Представлены теоретические и эмпирические исследования точности углового определения в треугольнике с использованием теоремы синусов, в случае, если угол близкий к прямому углу. Исследования выполнены для использования в геодезии (например, при определениях координат), где требуется определенный уровень точности. Для разных вариантов треугольников исследовано ряд моделей, на основе чего определено “безопасную” величину угла.

A special case of the triangle solution with the law of sines in geodetic application

T. Gargula

The paper presents theoretical and empirical analyses of the accuracy of an angle determination in a triangle with the law of sines if the angle is close to the right angle. The issue was considered in terms of occurrence of such cases in geodetic applications (e.g. in transfer of coordinates), where a certain level of accuracy of determination is required. A number of numerical tests were conducted for different triangle variants, which were then used in determination of a “safe” range of the angle (for geodetic applications). The theoretic analyses were based on the sine function graph for an angle whose value is close to 90°.