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Inverse Problem for Two-Dimensional Heat Equation with an Unknown Source

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Abstract—The paper establishes existence and unique conditions for an inverse problem with an unknown source. The unknown source is a polynom for two spatial variables with unknown coefficients depending on time.

Keywords—inverse problem, Green function, Volterra integral equations, unknown source

I. INTRODUCTION

The inverse problems of identification of the coefficients for parabolic equation were considered in various formulations in numerous works of national and foreign mathematicians.

The problem of determination of the unknown free term is the simplest among coefficient inverse problems. Particularly in this direction the following results were obtained. In paper [1] a free term of the heat equation was being looked up as a product of two unknown functions $f(t)g(x)$.

In [2] the unknown source has the form $f_1(x)g_1(t)+f_2(t)g_2(x)$, where $f_1(x)$, $f_2(t)$ have to be identified. For this problem, the local conditions of existence and uniqueness of the solution are set.

This problem was analyzed in paper [3], for the case of integrated overdetermination conditions, and contained numerous examples of its implementation.

Paper [4] also shows the numerical solution of simultaneous determination inverse problems of time-dependent coefficients of the younger members and the source of the parabolic equation.

In [5] the inverse problem for a weakly nonlinear ultraparabolic equation with three unknown functions depending on various arguments is considered. The conditions for the existence and the uniqueness of the generalized solution of this problem are established on some narrowed time interval. The components of the solution are from Sobolev spaces.

Paper [6] is dedicated to finding sources in a parabolic integral-differential equation in the form $\chi(\mathrm{t})\omega(\mathrm{x}), \chi(\mathrm{x_n})\omega(\overline{\mathrm{x}},\mathrm{t})$ and $\sum_{j=1}^{n}\omega_j\chi_j(x,t)$, where χ,χ_j are the specified functions.

In the paper [9] the problem for the two-dimensional parabolic equation with an unknown heterogeneous transient orthotropic thermal conductivity is studied. The author uses initial and Dirichlet boundary conditions and fluxes as overdetermination conditions. For the numerical solving of such problems with specific initial data, the finite-difference method is used.

To ensure the unique solvability of the inverse problem solution the measurement data represented by the heat fluxes are considered.

The inverse problems of identification of an unknown source were also considered in the works [10]-[13].

II. FORMULATING THE PROBLEM

A large number of technological processes in metallurgy occur at elevated temperatures. In such cases, it is often impossible to control the temperature during experiments. This happens when the heated object is of small geometric dimensions, such as a thin wire, thin rectangular domain or when the heating takes place in an environment that is not accessible for installing temperature sensors.

In recent years, in practice, new high-speed methods of heat treatment of metals have been widely used, the temperature distribution here can be controlled using a mathematical model, using solutions for inverse problems. This raises the problem of the existence and uniqueness of the solution for such problems.

Let's have a two-dimensional rectangular domain $(0<x<l, 0<y<h)$. Because of $u(x,y,t)$ we denote its temperature at the coordinate point (x,y) at the time t. It is necessary to organize such a process in a way that certain points of the rectangular domain ($(0,0), (0,h), (l,0), (l,h)$) have predetermined time-dependent temperature regimes.

This can be achieved due to the thermal effects of internal heat sources, that is, solving the problem of controlling the source in the heat equation. In fact, we need to investigate the inverse problem for the heat equation with an unknown free member, which depends on the two spatial variables x, y and time t.

The question of the complete determination of the source (the definition of a free member, which depends on spatial and time variables) remains open today.

A certain solution to complete definition of an unknown member of the parabolic equation is a polynomial approximation for its spatial variables with unknown coefficients that depend on time.

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This task of identification of the source of the polynomial is developed in this study. The approach to a complete determination of the unknown coefficients parabolic equation was tested in [7], [8], where the conditions of existence and uniqueness of the solution of the problem of identification of the major coefficient of the linear and quadratic functions of a spatial variable with unknown coefficients depending on the time variable were found.

III. Mathematical Model

In the domain $\Omega=\{(x,y,t):\ 0<x<l,\ 0<y<h,\ 0<t<T\}$ consider the parabolic equation

$$u_t = u_{xx}+u_{yy}+\sum_{i=0}^{1}\sum_{j=0}^{1}x^i y^j f_{ij}(t) \qquad (1)$$

with unknown coefficients $f_{ij}(t), i,j=\overline{0,1},$ with the initial condition

$$u(x,y,0)=\phi(x,y), \quad (x,y)\in[0,l]\times[0,h], \qquad (2)$$

the boundary conditions

$$u_x(0,y,t)=\mu_1(y,t),\ u_x(l,y,t)=\mu_2(y,t), \quad (y,t)\in[0,h]\times[0,T],$$

$$u_y(x,0,t)=\mu_3(x,t),\ u_y(x,h,t)=\mu_4(x,t),$$

$$(x,t)\in[0,l]\times[0,T] \qquad (3)$$

and the over determination conditions

$$u(0,0,t)=\nu_{00}(t), \qquad u(0,h,t)=\nu_{01}(t),$$

$$u(l,0,t)=\nu_{10}(t), \qquad u(l,h,t)=\nu_{11}(t),\ t\in[0,T]. \qquad (4)$$

The solution of the problem (1)-(5) are the functions $\left(u(x,y,t), f_{ij}(t)\right)\in\ C^{2,1}(\Omega)\times\left(C[0,T]\right)^4,\ i,j=\overline{0,1},$ that satisfy conditions (1)-(4).

IV. Existence and Uniqueness of the Solution for the Problem

Assuming that the output data possess the required smoothness and the conditions for their compliance, we reduce problem (1) - (5) to the system of equations for unknown functions $f_{ij}(t), i,j=\overline{0,1}.$

For this reason, let set $x=0,l$ and $y=0,h$, in the equation (1). In result we get the following correlations

$$\begin{aligned}
\nu'_{00}(t)&=u_{xx}(0,0,t)+u_{yy}(0,0,t)+f_{00}(t),\\
\nu'_{01}(t)&=u_{xx}(0,h,t)+u_{yy}(0,h,t)+f_{00}(t)+hf_{01}(t),\\
\nu'_{10}(t)&=u_{xx}(l,0,t)+u_{yy}(l,0,t)+f_{00}(t)+lf_{10}(t), \qquad (5)\\
\nu'_{11}(t)&=u_{xx}(l,h,t)+u_{yy}(l,h,t)+f_{00}(t)+\\
&\quad+hf_{01}(t)+lf_{10}(t)+lhf_{11}(t),
\end{aligned}$$

where $u(x,y,t)$ - the solution of the direct problem (1)-(3), which has the form

$$\begin{aligned}
u(x,y,t)&=\int_0^l d\xi\int_0^h \phi(\xi,\eta)G_2(x,y,t,\xi,\eta,0)d\eta+\\
&+\int_0^t d\tau\int_0^h \mu_2(\eta,\tau)G_2(x,y,t,l,\eta,\tau)\ -_1\\
&\quad-\mu_1(\eta,\tau)G_2(x,y,t,0,\eta,\tau)d\eta+\\
&+\int_0^t d\tau\int_0^l \mu_4(\xi,\tau)G_2(x,y,t,\xi,h,\tau)\ -_1\\
&\quad-\mu_3(\xi,\tau)G_2(x,y,t,\xi,0,\tau)d\eta+\\
&+\int_0^t\int_0^l\int_0^h\sum_{i=0}^{1}\sum_{j=0}^{1}\xi^i\eta^j f_{ij}(\tau)G_2(x,y,t,\xi,\eta,\tau)d\eta d\xi d\tau,
\end{aligned}$$

$G_2(x,y,t,\xi,\eta,\tau)$ -- Green function of the second kind.

Since the determinant of the system (5)

$$\Delta=\begin{vmatrix}1&0&0&0\\1&h&0&0\\1&0&l&0\\1&h&l&lh\end{vmatrix}=h^2l^2$$

is not a zero, the system of equations (5) can be reduced to the form (6):

$$f_{km}(t)=F_{km}(t)+\sum_{i=0}^{1}\sum_{j=0}^{1}P_{ij}^{km}(t)u_{xx}(l_i,h_j,t), \qquad (6)$$
$$k,m=\overline{0,1}$$

where $F_{km}(t), P_{ij}^{km}(t), i,j,k,m=\overline{0,1},$ - known functions, determined through input data, $l_0=0,\quad l_1=l,\quad h_0=0,\quad h_1=h.$

The system (6) is a system of Volterra integral equations (VIEs) of the second kind.

Using the theory of VIEs of the second kind, it can be stated that the solution for the system of equations (6) exists and it is unique only if the following conditions are satisfied:

$$\phi(x,y)\in C^{2,2}[0,l]\times[0,h],$$

$$\mu_i(x,t)\in C^{2,1}[0,l]\times[0,T],\ i=1,2,$$

$$\mu_j(y,t) \in C^{2,1}[0,h]\times[0,T],\ i=3,4,\ \nu_{ij}(t)\in C^1[0,T],$$

$$i, j = \overline{0,1}.$$

Based on the equivalence of the system of equations (6) and the problem (1) - (4), we conclude

THEOREM

The problem (1) - (4) has a unique solution if the conditions of smoothness (A) and harmonization $\phi'(0,y)=\mu_1(y,0)$, $\phi'(l,y)=\mu_2(y,0)$, $\phi'(x,0)=\mu_3(x,0)$, $\phi'(x,h)=\mu_4(x,0)$, $\phi(0,0)=\nu_{00}(0)$, $\phi(l,0)=\nu_{10}(0)$, $\phi(0,h)=\nu_{01}(0)$, $\phi(l,h)=\nu_{11}(0)$ are satisfied.

V. CONCLUSIONS

In this paper:

1) the mathematical model of the heating process in the thin rectangular plate with an available heating source with the unknown physical characteristics is constructed;

2) the obtained inverse problem for the heat equation is reduced to the system of the integral Volterra equations of the second kind;

3) basing on the theory of Volterra integral equations of the second kind there are established the conditions for the solution existence and uniqueness.

For the numerical solving of such problems, one can use the finite difference method, which is presented iteratively in the nonlinear minimization procedure.

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