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Modeling of Thermoviscoelasticity Time Harmonic Variational Problem for a Thin Wall Body

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***Abstract*—The paper presents the construction and analysis of vibration problem of thermoviscoelastic shells under the influence of non-stationary heat and under forced loads. The studied model was based on application of simplest finite element semidiscretization to mixed variational problem of dynamical thermoviscoelasticity. The problem in addition to the mutual influence of temperature field and stress field is also taken into account the viscoelastic properties of the material thin wall body. For assumptions quite suitable for applications we prove the well-posedness forthis model of time harmonic vibrations.**

Keywords—initial-boundary value problem, thermo-viscoelasticity, material with short-term memory, variational formulation, semidiscretization, well-posedness of problem, Galerkin discretization.

I. INTRODUCTION

Mathematical modeling methods of thin-walled structures that are under forced, temperature and electromagnetic loads are wide tools base of continuum mechanics and its engineering applications.

Last time, well developed analytical methods for solving this class of problems are actively complemented by methods of computational mathematics and computer simulation, the successful application of which often requires revision and supplementation of classical models, for example, shell theory, developing appropriate software. The filling of mechanics with an intensive influx of engineering problems, for example, with smart materials, makes studies in this field relevant and timely.

In authors` previous articles [2] a development and analysis of the dimension reducing methods for heat conduction problem and thermoelasticity problem for thin flexible bodies have been investigated.

In work [6] theory of thermoviscoelastic thin wall elements for dynamical problems was considered. In this article, similar techniques as in [7] are applied to the problem of forced vibrations of thermoviscoelastic shells [7].

II. PROBLEM STATEMENT

Let be the bounded connected domain $D \in \Re^n$ of points $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ with Lipschitz-continuous boundary $\partial D = S$, and $\mathbf{n} = \{n_i\}_{i=1}^n$ is unit outer normal vector $n_i = \cos(\mathbf{n}, x_i)$. Also let us consider time interval $[0,T]$, $0 < T < +\infty$. Notation $\{F_i(\mathbf{x},t)\}_{i=1}^3$ is a vector of volume mechanical forces, a vector of surface mechanical loads $\hat{\boldsymbol{\sigma}} = \{\hat{\sigma}_i(\mathbf{x},t)\}_{i=1}^3$ on the boundary $S_\sigma \subset S$, represents volume heat forces $g = g(\mathbf{x},t)$. Like in classic thermoelasticity problem, our goal is to find vector of elastic displacements $\mathbf{U} = \{U_i(\mathbf{x},t)\}_{i=1}^3$ and temperature increment $\theta(\mathbf{x},t)$, which satisfy the following equations in $D \times (0,T]$ (here and everywhere below the ordinary summation by repetitive indices is expected) [2],[3]:

$$\rho U_i'' - \partial_k \sigma_{ki} = \rho F_i, \tag{1}$$

$$c_\varepsilon \theta' - \partial_i(\lambda_{ij}\, \partial_j \theta) + \theta_0\, \beta_{ij}\, \partial_i U_i' = g\,, \tag{2}$$

The above expressions (1)-(2) are equation of motion, heat conduction equation, where $\partial_i := \partial v / \partial x_i$, $v' := \partial v / \partial t$, $v'' := \partial_t(\partial_t v)$. Below we will explain the meaning of each notation more thoroughly. Here $\boldsymbol{\sigma} = \left\{\sigma_{ij}\right\}_{i.j=1}^n$ is a stress tensor, which is defined by the following constitutive equation, namely hypothesis Duhamel-Neumann for material with short-term memory:

$$\begin{aligned}\sigma_{ij}(\mathbf{U},\theta) &:= \sigma_{ij}^e(\mathbf{U}) + \sigma_{ij}^{\mathrm{v}}(\mathbf{U}') + \sigma_{ij}^t(\theta) \\ &= c_{ijkm} E_{km}(\mathbf{U}) + a_{ijkm} E_{km}(\mathbf{U}') - \beta_{ij}\,\theta\,,\end{aligned} \tag{3}$$

Strain tensor $E_{ik}(\mathbf{U})$ is assumed to satisfy the relations:

$$E_{ik}(\mathbf{U}) := \frac{1}{2}(\partial_i U_k + \partial_k U_i). \tag{4}$$

Notation $\rho = \rho(\mathbf{x})$ is a mass density of thermoelastic material, $c_\varepsilon = c_\varepsilon(\mathbf{x})$ is its specific heat. Tensors c_{ijkm} and a_{ijkm} describe the thermoelasticity and viscosity properties of material with short-term memory. Also notation β_{ij} and λ_{ij} depicts a thermal stress coefficients tensor and a thermal conductivity coefficients tensor with the common properties of symmetry and ellipticity:

$$\begin{cases} c_{ijkm} = c_{jikm} = c_{kmij}, \quad a_{ijkm} = a_{jikm} = a_{kmij}, \\ c_{ijkm}\varepsilon_{ij}\varepsilon_{km} \ge c_0 \varepsilon_{ij}\varepsilon_{ij}, \quad c_0 = const > 0, \quad \forall \varepsilon_{ij} = \varepsilon_{ji} \in \mathbb{R}, \\ a_{ijkm}\varepsilon_{ij}\varepsilon_{km} \ge a_0 \varepsilon_{ij}\varepsilon_{ij}, \quad a_0 = const > 0, \quad \forall \varepsilon_{ij} = \varepsilon_{ji} \in \mathbb{R}; \end{cases} \tag{5}$$
$$\begin{cases} \lambda_{ij} = \lambda_{ji}, \quad \lambda_{ij}\xi_i\xi_j \ge \lambda_0, \quad \lambda_0 = const > 0, \\ \beta_{ij} = \beta_{ji}, \quad \beta_{ij}\xi_i\xi_j \ge \beta_0, \quad \beta_0 = const > 0 \quad \forall \xi_i \in \mathbb{R}. \end{cases}$$

To finalize the formulation of the initial boundary value problem of thermoelasticity, the system of partial differential equations (1), (2) is then complemented by boundary conditions

$$\begin{aligned} &\mathbf{U} = 0 \quad \text{in} \quad S_u \times [0,T], \quad S_u \subset S, \\ &\sigma_{ij} n_j = \hat{\sigma}_i \quad \text{in} \quad S_\sigma \times [0,T], \quad S_\sigma = S \setminus S_u, \end{aligned} \tag{6}$$

$$\begin{aligned} &\theta = 0 \quad \text{on} \quad S_\theta \times [0,T], \\ &-\lambda_{ij} n_i \partial_i \theta = \kappa\theta + \hat{q} \quad \text{on} \quad S_q \times [0,T], \quad S_q = S \setminus S_\theta, \end{aligned} \tag{7}$$

and the initial conditions

$$\mathbf{U}|_{t=0} = \mathbf{U}_0, \quad \partial_t \mathbf{U}|_{t=0} = \mathbf{V}_0, \quad \theta|_{t=0} = \theta_0 \quad \text{in} \quad D, \tag{8}$$

where κ is known heat transfer coefficient with the environment, θ_0 is a fixed initial temperature of the body. Vector $\hat{q}$ describes applied heat flux correspondingly.

I. Variational Problem of Thermoviscoelasticity

Let us introduce the spaces of admissible elastic displacements and temperature increments (relatively to the initial temperature T_0) respectively:

$$\mathbf{Y} = \left\{ \mathbf{V} \in [H^1(D)]^3 : \mathbf{V} = 0 \ \text{на} \ S_u \right\}, \mathbf{Z} = L^2(D),$$
$$G = \left\{ \xi \in H^1(D) : \xi = 0 \ \text{на} \ S_\theta \right\}, \qquad \mathbf{H} = Z^3.$$

Here symbol $H^m(D)$ means a standard Sobolev space.

Then the initial boundary value problem of thermoviscoelasticity (1)-(8), can be rewritten in the following variational formulation:

$$\begin{cases} \text{given } \mathbf{U}_0 \in \mathbf{Y}, \ \mathbf{V}_0 \in \mathbf{H}, \ \theta_0 \in Z; \\ \text{find } \{\mathbf{U}, \theta\} \in L^2(0,T; \mathbf{Y} \times G) \ \text{ such, as} \\ m(\mathbf{U}''(t), \mathbf{V}) + a(\mathbf{U}'(t), \mathbf{V}) + c(\mathbf{U}(t), \mathbf{V}) \\ \qquad - b(\theta(t), \mathbf{V}) = < l(t), \mathbf{V} >, \\ \Xi(\theta'(t), \xi) + \Lambda(\theta(t), \xi) + b(\xi, \mathbf{U}'(t)) = < r(t), \xi > \\ \qquad \forall t \in (0,T], \\ m(\mathbf{U}'(0) - \mathbf{V}_0, \mathbf{V}) = 0, \quad c(\mathbf{U}(0) - \mathbf{U}_0, \mathbf{V}) = 0 \forall \mathbf{V} \in \mathbf{Y}, \\ \qquad \Xi(\theta(0) - \theta_0, \xi) = 0 \qquad \forall \xi \in G. \end{cases} \tag{9}$$

The introduced bilinear and linear forms are as follows:

$$m(\mathbf{U},\mathbf{V}) := \iiint_D \rho \mathbf{U}.\mathbf{V} dD = \iiint_D \rho U_i V_i \, dD,$$
$$c(\mathbf{U},\mathbf{V}) := \iiint_D \sigma^e(\mathbf{U}) : E(\mathbf{V}) \, dD = \iiint_D \sigma^e_{ij}(\mathbf{U}) E_{ij}(\mathbf{V}) \, dD$$
$$a(\mathbf{U},\mathbf{V}) := \iiint_D \sigma^{\mathrm{v}}(\mathbf{U}) : E(\mathbf{V}) \, dD = \iiint_D \sigma^{\mathrm{v}}_{ij}(\mathbf{U}) E_{ij}(\mathbf{V}) \, dD,$$
$$b(\xi,\mathbf{V}) := \iiint_D \sigma^t(\xi) : E(\mathbf{V}) \, dD = \iiint_D \beta\xi \partial_i V_i \, dD, \ \forall \mathbf{U},\mathbf{V} \in \mathbf{Y},$$
$$\Xi(\theta,\xi) := \iiint_D c_\varepsilon \theta_0^{-1} \theta\xi \, dD,$$

$$\Lambda(\theta,\xi) := \iiint_D \theta_0^{-1}(\lambda_{ij}\nabla\theta).\nabla\xi dD + \iint_{S_q} \theta_0^{-1} \kappa\theta\xi dS,$$
$$< l, \mathbf{V} > := \iiint_D \rho \mathbf{F}.\mathbf{V} dD + \iint_{S_q} \hat{\boldsymbol{\sigma}}.\mathbf{V} dS, \quad \forall \mathbf{U},\mathbf{V} \in \mathbf{Y},$$
$$< r, \xi > := \iiint_D \theta_0^{-1} g \, \xi \, dD - \iint_{S_q} \theta_0^{-1} \hat{q} \xi dS \quad \forall \theta, \xi \in G.$$

Using Korn inequality and the symmetric and elliptic properties (5) we can define following norms on spaces Y and G

$$\begin{aligned} &\| \mathbf{U} \|_{\mathbf{H}} := \sqrt{m(\mathbf{U},\mathbf{U})}, \quad \| \mathbf{U} \|_{\mathbf{Y}} := \sqrt{c(\mathbf{U},\mathbf{U})}, \\ &||| \mathbf{U} |||_{\mathbf{Y}} := \sqrt{a(\mathbf{U},\mathbf{U})} \quad \forall \mathbf{U} \in \mathbf{Y} \quad \forall \theta \in G \\ &\| \theta \|_Z := \sqrt{\Xi(\theta,\theta)}, \quad \| \theta \|_G := \sqrt{\Lambda(\theta,\theta)} \end{aligned} \tag{10}$$

Also can be written the energy balance equation:

$$\frac{1}{2}\left[\|\mathbf{U}'(t)\|^2_{\mathbf{H}}+\|\mathbf{U}(t)\|^2_{\mathbf{Y}}+\|\theta(t)\|^2_{\mathbf{Z}}\right]+$$
$$\int_0^t\left[|||\,\mathbf{U}'(\tau)\,|||^2_{\mathbf{Y}}+\|\,\theta(\tau)\,\|^2_G\right]d\tau=$$
$$\frac{1}{2}\left[\|\,\mathbf{V}_0\,\|^2_{\mathbf{H}}+\|\,\mathbf{U}_0\,\|^2_{\mathbf{Y}}+\|\,\theta_0\,\|^2_{\mathbf{Z}}\right] \tag{11}$$
$$+\int_0^t\left[<l(\tau),\mathbf{V}'(\tau)>+<r(\tau),\theta(\tau)>\right]d\tau \quad \forall\tau\in(0,T].$$

Here $\frac{1}{2}\left[\|\,\mathbf{U}'(t)\,\|^2_{\mathbf{H}}+\|\,\mathbf{U}(t)\,\|^2_{\mathbf{Y}}+\|\,\theta(t)\,\|^2_{\mathbf{Z}}\right]$ determines the instant total energy value, $\int_0^t\left[|||\,\mathbf{U}'(\tau)\,|||^2_{\mathbf{Y}}+\|\,\theta(\tau)\,\|^2_G\right]d\tau$ determines dissipation of energy was caused by viscosity and temperature field of an elastic body, $\frac{1}{2}\left[\|\,\mathbf{V}_0\,\|^2_{\mathbf{H}}+\|\,\mathbf{U}_0\,\|^2_{\mathbf{Y}}+\|\,\theta_0\,\|^2_{\mathbf{Z}}\right]$ initial energy value, $\int_0^t\left[<l(\tau),\mathbf{V}'(\tau)>+<r(\tau),\theta(\tau)>\right]d\tau$ an influx of energy.

Formulated in accordance with the problem (1) – (8), the variational task of the dynamic thermoviscoelasticity of an elastic body, taking into account the corresponding linear elastic-viscous properties of the material and the energy balance equation (1.10), will be the basis for investigations of thermoviscoelastic processes in thin-walled bodies.

III. Partially Discretized Variational Problem of Thermoviscoelasticity for a Thin Wall Body

Let an elastic body $D\in\Re^3$ referred to fixed curvilinear orthogonal coordinate system $(\alpha_1,\alpha_2,\alpha_3)$ (fig. 1) as follows:

$$D:=\{\mathbf{r}=(\boldsymbol{\alpha},\alpha_3)\in\mathbb{R}^3:\boldsymbol{\alpha}=(\alpha_1,\alpha_2)\in\Omega,$$
$$\alpha_3\in(-\tfrac{1}{2}h,+\tfrac{1}{2}h)\}=\Omega\times(-\tfrac{1}{2}h,+\tfrac{1}{2}h),$$

where thickness $h=const>0$ is substantially smaller compared to other space dimensions, $h/diam\Omega\ll 1$. The body of such kind we shall name shell, its set $\Omega=\{\mathbf{r}=(\boldsymbol{\alpha},0)\in D\}$ will named the middle surface of shell and denote its contour through $\Gamma=\partial\Omega$. In this coordinate system a surface element $d\Omega$ and a volume element dD of the body are defined as:

$$d\Omega=H_1H_2d\boldsymbol{\alpha},\qquad dD=H_1H_2H_3d\boldsymbol{\alpha}d\alpha_3=d\Omega d\alpha_3, \tag{12}$$

$$H_i=A_i(1+\alpha_3k_i),\quad H_3=A_3\equiv 1,\quad i=1,2. \tag{13}$$

Here $A_i=A_i(\boldsymbol{\alpha})$ and $k_i=k_i(\boldsymbol{\alpha})$ – coefficients of the first quadratic form and the principal curvatures of the surface Ω [4]. Notes $\Omega_\pm=\Omega\times\{\pm h/2\}$ are facial surfaces and $\Sigma=\Gamma\times(-h/2,h/2)$ – lateral surface, then $S=\Omega_+\cup\Omega_-\cup\Sigma$. Assume the surface of the body is divided into parts nonzero measure as follows

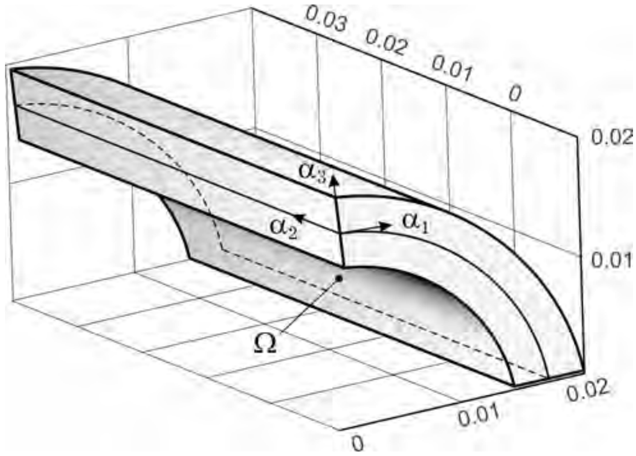

Fig. 1. – Domain D and mid-surface Ω referred to fixed curvilinear coordinate system $(\alpha_1,\alpha_2,\alpha_3)$.

$$S_u=S_\theta=\Sigma:=\left\{\mathbf{r}\in D:\ \boldsymbol{\alpha}\in\Gamma=\partial\Omega,\ |\alpha_3|\le\tfrac{1}{2}h\right\},$$
$$S_\sigma=S_q=\Omega_+\cup\Omega_-,\quad \Omega_\pm:=\left\{\mathbf{r}\in\bar{D}:\ \boldsymbol{\alpha}\in\Omega,\ \alpha_3=\pm\tfrac{1}{2}h\right\}.$$

By the Timoshenko-Mindlin hypotheses [5] we shall assume that a displacement vector $\mathbf{U}=\{U_i(\mathbf{r},t)\}_{i=1}^3$ and temperature $\theta=\theta(\mathbf{r},t)$ can approximated by the linear combinations of a functions $\mathbf{s}=(\mathbf{u}(\boldsymbol{\alpha},t),\boldsymbol{\gamma}(\boldsymbol{\alpha},t))$ and $\boldsymbol{\theta}=(\theta_1(\boldsymbol{\alpha},t),\theta_2(\boldsymbol{\alpha},t))$ such that

$$\mathbf{U}(\mathbf{r},t)\cong\mathbf{u}(\boldsymbol{\alpha},t)+\alpha\boldsymbol{\gamma}(\boldsymbol{\alpha},t),$$
$$\theta(\mathbf{r},t)\cong\theta(\boldsymbol{\alpha},t)+\alpha\theta(\boldsymbol{\alpha},t)\quad\forall(\boldsymbol{\alpha},\alpha)\in D.$$

Here $\mathbf{u}=\{u_i(\boldsymbol{\alpha},t)\}_{i=1}^3$ and $\theta_1=\theta_1(\boldsymbol{\alpha},t)$ are approximations of the displacement vector and temperature on the middle surface,

$$\boldsymbol{\gamma}(\boldsymbol{\alpha},t)\cong\partial\mathbf{U}(\boldsymbol{\alpha},0,t)/\partial\alpha_3,$$
$$\theta_2(\boldsymbol{\alpha},t)\cong\partial\theta(\boldsymbol{\alpha},0,t)/\partial\alpha_3,\quad\forall(\boldsymbol{\alpha},t)\in\bar{\Omega}\times[0,T].$$

As results of partially discretization after the thickness variable of the problem equations (9) we obtained a variation formulation problem for thermoelastic shells in the terms of the displacements vector $\mathbf{s}=(\mathbf{s}_1,\mathbf{s}_2)=(\mathbf{u}(\boldsymbol{\alpha},t),\boldsymbol{\gamma}(\boldsymbol{\alpha},t))$ and temperature vector $\boldsymbol{\theta}=(\theta(\boldsymbol{\alpha},t),\theta(\boldsymbol{\alpha},t))$:

$$
\left\{
\begin{array}{l}
\text{given } \mathbf{s}_0 \in W_h,\ \mathbf{v}_0 \in \mathbf{H},\ \boldsymbol{\theta}_0, \mathbf{g} \in \mathbf{Z},\ \mathbf{f} \in \mathbf{H}; \\
\text{find } \ \boldsymbol{\psi} = \{\mathbf{s}\ , \boldsymbol{\theta}\ \} \in L^2(0,T; W_h \times Q_h) \text{ such,as} \\
m_\Omega(\mathbf{s}''(t), \mathbf{v}) + a_\Omega(\mathbf{s}'(t), \mathbf{v}) + c_\Omega(\mathbf{s}(t), \mathbf{v}) \\
\qquad - b_\Omega(\boldsymbol{\theta}(t), \mathbf{v}) = < l(t), \mathbf{v} >, \\
\Xi_\Omega(\boldsymbol{\theta}'(t), \boldsymbol{\xi}) + \Lambda_\Omega(\boldsymbol{\theta}(t), \boldsymbol{\xi}) \qquad\qquad \forall t \in (0,T], \\
\qquad + b_\Omega(\boldsymbol{\xi}, \mathbf{s}'(t)) = < r(t), \boldsymbol{\xi} > \\
m_\Omega(\mathbf{s}'(0) - \mathbf{v}_0, \mathbf{v}) = 0, \qquad c_\Omega(\mathbf{s}(0) - \mathbf{s}_0, \mathbf{v}) = 0 \\
\Xi_\Omega(\boldsymbol{\theta}(0) - \boldsymbol{\theta}_0, \boldsymbol{\xi}) = 0 \qquad \forall \mathbf{v} \in W_h, \qquad \forall \boldsymbol{\xi} \in Q_h.
\end{array}
\right. \tag{14}
$$

We used the follows introduced spaces:

$$
W_h = \{\mathbf{w} \in [H^1(\Omega)]^6 : \mathbf{w} = 0 \ \text{на } S_u\},
$$
$$
Q_h = \{\xi \in [H^1(\Omega)]^2 : \xi = 0 \ \text{на } S_\theta\}.
$$

The bilinear and linear forms are defined as:

$$
\begin{array}{l}
m_\Omega(\mathbf{s}, \mathbf{v}) = \rho \sum\limits_{i,j=1}^{2} \iint\limits_{\Omega} \phi^{i+j-2} \mathbf{s}_i . \mathbf{v}_j A_1 A_2 d\boldsymbol{\alpha}, \\
a_\Omega(\mathbf{s}, \mathbf{v}) = \iint\limits_{\Omega} (\mathbf{Cs}) . \left(\tilde{\mathbf{B}} \mathbf{C} \mathbf{v}\right) A_1 A_2 \ d\boldsymbol{\alpha}, \\
c_\Omega(\mathbf{s}, \mathbf{v}) = \iint\limits_{\Omega} (\mathbf{Cs}) . \left(\mathbf{B} \mathbf{C} \mathbf{v}\right) A_1 A_2 \ d\boldsymbol{\alpha} \\
\qquad\qquad \forall \mathbf{s} = (\mathbf{s}_1, \mathbf{s}_2), \ \mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) \in W_h, \\
b_\Omega(\boldsymbol{\theta}, \mathbf{v}) = \beta \iint\limits_{\Omega} \boldsymbol{\Phi}(\boldsymbol{\theta}) . (\mathbf{C} \mathbf{v}) A_1 A_2 d\boldsymbol{\alpha}, \\
\Xi_\Omega(\boldsymbol{\theta}, \boldsymbol{\xi}) = \theta_0^{-1} \sum\limits_{i,j=1}^{2} \iint\limits_{\Omega} \phi^{i+j-2} \theta_i \xi_j A_1 A_2 d\boldsymbol{\alpha}, \\
\Lambda_\Omega(\boldsymbol{\theta}, \boldsymbol{\xi}) = \lambda_\Omega(\boldsymbol{\theta}, \boldsymbol{\xi}) + \kappa_\Omega(\boldsymbol{\theta}, \boldsymbol{\xi}), \\
\qquad\qquad \forall \boldsymbol{\theta} = (\theta_1, \theta_2), \ \boldsymbol{\xi} = (\xi_1, \xi_2) \in Q_h, \\
< r, \boldsymbol{\xi} > := -\theta_0^{-1} \iint\limits_{\Omega} \left\{ (q^+ + q^-) \xi_1 \right. \\
\qquad + \dfrac{h}{2} (q^+ - q^-) \left((k_1 + k_2) \xi_1 + \xi_2 \right) \left. \right\} A_1 A_2 d\boldsymbol{\alpha} \\
\qquad + \Xi_\Omega(c_\varepsilon^{-1} \mathbf{g}(t), \boldsymbol{\xi}) \quad \forall\ \boldsymbol{\xi} = (\xi_1, \xi_2) \in Q_h; \\
< l, \mathbf{v} > := -\sum\limits_{i,j=1}^{2} \iint\limits_{\Omega} (\hat{\boldsymbol{\sigma}}^+ + \hat{\boldsymbol{\sigma}}^-) \left\{ [1 + \tfrac{1}{2} h (1 + k_1 + k_2)] \mathbf{v}_1 \right. \\
\qquad - \tfrac{1}{2} h \mathbf{v}_2 \left. \right\} A_1 A_2 d\boldsymbol{\alpha} + m_\Omega(\mathbf{f}(t), \mathbf{v}) \quad \forall \mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) \in W_h.
\end{array} \tag{15}
$$

Here $\mathbf{C} = \left\{ C_{ij} \right\}_{i,j=1}^{6}$, $\mathbf{B} = \left\{ B^{ij}(\boldsymbol{\theta}) \right\}_{i,j=1}^{11}$, $\boldsymbol{\Phi}(\boldsymbol{\theta}) = \left\{ \Phi^i(\boldsymbol{\theta}) \right\}_{i=1}^{11}$, β ar

e data presented in [1], heat flux data q^+, q^- are given on Ω_+, Ω_-, also surface loads $\hat{\boldsymbol{\sigma}}(\mathbf{r}, t)$ are described such as

$$
\hat{\boldsymbol{\sigma}}(\mathbf{r}, t) = \{\hat{\sigma}_i(\boldsymbol{\alpha}, \alpha_3, t)\}_{i=1}^{3} =
$$
$$
= \left\{
\begin{array}{l}
\boldsymbol{\sigma}^+(\boldsymbol{\alpha}, t) = \{\sigma_i^+(\boldsymbol{\alpha}, t)\}_{i=1}^{3}, \ if \ \ \boldsymbol{\alpha} \in \Omega_+, \\
\boldsymbol{\sigma}^-(\boldsymbol{\alpha}, t) = \{\sigma_i^-(\boldsymbol{\alpha}, t)\}_{i=1}^{3}, \ if \ \ \boldsymbol{\alpha} \in \Omega_-.
\end{array}
\right.
$$

$$
\lambda_\Omega(\boldsymbol{\theta}, \boldsymbol{\xi}) = \theta_0^{-1} \sum_{i,j=1}^{2} \iint\limits_{\Omega} \lambda \left[\sum_{k=1}^{2} \frac{\chi_k^{i+j-2}}{A_k^2} \frac{\partial \theta_i}{\partial \alpha_k} \frac{\partial \xi_j}{\partial \alpha_k} \right.
$$
$$
\left. + (ij - i - j + 1) \phi^{i+j-4} \theta_i \xi_j \right] A_1 A_2 d\boldsymbol{\alpha},
$$
$$
\kappa_\Omega(\boldsymbol{\theta}, \boldsymbol{\xi}) = \left\{ (\kappa^+ + \kappa^-) \theta_1 \xi_1 \right.
$$
$$
\left. + (\kappa^+ - \kappa^-) \frac{h}{2} [(k_1 + k_2) \theta_1 \xi_1 + (\theta_1 \xi_2 + \theta_2 \xi_1)] \right\} \iint\limits_{\Omega} A_1 A_2 d\boldsymbol{\alpha},
$$

$$
\begin{array}{l}
\phi^n := \displaystyle\int_{-h/2}^{h/2} \alpha_3^n (1 + \alpha_3 k_1)(1 + \alpha_3 k_2) d\alpha_3, \\
\chi_m^n = \displaystyle\int_{-h/2}^{h/2} (\alpha_3)^n \frac{(1 + \alpha_3 k_1)(1 + \alpha_3 k_2)}{(1 + \alpha_3 k_m)^2} d\alpha_3, \ m = 1, 2.
\end{array} \tag{16}
$$

Here κ^+, κ^- are the heat transfer coefficients on the surfaces Ω_+, Ω_-, respectively.

Details of the construction of the problem (14) see [1].

IV. VIBRATION VARIATIONAL PROBLEM STATEMENT

We suppose that the harmonic loadings with angular frequency ω are applied to the thin shell

$$
\begin{array}{l}
l(t) = l_c \cos \omega t + l_s \sin \omega t, \\
r(t) = r_c \cos \omega t + r_s \sin \omega t, \quad \forall t \in (0, T].
\end{array} \tag{17}
$$

Then the approximate solutions of problem (14) can be looked for in the form of the following expansions:

$$
\begin{array}{l}
\mathbf{s}(\boldsymbol{\alpha}, t) = \mathbf{s}_c(\boldsymbol{\alpha}) \cos \omega t + \mathbf{s}_s(\boldsymbol{\alpha}) \sin \omega t, \\
\boldsymbol{\theta}(\boldsymbol{\alpha}, t) = \boldsymbol{\theta}_c(\boldsymbol{\alpha}) \cos \omega t + \boldsymbol{\theta}_s(\boldsymbol{\alpha}) \sin \omega t,
\end{array} \tag{18}
$$

where $\mathbf{s}_c(\boldsymbol{\alpha})$, $\mathbf{s}_s(\boldsymbol{\alpha})$, $\boldsymbol{\theta}_c(\boldsymbol{\alpha})$, $\boldsymbol{\theta}_s(\boldsymbol{\alpha})$ are the unknown amplitudes of vector of mechanical displacements and temperature respectively.

Substituting expressions (17) and (18) into variational problem (15) and neglecting its initial conditions, we obtain the variational problem for force harmonic vibrations of thermo-elastic thin shell:

$$\begin{cases} \text{given } \omega > 0, (l_1, l_2, r_1, r_2) \in \mathbf{W}' = \mathbf{\Phi}' \times \mathbf{\Phi}', \mathbf{\Phi}' = W_h' \times Q_h'; \\ \text{find } \quad \boldsymbol{\psi} = \{\mathbf{s}_c, \boldsymbol{\theta}_c, \mathbf{s}_s, \boldsymbol{\theta}_s\} \in \mathbf{W} = \mathbf{\Phi} \times \mathbf{\Phi}) \\ \text{such that } \forall \{\mathbf{v}_c, \xi_c, \mathbf{v}_s, \xi_s\} \in \mathbf{W} \\ -\omega^2 m_\Omega(\mathbf{s}_c, \mathbf{v}_s) + \omega a_\Omega(\mathbf{s}_s, \mathbf{v}_s) + c_\Omega(\mathbf{s}_c, \mathbf{v}_s) \\ \qquad - b_\Omega(\boldsymbol{\theta}_c, \mathbf{v}_s) = < l_c, \mathbf{v}_s >, \\ -\omega^2 m_\Omega(\mathbf{s}_s, \mathbf{v}_c) + \omega a_\Omega(\mathbf{s}_c, \mathbf{v}_c) + c_\Omega(\mathbf{s}_s, \mathbf{v}_c) \\ \qquad - b_\Omega(\boldsymbol{\theta}_s, \mathbf{v}_c) = < l_s, \mathbf{v}_c >, \\ \omega \Xi_\Omega(\boldsymbol{\theta}_s, \xi_c) + \Lambda_\Omega(\boldsymbol{\theta}_c, \xi_c) + \omega b_\Omega(\xi_c, \mathbf{s}_s) = < r_c, \xi_c > \\ -\omega \Xi_\Omega(\boldsymbol{\theta}_c, \xi_s) + \Lambda_\Omega(\boldsymbol{\theta}_s, \xi_s) - \omega b_\Omega(\xi_s, \mathbf{s}_c) = < r_s, \xi_s >. \end{cases} \tag{19}$$

Having added all the equations of the problem (19) we introduce the linear form $\mathrm{X}_\omega : \mathbf{W} \to \Re$:

$$\begin{aligned} < \mathrm{X}_\omega, \mathbf{w} > :&= < l_c, \mathbf{v}_s > - < l_s, \mathbf{v}_c > \\ &+ \omega^{-1} \left(< r_c, \xi_c > + < r_s, \xi_s > \right), \\ &\forall\ \mathbf{w} = (\mathbf{v}_c, \xi_c, \mathbf{v}_s, \xi_s) \in \mathbf{W}, \end{aligned} \tag{20}$$

and the bilinear form $\Pi_\omega : \mathbf{W} \times \mathbf{W} \to \Re$:

$$\begin{aligned} \Pi_\omega(\boldsymbol{\psi}, \mathbf{w}) &= -\omega^2 \left[m_\Omega(\mathbf{s}_c, \mathbf{v}_s) - m_\Omega(\mathbf{s}_s, \mathbf{v}_c) \right] \\ &+ \omega \left[a_\Omega(\mathbf{s}_c, \mathbf{v}_c) + a_\Omega(\mathbf{s}_s, \mathbf{v}_s) \right] \\ &+ \left[c_\Omega(\mathbf{s}_c, \mathbf{v}_s) - c_\Omega(\mathbf{s}_s, \mathbf{v}_c) \right] \\ &+ \left[\Xi_\Omega(\boldsymbol{\theta}_c, \xi_s) + \Xi_\Omega(\boldsymbol{\theta}_s, \xi_c) \right] \\ &- \left[b_\Omega(\boldsymbol{\theta}_c, \mathbf{v}_s) - b_\Omega(\boldsymbol{\theta}_s, \mathbf{v}_c) \right] \\ &+ \left[b_\Omega(\xi_c, \mathbf{s}_s) - b_\Omega(\xi_s, \mathbf{s}_c) \right] \\ &+ \omega^{-1} \left[\Lambda_\Omega(\boldsymbol{\theta}_c, \xi_c) + \Lambda_\Omega(\boldsymbol{\theta}_s, \xi_s) \right] \\ &\quad \forall \boldsymbol{\psi} = \{\mathbf{s}_c, \boldsymbol{\theta}_c, \mathbf{s}_s, \boldsymbol{\theta}_s\} \in \mathbf{W}, \\ &\quad \forall\ \mathbf{w} = (\mathbf{v}_c, \xi_c, \mathbf{v}_s, \xi_s) \in \mathbf{W}. \end{aligned} \tag{21}$$

Then variational problem for forced harmonic vibrations of the thermoviscoelastic thin wall body can be rewritten as follows:

$$\begin{cases} \text{given } \omega > 0, \quad < \mathrm{X}_\omega, \mathbf{w} > \in \mathbf{W}' = \mathbf{\Phi}' \times \mathbf{\Phi}'; \\ \text{find } \quad \boldsymbol{\psi} = \{\mathbf{s}_c, \boldsymbol{\theta}_c, \mathbf{s}_s, \boldsymbol{\theta}_s\} \in \mathbf{W} = \mathbf{\Phi} \times \mathbf{\Phi} \text{ such that} \\ \Pi_\omega(\boldsymbol{\psi}, \mathbf{w}) = < \mathrm{X}_\omega, \mathbf{w} > \quad \forall\ \mathbf{w} = (\mathbf{v}_c, \xi_c, \mathbf{v}_s, \xi_s) \in \mathbf{W}. \end{cases} \tag{22}$$

V. Well-Posedness of the Variational Problem

Let us introduce a scalar product on the space W in the following way:

$$\begin{aligned} ((\mathbf{y}, \mathbf{w})) &= a_\Omega(\mathbf{s}_c, \mathbf{v}_c) + a_\Omega(\mathbf{s}_s, \mathbf{v}_s) + \\ &\quad \Lambda_\Omega(\boldsymbol{\theta}_c, \xi_c) + \Lambda_\Omega(\boldsymbol{\theta}_s, \xi_s) \\ &\forall \mathbf{y} = \{\mathbf{s}_c, \boldsymbol{\theta}_c, \mathbf{s}_s, \boldsymbol{\theta}_s\} \in \mathbf{W}, \\ &\quad \forall\ \mathbf{w} = (\mathbf{v}_c, \xi_c, \mathbf{v}_s, \xi_s) \in \mathbf{W}. \end{aligned} \tag{23}$$

And we introduce a norm generated by the scalar product (23):

$$||| \mathbf{y} |||^2 = ((\mathbf{y}, \mathbf{y})) \quad \forall \mathbf{y} \in \mathbf{W}. \tag{24}$$

Then we can easily notice the following estimations:

$$\begin{aligned} &\Pi_\omega(\mathbf{y}, \mathbf{w}) \le M_c(\omega) ||| \mathbf{y} ||| \cdot ||| \mathbf{w} |||. \\ &M_c(\omega) = C \max\{\omega^{-1}, 1, \omega, \omega^2\} \quad \forall\ \mathbf{y}, \mathbf{w} \in \mathbf{W}, \end{aligned} \tag{25}$$

$$\begin{aligned} &|< \mathrm{X}_\omega, \mathbf{w} >| \le M_s(\omega) \| \mathrm{X}_\omega \| \cdot ||| \mathbf{w} |||. \\ &M_s(\omega) = C \max\{\omega^{-1}, 1\} \quad \forall\ \mathbf{w} \in \mathbf{W}. \end{aligned} \tag{26}$$

Here and everywhere the symbol C – a positive constant value, independent on solutions of variational problem (22).

Now for confirm **W**-ellipticity of the bilinear form $\Pi_\omega : \mathbf{W} \times \mathbf{W} \to \Re$ we consider the expression for $\Pi_\omega(\mathbf{w}, \mathbf{w})$

$$\begin{aligned} \Pi_\omega(\mathbf{w}, \mathbf{w}) &= \omega \left[a_\Omega(\mathbf{s}_c, \mathbf{s}_c) + a_\Omega(\mathbf{s}_s, \mathbf{s}_s) \right] \\ &+ \omega^{-1} \left[\Lambda_\Omega(\boldsymbol{\theta}_c, \boldsymbol{\theta}_c) + \Lambda_\Omega(\boldsymbol{\theta}_s, \boldsymbol{\theta}_s) \right] \\ &\ge \omega \left[a_\Omega(\mathbf{s}_c, \mathbf{s}_c) + a_\Omega(\mathbf{s}_s, \mathbf{s}_s) \right] \\ &+ \omega^{-1} \left[\Lambda_\Omega(\boldsymbol{\theta}_c, \boldsymbol{\theta}_c) + \Lambda_\Omega(\boldsymbol{\theta}_s, \boldsymbol{\theta}_s) \right] \\ &\ge \eta(\omega) ||| \mathbf{w} |||^2, \quad \eta(\omega) = \min\{\omega^{-1}, \omega\} \\ &\qquad \forall\ \mathbf{w} = (\mathbf{v}_c, \xi_c, \mathbf{v}_s, \xi_s) \in \mathbf{W}. \end{aligned} \tag{27}$$

Since the statements (25)-(27) are proofed and they are actually the conditions of Lions-Lax-Milgram theorem, the following theorem is then correct:

Theorem 6.1. For each $\omega > 0$ the variational problem (22) has a unique solution $\boldsymbol{\psi} \in \mathbf{W}$, which satisfies the relation:

$$||| \boldsymbol{\psi} ||| \le \eta^{-1}(\omega) M_s(\omega) \| \mathrm{X}_\omega \|_* . \tag{28}$$

VI. Galerkin Discretization

Standard Galerkin scheme was used for solving of variational problem (22). We chose some finite-dimensional subspace $\mathbf{W}_h = \mathbf{\Phi}_h \times \mathbf{\Phi}_h$, $\mathbf{\Phi}_h \subset \mathbf{\Phi}$, $\dim \mathbf{W}_h = N(h) < +\infty$. Thus, the Galerkin-discretized variational problem (23) looks in the following way:

$$\begin{cases} \text{given } \omega > 0, \ \mathrm{X}_\omega \in \mathbf{W}', \mathbf{W}_h \subset \mathbf{W}, \dim \mathbf{W} < +\infty,; \\ \text{find } \quad \boldsymbol{\psi}_h = \{\mathbf{s}_{ch}, \boldsymbol{\theta}_{c_h}, \mathbf{s}_{s_h}, \boldsymbol{\theta}_{sh}\} \in \mathbf{W}_h \text{ such that} \\ \Pi_\omega(\boldsymbol{\psi}_h, \boldsymbol{\varphi}) = < \mathrm{X}_\omega, \boldsymbol{\varphi} > \quad \forall\ \boldsymbol{\varphi} \in \mathbf{W}_h. \end{cases} \tag{29}$$

We can say the problem (23) is well-posed same as the problem (29). In the space **W** we select some basic functions $\{\mathbf{w}\}_{i=1}^{\infty}$. For each natural number $m \ge 0$, $h = 1/m$ a sequence of approximation spaces $\mathbf{W}_h$ and operators of orthogonal projection $Pr_h : \mathbf{W} \to \mathbf{W}_h$ are defined so that a set $\{\mathbf{w}\}_{i=1}^{m}$ is a basis of $\mathbf{W}_h$, and $((\boldsymbol{\psi} - Pr_h\boldsymbol{\psi}, \mathbf{w})) = 0 \quad \forall \boldsymbol{\psi} \in \mathbf{W}, \forall \mathbf{w}_h \in \mathbf{W}_h$. Now variational problem (22) is replaced by a sequence of the following problems:

$$\begin{cases} \text{given } \omega > 0,\ \mathrm{X}_\omega \in \mathbf{W}',\ h > 0, \\ \mathbf{W}_h \subset \mathbf{W}, \dim \mathbf{W} = m < +\infty; \\ \text{find } \ \boldsymbol{\psi}_h = \{\mathbf{s}_{ch}, \boldsymbol{\theta}_{c_h}, \mathbf{s}_{s_h}, \boldsymbol{\theta}_{sh}\} \in \mathbf{W}_h \text{ such that} \\ \Pi_\omega(\boldsymbol{\psi}_h, \boldsymbol{\varphi}) = <\mathrm{X}_\omega, \boldsymbol{\varphi}> \quad \forall\ \boldsymbol{\varphi} \in \mathbf{W}_h. \end{cases} \tag{30}$$

Theorem 5.1. Let be $\forall \boldsymbol{\psi} \in \mathbf{W}$ a solution of problem (22) with parameter $\omega > 0$. Then a sequence of Galerkin approximations $\forall \boldsymbol{\psi}_h \in \mathbf{W}$ is unambiguously defined by the solutions of the problems (30) and has the following properties:

$$|||\ \boldsymbol{\psi} - \boldsymbol{\psi}_h\ ||| \le \eta^{-1} M_c(\omega) \inf_{\forall \mathbf{w} \in \mathbf{W}_h} |||\ \boldsymbol{\psi} - \mathbf{w}\ |||\ \forall h > 0; \tag{31}$$

$$\lim_{h \to 0} |||\ \boldsymbol{\psi} - \boldsymbol{\psi}_h\ ||| = 0. \tag{32}$$

Proof. The correctness can be done like in [7]. Sinse for the inequality (31)

$$\Pi_\omega(\boldsymbol{\psi} - \boldsymbol{\psi}_h, \mathbf{w}) = 0 \quad \forall \mathbf{w} \in \mathbf{W}_h$$

and the estimation

$$\begin{aligned} a\,|||\ \boldsymbol{\psi} - \boldsymbol{\psi}_h\ |||^2 &\le \Pi_\omega(\boldsymbol{\psi} - \boldsymbol{\psi}_h, \boldsymbol{\psi} - \boldsymbol{\psi}_h) = \Pi_\omega(\boldsymbol{\psi} - \boldsymbol{\psi}_h, \boldsymbol{\psi} - \mathbf{w}) \\ &\le M_c(\omega)\,|||\ \boldsymbol{\psi} - \boldsymbol{\psi}_h\ |||\ |||\ \boldsymbol{\psi} - \mathbf{w}\ |||\ \forall \mathbf{w} \in \mathbf{W}_h. \end{aligned}$$

Taking into account the density of sequence of spaces $\{\mathbf{W}_h\}$ in the separable space $\mathbf{W}$:

$$\lim_{h \to 0} |||\ \boldsymbol{\psi} - Pr_h\mathbf{w}\ ||| = 0\ \forall \mathbf{w} \in \mathbf{W}. \tag{34}$$

Therefore, basing on the equality

$$\inf_{\forall \mathbf{w} \in \mathbf{W}_h} |||\ \boldsymbol{\psi} - \mathbf{w}\ ||| = |||\ \boldsymbol{\psi} - Pr_h\boldsymbol{\psi}\ ||| \tag{35}$$

and inequality (31) we can conclude the correctness of (32), when $\omega > 0$.

VII. Numerical Experiments

Below we present some results of our numerical experiments on computations of eigenvalue problem for our semidiscreted model. We consider a circular cylindrical shell made of homogenous material with radius radius R=10 m and length L =10 m and which is under constant temperature. Young's modulus of shell material is equa to 1 Pa, Poisson's coefficient is 0.3, and mass density is 1 kg/ m3.

Boundary conditions are following type:

$$u_2 = \gamma_2 = u_3 = \gamma_3 = 0, \quad on\ \ \alpha_1 = 0, \alpha_1 = L;$$
$$u_1 = \gamma_1 = u_3 = \gamma_3 = 0 \quad on\ \alpha_2 = 0, \alpha_2 = \pi/8.$$

The first column of the Table includes the number of quadratic finite element mesh , the second and third columns include the computed eigenvalues $\omega^2 \cdot 10^3$ and their relative errors δ taking from [1]. Same our results are in the two last columns of the Table.

Mesh	$\omega^2 \cdot 10^3$ [1])	δ, %	$\omega^2 \cdot 10^3$	δ, %
3×3	0,3305068	11,7	0,3583986	11,9
4×4	0,3024595	2,23	0,3401321	6,2
5×5	0,2974929	0,55	0,3288990	2,7

VIII. Conclusion

The partially variational problem for a thin wall body was constructed on base the dynamic coupled three-dimensional problem. Under the assumptions about harmonic vibration with known angular frequency we have formulated the corresponding variational problem and then we proved its well posedness. These results shows that we can use well known finite element approximations for Sobolev spaces and obtain the convergence rate its errors.

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