УДК 517.53

Presence of function by its asymptotic decomposition

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The series $\sum_{n=1}^{\infty} a_n z^n$ is an asymptotic decomposition for $g(z)$ when $z \to 0$ if for every N

$$\lim_{z\to 0} \frac{g(z) - \sum_{n=1}^{N} a_n z^n}{z^N} = 0 \ \text{ for every } N. \tag{1}$$

Denote $z = \frac{1}{\lambda}$, $\lambda \to \infty$, then from (1) we have

$$\lim_{z\to 0} \left[\left(g\left(\frac{1}{\lambda}\right) - \sum_{n=1}^{N} \frac{a_n}{\lambda^N} \right) \lambda^N \right] = 0.$$

Let $f(\lambda) = g(\frac{1}{\lambda})$ then

$$\lim_{\lambda\to\infty} \left(f(\lambda) - \sum_{n=1}^{N} \frac{a_n}{\lambda^N} \right) \lambda^N = 0.$$

We introduce the following notations $g(z) \sim \sum_{n=1}^{N} a_n z^n$, $z \to 0$,

$$f(\lambda) = g\left(\frac{1}{\lambda}\right) \sim \sum_{n=1}^{N} \frac{a_n}{\lambda^n} \ , \ \lambda \to \infty. \tag{2}$$

We denote $c_k(f) = \sup\limits_{x>0}(x^k|f(x)|)$, and let $E \subset L^2(0;\infty)$ be the linear subspace

$$E = \{f \in L^2(0;\infty); c_k(f) < \infty, k = 1, 2, \ldots\}.$$

By the same symbol $E \subset l^2[1;\infty)$ we denote the subspace of number sequences $\{h(N), N = = 1, 2, \ldots\}$ such that

$$c_k(h(*)) = \sup_N (N^k(h(N))), \ k = 1, 2, \ldots .$$

In other words if $f \in E \subset L^2(0;\infty)$ and $h(*) \in E \subset l^2[1;\infty)$ then

$$|f(x)| \leqslant \frac{c_k}{x^k} \ , \ \ x \in (0;\infty) \ \text{ and } \ |h(N)| < \frac{c_k}{x^k} \ , \ k, N = 1, 2, \ldots . \tag{3}$$

Definition 1. For the function f the symbol $f(x) \xrightarrow{E} 0$, $x \to \infty$, signify $f \in$

$\in E \subset L^2(0;\infty)$. By analogy, for the sequence $h(*)$, the symbol $h(N) \overset{E}{\to} 0$, $N \to \infty$, signify $h(*) \in E \subset l^2[1;\infty)$.

Definition 2. We say that sequence x_n is E-convergent to the number x if $\{x_N - -x\} \in E$.

In view of (3) there exists $c_k > 0$ such that

$$|x_N - x| < \frac{c_k}{N^k} \ , \ k, N = 1, 2, \ldots .$$

Note that if E-limit is equal to x (signifying is $\lim\limits_{N\to\infty} x_N \overset{E}{=} x$), then $\lim\limits_{N\to\infty} x_N = = x$ but not inversely.

Definition 3. The sequence of asymptotic decomposition (2)

$$f_N(\lambda) \sim \frac{a_{1N}}{\lambda} + \ldots + \frac{a_{lN}}{\lambda^l} + \ldots, \quad \lambda > 1, \ N = 1, 2, \ldots \quad , \tag{4}$$

is called uniformly bounded if there exist constants M_l, $l = 1, 2, \ldots$ that are independent from N such

$$sup_{\lambda>1} \left| \lambda^{l+1} \left(f_N(\lambda) - \left(\frac{a_{1N}}{\lambda} + \ldots + \frac{a_{lN}}{\lambda^l} \right) \right) \right| \leqslant M_l \ , \ l = 1, 2, \ldots$$

for all $N = 1, 2, \ldots$.

Theorem . If the sequence of the asymptotic decompositions (4) is uniformly bounded and there exist two E-limits $f(\lambda) = \lim\limits_{N\to\infty} f_N(\lambda)$ and $a_l = \lim\limits_{N\to\infty} a_{lN}$ then the function $f(\lambda)$admits the following asymptotic decomposition

$$f(\lambda) \sim \frac{a_1}{\lambda} + \ldots + \frac{a_l}{\lambda^l} \ . \tag{5}$$

Example. Let $f(x) = e^{-x}$, $c_n(f) = \sup\limits_{x>0}(x^n e^{-x}) = \sup e^{\ln(x^n e^{-x})} = \sup e^{nln-x}$,

$$(e^{nln-x})' = e^{nlnx-x} \left(\frac{n}{x} - 1 \right) = 0 \ , \ x_{\max} = n, \ (x^n e^{-x})_{\max} = n^n e^{-n} \ .$$

Conclusion. We may pass to limit term by term in a sequence of asymptotic decomposition (see (4),(5)).