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Properties of Fibonacci and Lucas polynomials

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The k-Fibonacci and k-Lucas polynomials [2] are the natural extension of the k-Fibonacci and k-Lucas numbers and many of their properties admit a straightforward proof. The Fibonacci sequence and the golden ratio have appeared in many fields of science including high energy physics, cryptography and coding [1, 5].

Definition 1. The Fibonacci polynomial $F_n(x)$ is defined recurrently relation

$$F_{n+1}(x) = xF_n(x) + F_{n-1}(x) \tag{1}$$

with $F_0(x) = 0,\ F_1(x) = 1$ **for** $n \geqslant 1$.

Fibonacci polynomials for negative subscripts are defined as $F_{-n}(x) = (-1)^{n+1}F_n(x)$ for $n \geqslant 1$.

Definition 2. The Lucas polynomial $L_n(x)$ is defined by the relation

$$L_{n+1}(x) = xL_n(x) + L_{n-1}(x) \tag{2}$$

with $L_0(x) = 2,\ L_1(x) = x$ **for** $n \geqslant 1$ **and** $L_n(x) = F_{n+1}(x) + F_{n-1}(x)$ **for** $n \in \mathbb{Z}$.

If $x = 1$, the classic Fibonacci and Lucas sequences are obtained from (1), (2) [3–5].

Lemma. *If X is a square matrix with $X^2 = xX + I$, then $X^n = F_n(x)X + F_{n-1}(x)I$ for all $n \in \mathbb{Z}$.*

Theorem 1. *Let* $Q(x) = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}$. *Then*

1) $Q(x)^n = \begin{pmatrix} F_{n+1}(x) & F_n(x) \\ F_n(x) & F_{n-1}(x) \end{pmatrix}$ *for all* $n \in \mathbb{Z}$;

2) $\det Q(x)^n = (-1)^n$ *(Cassini's identity).*

Theorem 2. *Let* $R(x) = \begin{pmatrix} x & 2 \\ 2 & -x \end{pmatrix}$. *Then*

1) $Q(x)R(x) = R(x)Q(x)$;

2) $Q(x)^n R(x) = \begin{pmatrix} L_{n+1}(x) & L_n(x) \\ L_n(x) & L_{n-1}(x) \end{pmatrix}$ *for all* $n \in \mathbb{Z}$;

3) $\det(Q(x)^n R(x)) = (-1)^{n+1}(x^2 + 4)$ *(Cassini's identity).*

Theorem 3. *The n-th Fibonacci polynomial may be written as* $F_n(x) = \frac{\sigma^n - (-\sigma)^{-n}}{\sigma + \sigma^{-1}}$ *being* $\sigma = \frac{x + \sqrt{x^2+4}}{2}$ *(Binet's formula).*

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