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P. Tymoshchuk
Lviv Polytechnic National University

PARALLEL SORTING BASED ON IMPULSE K-WINNERS-TAKE-ALL NEURAL NETWORK

A continuous-time *K*-winners-take-all (KWTA) neural network (NN) which is capable of selecting the largest *K* of *N* inputs, where $1 £ K < N$ is described. The network is modeled by a state equation with a discontinuous right-hand side and by an output equation. The state equation contains an impulse train defined by a sum of Dirac delta functions. The main advantage of the network comparatively to other close analogs is widening convergence speed limitations. The network is applied for parallel sorting. Theoretical results are derived and illustrated with computer simulation example that demonstrates the network's performance.

Key words: continuous-time, *K*-winners-take-all (KWTA) neural network (NN), state equation with a discontinuous right-hand side, impulse train, Dirac delta function, parallel sorting.

ПАРАЛЕЛЬНЕ СОРТУВАННЯ НА ОСНОВІ ІМПУЛЬСНОЇ НЕЙРОННОЇ МЕРЕЖІ ТИПУ“K-WINNERS-TAKE-ALL”

Описано нейронну мережу (НМ) неперервного часу типу “K-winners-take-all” (KWTA), яка ідентифікує найбільші K з-поміж N входів, де керуючий сигнал 1£K<N. Мережа описується рівнянням стану з розривною правою частиною і вихідним рівнянням. Рівняння стану містить шлейф імпульсів, які описуються сумою дельта-функцій Дірака. Головною перевагою мережі порівняно з іншими близькими аналогами є відсутність обмежень на швидкість збіжності. Наведено застосування мережі для швидкого паралельного сортування. Отримані теоретичні результати проілюстровано прикладом комп'ютерного моделювання, який демонструє ефективність мережі.

Ключові слова: мережа неперервного часу, нейронна мережа (НМ) типу “K-winners-take-all” (KWTA), рівняння стану з розривною правою частиною, дельта-функція Дірака, паралельне сортування.

1. Introduction

K-winners-take-all (KWTA) neural networks (NNs) choose the *K* largest out of a set of *N* inputs, where $1 £ K < N$ is a positive integer [1–3]. In the particular case when $K = 1$, the KWTA network is a winner-takes-all (WTA) network [4], [5].

KWTA NNs have numerous applications, in particular, in signal and data processing, for pattern recognition, in making decision, in competitive learning, and in sorting. The KWTA networks are used in vision systems and telecommunications, for decoding, filtering, image processing, classification, clustering, navigation of mobile robots, and for feature extraction. KWTA tools are employed for networking cognitive phenomena and impulse neural networks [6].

There are different NNs for solving the WTA and KWTA problems. For instance, a design, fabrication and verification of WTA mechanism in series of CMOS integrated circuits is given in [7]. Continuous-time KWTA NNs realized in analog hardware are faster, more power-efficient and more compact compared to digital realizations [8, 9].

Neurons can be modeled by using impulses [10]. Impulse can be described using Dirac delta function or exponential delay. Dirac delta functions and exponential delays are applied for simulating membrane potential of neuron. Sums of exponential delays between pulses and delta functions are employed for describing the level of activity of neuron that averages influences of postsynaptic pulses on neuron. Neurons with signals in the form of impulses are applied for psychophysical phenomena modeling in cortex plastic networks. Computations based on impulses of central neural system comparatively to traditional approaches can have significant advantages of performance in the process of solving many high dimensional tasks.

In this paper, an impulse train defined by the sum of Dirac delta functions is used in continuous-time KWTA NN. As a result, in contrast to other close analogs for which the state variable trajectory of the network to the KWTA operation has continuous nonlinear, continuous piecewise-linear or continuous linear shape, the state variable trajectory of the network to the KWTA operation has stepping form. Therefore, the theoretical speed of convergence of the network state variable trajectories to the WTA operation is approached to infinity if the period of firing impulses goes to zero. This implies that the NN is capable to select instantaneously without transient dynamics, the *K* largest out of N inputs. This is the main advantage of the network. Application of the NN for fast and accurate parallel sorting is presented. Computer simulations illustrating and confirming theoretical results are given.

2. Problem formulation of the network design

Consider a vector of inputs $\boldsymbol{a} = \left(a_{n_1}, a_{n_2}, ..., a_{n_N}\right)^T \in \Re^n$, $1 < N < \infty$ with unknown finite value elements. The inputs are different and can be positioned in a descending order of value satisfying the following inequalities:

$$\infty > a_{n_1} > a_{n_2} > ... > a_{n_N} > -\infty, \tag{1}$$

where $n_1, n_2, ..., n_N$ are the unknown numbers of the first maximal input, the second maximal input and so on up to the *N*th maximal input. It is required to design the NN that can perform a parallel sorting based on KWTA network without transient dynamics, i.e. instantaneously. The KWA NN should process the input vector $\boldsymbol{a}$ to obtain a corresponding output vector $\boldsymbol{b} = \left(b_{n_1}, b_{n_2}, ..., b_{n_N}\right)^T$ such that the following KWTA property is met [1], [6]:

$$b_{n_i} > 0, i = 1, 2, ..., K;\ b_{n_j} < 0,\ j = K+1, K+2, ..., N. \tag{2}$$

It has to be also possible to determine the KWTA operation as follows [11]:

$$d_{n_i} = 1, i = 1, 2, ..., K;\ d_{n_j} = 0, j = K+1, K+2, ..., N. \tag{3}$$

Note that in the case of using outputs (3) of the network only *K* winners out of *N* inputs are selected. No information is available on the arranging of inputs by value that can be used further, for example, for solving problems of clustering, classification, etc. [6].

3. Impulse continuous-time KWTA neural network

Let us consider the continuous-time KWTA NN presented in [6] and described by the following state equation:

$$\frac{dx}{dt} = rD(x)\sum_{l=0}^{m} \delta(t - t_l) \tag{4}$$

and by output equation

$$b_{n_k} = a_{n_k} - x, k = 1, 2, ..., N, \tag{5}$$

where

$$D(x) = \sum_{k=1}^{N} S_k(x) - K \tag{6}$$

is a function of difference between obtained and required quantities of positive outputs,

$$S_k(x)=\begin{cases}1, & if \quad a_{n_k}-x>0;\\ 0, & otherwise\end{cases} \tag{7}$$

is a step function,

$$\delta(t-t_l)=\begin{cases}+\infty, & if \quad t=t_l;\\ 0, & if \quad t\neq t_l\end{cases} \tag{8}$$

is an impulse shaped as the Dirac delta function, $\sum_{l=1}^{m}\delta(t-t_l)$ is a train of impulses, t_l is an instant of time of firing impulses, m is a number of impulses necessary for reaching a convergence of search process to the KWTA operation, and r is a resolution of the network.

In contrast to other close analogs with continuous nonlinear trajectory of state variable *x*, the network has stepping state trajectory of *x*. Therefore, it can reach theoretically arbitrary finite speed of processing inputs defined by the period of firing impulses. If this period approaches zero, the network processing time of inputs also goes to zero. This is the main advantage of the network. Practical speed of processing inputs of the network is limited by bounds of its software or hardware realization. In particular, the network software realization is has limited accuracy of computation. If the network is realized in hardware, the restrictions are finite speed of comparator, non-idealities of integrator, mismatch, etc.

4. Parallel sorting based on the network

Henceforward, we will refer to the network with train of impulses for special values of *K*. In particular, we will apply #WTA network for *K* = # for arbitrary special quantity desired. As known, sorting is a basic operation of processing data. If parallel sorting is used, an order of sorting can be represented as a permutation matrix. In such a matrix, "*1*" in the row marked by a_i and column labeled by g_j can be determined as the *i*th item in an unsorted list and *j*th item in a sorted list. The corresponding matrix of permutation represents an unsorted list and its ordered list. The matrix of permutation can be converted to the sorting matrix. Let us modify algebraic expressions of the matrix for the results of sorting used in [12] and present them as follows:

$$c_1=a^TS^1\ ,\ c_{K+1}=a^T\left(S^{K+1}-S^K\right), \tag{9}$$

where the elements of *K*th column $S^K=\left[S_1^K,S_2^K,...,S_N^K\right]^T$, $K=1,2,\ .\ .\ .\ ,N\text{-}1$ of the matrix of sorting are determined by the step functions (7) by using the state equation (4) of the impulse KWTA network. As $S^N=[1,1,...,1]^T$, we apply *N-1* equations (7) and each equation computes one column of the matrix of sorting from left to right with *k* increasing from *1* to *N-1*. Therefore, only *N-1* neurons are required compared to the other analog networks of sorting with N^2 neurons. In particular, the 1WTA network is employed to define the maximal item of the list. The 1WTA network and 2WTA network are used for computing the second element in the list in parallel mode without recalculating the first element. The 2WTA network and 3WTA network are applied for determining the third element in the list in parallel without recalculating the second element and so on. As such, the entire list of *N* elements can be sorted by applying the KWTA, *K=1,2,…,N-1* networks without the need to calculate the last element by giving $S^N=[1,1,...,1]^T$.

Example. Let it be required to sort elements of vector of inputs $\boldsymbol{a}=(1.4,3.1,2.3,9.2,10,7.6,$ $5.7,4.8,6.9,8.5)^T$.

by using the KWTA neural network modeled by the state equation (4). In this case, applying nine networks modeled by (4), only nine neurons are required in contrast to *100* neurons in the analog network of sorting in [13]. Let us use a 1.81 GHz desktop PC and Euler solver of non-stiff ordinary differential equations (ODE1) with automatic choice of the fixed-step size and sample time *1e-15* in the Simulink environment of Matlab program. We implement an impulse source by sequential connection of impulse generator, differentiator, and absolute value blocks. Fig. 1 presents the trajectories of the state variables in the nine networks modeled by (4) for parallel sorting, computed for $r = 0.5$, $x_0 = 0$, $t = 0.1$ *ps*. The next correct sorting of the inputs is achieved after the time of convergence that is not larger than *1 ps*, i. e. after extremely small time of convergence: $c_1 = 10$, $c_2 = 9.2$, $c_3 = 8.5$, $c_4 = 6.9$, $c_5 = 5.7$, $c_6 = 4.8$, $c_7 = 3.9$, $c_8 = 3.1$, $c_9 = 2.3$, $c_{10} = 1.4$. Note that this time is by six order less than that of the predecessor described in [3].

Conclusion

This paper describes a continuous-time KWTA NN with impulse train. The network can define *K* largest among arbitrary unknown finite value *N* distinct inputs located in an unknown range, where $1 \le K < N$. In contrast to other close competitors, the state variable trajectories of the network piecewise-constant, i. e. stepping form. Therefore, working mode of the network can be reached theoretically instantaneously since the period of firing impulses approximates zero. Thus, the network is capable to determine the *K* maximal out of N inputs without transient dynamics. Application of the network for high speed and accurate parallel sorting is given.

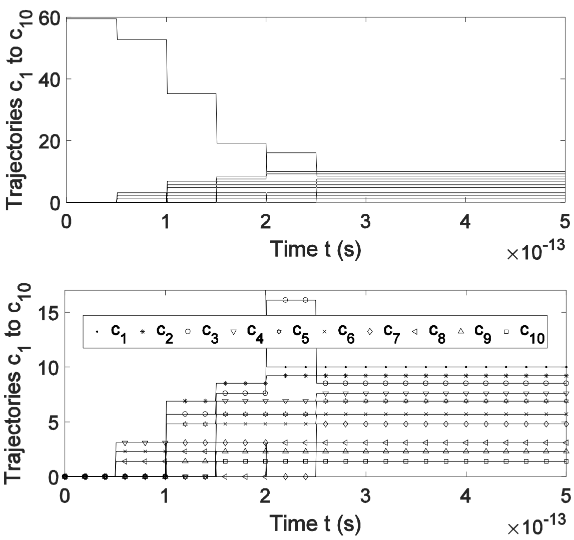

Fig. 1. An illustration of Example, showing trajectories of the states of five networks described by equation (4) for parallel sorting with inputs $\mathbf{a}=[1.4,3.1,2.3,9.2,10,7.6,5.7,4.8,6.9,8.5]^T$*, i.e., N=10, K=1,2,3,4,5,6,7,8,9 and* $x_0 = 0$.

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