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Determination of Approaches for Project Costs Minimization with Use of Dual Problems

S. Chernov[1], S. Titov[1], L. Chernova[1], N. Kunanets[2], L. Chernova[1]

[1] Project Management Department
Admiral Makarov National University of Shipbuilding
Mykolaiv, Ukraine
19chsk56@gmail.com, ss1-ss10@ukr.net
[2] Lviv Polytechnic National University; 12, S. Bandery Str., Lviv, Ukraine
nek.lviv@gmail.com

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Abstract. For determining ways of company development, ensuring the growth of profit in manufacture and sales of certain products, it has been proposed to use an algorithm of constructing a problem being inverse to primal-dual one, for minimization of the project costs. The primal and the inverse problems contribute to improving the efficiency of calculation when determining approaches for minimization of costs. This pair of problems is mutually conjugate. The proposed rigorous approach to obtaining the algorithm of constructing a dual problem is based on the following statement: a problem being inverse to a dual one is a primal (original) problem. The authors have proposed and rigorously proven the algorithm of a general approach to the construction of conjugate problem pairs. Formalization of the algorithm developed allows obtaining easily correct pairs of known dual problems. This permitted proposing and proving the truth of the algorithm of constructing a dual problem for the arbitrary form of a primal problem representation.

Key words: linear optimization, primal problem, dual problem, duality, objective function, constraint system, pairs of dual problems

INTRODUCTION

In the information society, assurance of efficient activities of a company or enterprise is based on highly professional project management, provided for the creation of up-to-date techniques for successful management and consolidation of manpower, finance, material and equipment for the project implementation success. The methodology is based on using modern methods, means and technology of management for achieving relevant results in the calculation of the scope of works, their cost and period.

For determining the project implementation status, a number of values are to be analyzed to define the project content and success during its lifecycle. The project management function is of great importance and consists of planning resources, estimating the cost in progress of the project implementation, forecasting the profit and taking decisions when the expense becomes excessive. The main task of the project cost management is receiving a forecasted profit from its implementation. Various methods shall be used for making adequate calculations. We are going to use the duality principle that is linked in its methodology to the dependent inequality systems theory.

THE STATE OF THE ART REVIEW

Modern techniques of project activities became a subject of many researches. Kevin R. Callahan provides techniques of a highly professional level of project management, preconditions for implementation of modern project management methods, improvement of the existing project`s rate of success [1]. Paul C. Dinsmore, Jeannette Cabanis-Brewin analyze the development of project management from the historical point of view, in view of the professional culture, specialist’s approach and career growth. The authors consider the matters that became the key matters in project management, particularly: the measurement of productivity, selection of project portfolio, formation of corporate systems, organizational culture and structures [2].

Thomas W. Grisham provides analysis of results of studying the best professional approaches to implementation of international projects emphasizing the importance of leadership skills and modern approaches to the formation of virtual teamwork assuring the success of an international project, organization of such business processes as planning, monitoring, and control [3].

Project Management Institute analyzes organizational processes and ways of acquiring professionalism in project management, particularly, development of standards, scientific researching, holding conferences and training workshops, as well as the issuance of qualification in project management [4].

S. D. Bushuev and coauthors analyze global processes on project management with trends in their development running in various branches of knowledge and having an effect on the development of information and communication technology and competences on program and project portfolio management. The work provides results of studying the mechanisms for assessing competences of project managers based on global trends and the flexible Agile methodology. The authors' opinion is that the global trends contributing to the formation of competences on project management are related to the global acceleration, digitalization of the society, development of cloud and fog technology, active implementation of intellectual systems, move from "rational" to "behavioral" economics. Combining these trends allows creating a new platform to assure efficient application of up-to-date methodologies, assessment of project managers' competence [5].

E. B. Danchenko provides a model of indicators that permit analyzing the project deviation based on the integrated management conceptual model (PDIM). The author has analyzed the reasons for such deviations and analyzed in detail the shortcomings creating risks of deviations appearance based on the known IPMA Delta model, construction of cognitive maps and the project systemic interrelation matrix.

The research proposes an indicative model of deviations in projects contributing to more exact identification of the place of negative deviations in the project [6]. However, the publications pay rather little attention to minimization of project costs.

The analysis of the results of applying the proposed algorithms of solving a linear optimization problem to forecast the project cost and determine the ways for its minimization are the objectives of this article.

STATEMENT OF BASIC MATERIAL

The classical manner of the project management methodology realization includes several stages: project environment, objectives and tasks formulation, project implementation strategy development, project planning, immediate technical execution of the project plan items, plan-relevant project fulfillment control. The project management objective is to meet the estimated cost, time and quality of the project implementation. For minimizing the project costs, using the resources management technology holding the expenses within the planned amount, it has been proposed to use the dual problem construction algorithm.

As a rule, the existing methods of transition from primal to the dual linear optimization problem have an expressly economic content. Therefore, they can hardly be used in project management models. Obtaining a formal algorithm of transition to the dual problem and strict proving of these rules constitute the main objectives of this research.

Determination of duality for a standard form linear optimization problem.

Let a primal (original) linear optimization problem be represented in the standard recording form [7,8].

Let the following LO problem be called standard problem:

$$\mathbf{I}:\quad \begin{aligned} &\mathrm{W}_I = \sum_{j=1}^{n} c_j x_j \to \max,\\ &\Omega_I : \sum_{j=1}^{n} a_{ij} x_j \le b_i,\ i = 1,\ldots,m,\\ &x_j \ge 0,\ j = 1,\ldots,n, \end{aligned}$$

or represented as a matrix:

$$\mathbf{I}:\quad \begin{aligned} &\mathrm{W}_I = CX \to \max,\\ &\Omega_I : AX \le B,\\ &X \ge 0. \end{aligned}$$

A problem as given below shall be called dual or conjugate to it:

$$\mathbf{II}:\quad \begin{aligned} &\mathrm{W}_{II} = \sum_{i=1}^{m} b_i y_i \to \min,\\ &\Omega_{II} : \sum_{i=1}^{m} a_{ij} y_i \ge c_j,\ \ j = 1, 2, \ldots, n,\\ &y_i \ge 0,\quad i = 1, 2, \ldots, m, \end{aligned}$$

or represented as a matrix:

$$\mathbf{II}:\quad \begin{aligned} &\mathrm{W}_{II} = Y B - \min,\\ &\Omega_{II} : Y A \ge C^T,\\ &Y \ge 0. \end{aligned}$$

To simplify, let us introduce and comment the following symbols:

$\mathbf{c} = C = c = [\,c_1, c_2, \ldots, c_n\,],\ C \in \mathbf{R}^n$ are the coefficients of the target function W_I of the primal LO problem,

$$\mathbf{x} = X = x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [\,x_1, x_2, \ldots, x_n\,]^T,\ X \in \mathbf{R}^n$$

is the variable (unknown) values (plan) of the primal LO problem; **I** – primal problem symbol; **II** – dual problem symbol,

$$A = [\,a_{ij}\,]_{(m\times n)} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3j} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

is a matrix of the primal problem constraint system coefficients

$$\mathbf{b} = B = b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = [\,b_1, b_2, \ldots, b_m\,]^T, B \in \mathbf{R}^m$$

is the coefficients of the primal problem constraint system right-hand sides,

$$\mathbf{y} = Y = y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = [\,y_1, y_2, \ldots, y_m\,]^T,\ Y \in \mathbf{R}^m$$

are the variable (unknown) values of dual LO problem.

Let us introduce covariant and contravariant vector systems for consideration [9]

$$\mathbf{a}_j = a_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix} = [\,a_{1j}, a_{2j}, \ldots, a_{nj}\,]^T \in \mathbf{R}^m,\ j = 1, 2, \ldots, m$$

are the column vectors (covariant vectors) of matrix A in constraint system Ω_I of the primal problem,

$$\mathbf{a}^i = a^i = [\,a_{i1}, a_{i2}, \ldots, a_{in}\,] \in \mathbf{R}^n,\ i = 1, 2, \ldots, n$$

are the row vectors (contravariant vectors) of matrix A in constraint system Ω_I of the primal problem.

In this case, matrix A of the system coefficients can be represented in vector format:

$$A = \begin{bmatrix} \mathbf{a}^1 \\ \mathbf{a}^2 \\ \vdots \\ \mathbf{a}^m \end{bmatrix} = [\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^m]^{\mathrm{T}} = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n] \in \mathbf{R}^m \otimes \mathbf{R}^n,$$

and the pair of dual problems has the third recording form:

$$\mathrm{W}_I = (\mathbf{c}, \mathbf{x}) \to \max,$$
$$\Omega_I : (\mathbf{a}_j, \mathbf{x}) \le \mathbf{b},\ \mathbf{x} \ge 0,$$

is a primal problem,

$$\mathrm{W}_{II} = (\mathbf{b}, \mathbf{y}) \to \min,$$
$$\Omega_{II} : (\mathbf{a}^i, \mathbf{y}) \ge \mathbf{c},\ \ \mathbf{y} \ge 0.$$

is a dual problem to the primal one, given above.

As a result, we finally have three forms of recording the definition of a problem being dual to the standard LO problem:

$$\mathbf{I}: \begin{array}{l} \mathrm{W}_I = \sum_{j=1}^{n} c_j x_j \to \max, \\ \Omega_I : \sum_{j=1}^{n} a_{ij} x_j \le b_i,\ i = 1, \ldots, m, \\ x_j \ge 0,\ j = 1, \ldots, n, \end{array} \xrightarrow{def\ \ Dual} \mathbf{II}: \begin{array}{l} \mathrm{W}_{II} = \sum_{i=1}^{m} b_i y_i \to \min, \\ \Omega_{II} : \sum_{i=1}^{m} a_{ij} y_i \ge c_j,\ j = 1, 2, \ldots, n, \\ y_i \ge 0,\ \ i = 1, 2, \ldots, m, \end{array}$$

$$\mathbf{I}: \begin{array}{l} \mathrm{W}_I = CX \to \max, \\ \Omega_I : AX \le B, \\ X \ge 0, \end{array} \xrightarrow{def\ \ Dual} \mathbf{II}: \begin{array}{l} \mathrm{W}_{II} = Y\,B - \min, \\ \Omega_{II} : Y\,A \ge C^T, \\ Y \ge 0, \end{array}$$

$$\mathbf{I}: \begin{array}{l} \mathrm{W}_I = (\mathbf{c}, \mathbf{x}) \to \max, \\ \Omega_I : (\mathbf{a}_j, \mathbf{x}) \le \mathbf{b}, \\ \mathbf{x} \ge 0, \end{array} \xrightarrow{def\ \ Dual} \mathbf{II}: \begin{array}{l} \mathrm{W}_{II} = (\mathbf{b}, \mathbf{y}) \to \min, \\ \Omega_{II} : (\mathbf{a}^i, \mathbf{y}) \ge \mathbf{c}, \\ \mathbf{y} \ge 0. \end{array}$$

The conjugacy or duality of the given definition will be grounded with a certain sequence of operations which in case of cyclic application have to result in a primal problem, i.e.

$$\mathrm{I} \xrightarrow{def\ \ Dual} \mathrm{II} \xrightarrow{def\ \ Dual} \mathrm{I},$$

where $\xrightarrow{def\ \ Dual}$ is a set of rules for transition to the dual problem.

Careful analysis of definition of a problem being dual to the primal problem represented in the standard form allows determining the operations necessary for transition to the dual problem $\xrightarrow{def\ \ Dual}$:

- the extreme requirements of the primal and dual problems target functions are opposite in their meaning:
  $$\mathrm{W}_I \to \max \xrightarrow{def\ \ Dual} \mathrm{W}_{II} \to \min;$$
- for a problem on max target function, the inequalities present in the constraint system are to have the sign $\le$;
  $$\sum_{j=1}^{n} a_{ij} x_j \le b_i,$$
- the dual problem target function coefficients are components of the vector of the right-hand sides of the primal LO problem constraint system;
- matrix A^{T} of the constraint system of dual problem Ω_{II} is transposed to matrix A of the constraint system of the primal problem Ω_I (This is actually true because $YA = A^T Y^T$);
- the right-hand sides of the constraint system of dual problem $\Omega_{II} : (\mathbf{a}^i, \mathbf{y}) \ge \mathbf{c}$ are coefficients of the target function $\mathrm{W}_I = (\mathbf{c}, \mathbf{x}) \to \max$ of the primal problem;
- each constraint – inequality of the primal problem constraint system is associated with a nonnegative dual unknown
  $$\Omega_I : \sum_{j=1}^{n} a_{ij} x_j \le b_i \xrightarrow{def\ \ Dual} y_j \ge 0,\ i = 1, \ldots, m;$$
- each nonnegative unknown of the primal LO problem is associated with a constraint – inequality of dual
  $$x_j \ge 0 \xrightarrow{def\ \ Dual} \Omega_{II} : \sum_{i=1}^{m} a_{ij} y_i \ge c_j,\ \ j = 1, 2, \ldots, n\cdot$$

We should mention that different recording forms of linear optimization problems are equivalent – they keep the set of solutions. This can be achieved provided that methods of equivalent transformation are used for transition from one form of problems to the other.

Thus, a certain equation of the linear optimization problem constraint system is equivalent to the system of two inequalities:

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \Leftrightarrow \begin{cases} \sum_{j=1}^{n} a_{ij} x_j \le b_i, \\ -\sum_{j=1}^{n} a_{ij} x_j \le -b_i. \end{cases}$$

Variables with the arbitrary sign can be represented as a difference of 2 nonnegative variables:

$$x_j = u_j - v_j, \quad u_j \ge 0, \ v_j \ge 0.$$

The transition from constraints – inequalities to constraints – equations is made by adding a nonnegative (balancing) variable:

$$\sum_{j=1}^{n} a_{ij} x_j \le b_j \Rightarrow \sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i, \quad x_{n+i} \ge 0, \quad i = 1, \mathbf{K}, k.$$

For simpler conversion of linear optimization problems into various recording forms, the transition from maximization to the minimization of the target function and the transition the other way around is in use as well:

$$W_I = \sum_{j=1}^{n} c_j x_j \to \max \Leftrightarrow W_I = -\sum_{j=1}^{n} c_j x_j \to \min.$$

Let us ascertain that the entered operations and transformations form the chain of conjugacy for the pair of problems given above

$$\text{I} \xrightarrow{def \ \ Dual} \text{II} \xrightarrow{def \ \ Dual} \text{I}.$$

Let us represent the dual problem as a maximum problem and, using the rules of transition and equivalent transformations, prove the conjugacy of the pair of problems – a problem being dual to the dual problem makes an original primal problem.

$$\mathbf{II}: \begin{array}{l} W_{II} = -YB - \max, \\ \Omega_{II}: -YA \le -C^T, \\ Y \ge 0, \end{array} \xrightarrow{def \ \ Dual} \mathbf{I}: \begin{array}{l} W_I = -(C^T)^T X - \min, \\ \Omega_I: -AX \le -B, \\ X \ge 0, \end{array} \Leftrightarrow$$

c

$$\Leftrightarrow \mathbf{I}: \begin{array}{l} W_I = CX - \max, \\ \Omega_I: AX \le B, \\ X \ge 0. \end{array}$$

Thus, it has been confirmed that the possibility of reducing one LO problem pairs to the other by definition is the main feature of their duality.

Model example No. 1

If the project cost is provided as an equation system

$$W_I = -4x_1 + 2x_2 - 3x_3 - 8x_4 + x_5 - x_6 + 5x_7 - \max,$$

$$\mathbf{I}: \Omega_I: \begin{cases} 3x_1 - 4x_3 - 7x_4 + 5x_5 - 4x_6 - x_7 \le 12, \\ x_1 - x_2 + 3x_3 - 2x_4 + 7x_6 + 9x_7 \le 3, \\ 5x_1 - 3x_2 + 8x_4 - 5x_5 + 9x_6 + 4x_7 \ge 2, \\ 7x_1 - 5x_2 - 9x_3 + x_4 - 3x_5 - x_7 \ge 3, \end{cases}$$

$$x_j \ge 0, \quad j = 1, \mathbf{K}, 7.$$

To calculate its minimization, we construct a dual problem.

We prepared the constraint system. For a maximum problem, there should be inequalities in a form of $\le$. In view of this, we need to change the sign of the third and the fourth inequalities for the opposite one through multiplying by (-1).

$$W_I = -4x_1 + 2x_2 - 3x_3 - 8x_4 + x_5 - x_6 + 5x_7 - \max,$$

$$\mathbf{I}: \Omega_I: \begin{cases} 3x_1 - 4x_3 - 7x_4 + 5x_5 - 4x_6 - x_7 \le 12, \\ x_1 - x_2 + 3x_3 - 2x_4 + 7x_6 + 9x_7 \le 3, \\ -5x_1 + 3x_2 - 8x_4 + 5x_5 - 9x_6 - 4x_7 \le -2, \\ -7x_1 + 5x_2 + 9x_3 - x_4 + 3x_5 + x_7 \le -3, \end{cases}$$

$$x_j \ge 0, \quad j = 1, \mathbf{K}, 7.$$

Table 1 is suitable for making transition to the dual problem.

Following the algorithm of transition to the dual problem, we fill in Table 1. In our case of a standard representation of the primal problem, we have the following dual problem:

$$W_{II} = 12y_1 + 3y_2 - 2y_3 - 3y_4 \to \min,$$

$$\mathbf{II}: \Omega_{II}: \begin{cases} 3y_1 + y_2 - 5y_3 - 7y_4 \ge -4, \\ -y_2 + 3y_3 + 5y_4 \ge 2, \\ -4y_1 + 3y_2 + 9y_4 \ge -3, \\ -7y_1 - 2y_2 - 8y_3 + y_4 \ge -8, \\ 5y_1 + 5y_3 + 3y_4 \ge -1, \\ -4y_1 + 7y_2 - 9y_3 \ge -1, \\ -y_1 + 9y_2 - 4y_3 + y_4 \ge 5, \end{cases}$$

$$y_i \ge 0, \quad i = 1, \mathbf{K}, 4.$$

To obtain the full range of operations on transition to dual problems, let us consider the following LO problems pair and prove their conjugacy.

TABLE 1. MAKING TRANSITION TO THE DUAL PROBLEM

Y\X	$x_1 \ge 0$	$x_2 \ge 0$	$x_3 \ge 0$	$x_4 \ge 0$	$x_5 \ge 0$	$x_6 \ge 0$	$x_7 \ge 0$	?	B
$y_1 \ge 0$	3	0	-4	-7	5	-4	-1	$\le$	12
$y_2 \ge 0$	1	-1	3	-2	0	7	9	$\le$	3
$y_3 \ge 0$	-5	3	0	-8	5	-9	-4	$\le$	-2
$y_4 \ge 0$	-7	5	9	1	3	0	1	$\le$	-3
?	$\ge$	$\ge$	$\ge$	$\ge$	$\ge$	$\ge$	$\ge$		
C	-4	2	-3	-8	1	-1	5		

$$\mathbf{I}: \begin{array}{l} W_I = CX - max, \\ \Omega_I: AX = B, \end{array} \xrightarrow{def \ \ Dual} \mathbf{II}: \begin{array}{l} W_I = YB - \min, \\ \Omega_{II}: YA = C^T. \end{array}$$

$$\mathbf{I}:\ \begin{array}{l} W_I = CX - \max, \\ \Omega_I : AX = B, \end{array} \xrightarrow{X = X'' - X',\, X''\ge 0,\, X' \ge 0} \mathbf{II}:\ \begin{array}{l} W_I = C(X'' - X') - \max, \\ \Omega_I : \begin{cases} A(X'' - X') \le B, \\ -A(X'' - X') \le -B, \end{cases} \\ X'' \ge 0, X' \ge 0, \end{array}$$

$$\xrightarrow{def\quad Dual} \mathbf{II}:\ \begin{array}{l} W_{II} = (Y'' - Y')B - \min, \\ \Omega_{II} : \begin{cases} (Y'' - Y')A \ge C^T, \\ (Y'' - Y')(-A) \ge -C^T, \end{cases} \\ Y'' \ge 0, Y' \ge 0, \end{array} \Leftrightarrow \mathbf{II}:\ \begin{array}{l} W_I = YB - \min, \\ \Omega_{II} : YA = C^{\mathrm{T}}. \end{array}$$

The proven duality of this problems pair allows formulating the results of defining the duality for a standard LO problem:

- each constraint – equation of the primal problem is associated with a dual unknown with the arbitrary sign;

$$\mathbf{I}:\ \Omega_I : AX = B \xrightarrow{def\quad Dual} \mathbf{II}:\ Y.$$

(Y is a dual variable with the arbitrary sign)

- each primal problem variable with the arbitrary sign is associated with a constraint – equation of dual problem;

$$\mathbf{I}:\ X \xrightarrow{def\quad Dual} \mathbf{II}:\ \Omega_{II} : YA = C^{\mathrm{T}}.$$

(X is a primal problem variable with the arbitrary sign)

Using similar transformations, it is possible to prove conjugacy of the basic nonsymmetrical forms of dual problem pairs

$$\mathbf{I}:\ \begin{array}{l} W_I = CX - \max, \\ \Omega_I : AX \le B, \end{array} \xrightarrow{def\quad Dual} \mathbf{II}:\ \begin{array}{l} W_{II} = YB - \min, \\ \Omega_{II} : YA = C^{\mathrm{T}}, \\ Y \ge 0. \end{array}$$

$$\mathbf{I}:\ \begin{array}{l} W_I = CX - \max, \\ \Omega_I : AX = B, \\ X \ge 0, \end{array} \xrightarrow{def\quad Dual} \mathbf{II}:\ \begin{array}{l} W_{II} = YB - \min, \\ \Omega_{II} : YA \ge C^{\mathrm{T}}, \end{array}$$

$$\mathbf{I}:\ \begin{array}{l} W_I = CX - \min, \\ \Omega_I : AX \ge B, \end{array} \xrightarrow{def\quad Dual} \mathbf{II}:\ \begin{array}{l} W_{II} = YB - \max, \\ \Omega_{II} : YA = C^{\mathrm{T}}, \\ Y \ge 0. \end{array}$$

Model examples No. 2, 3, 4

No. 2 *The following primal linear optimization problem*

$$W_I = 4x_1 + 9x_2 + x_3 + 2x_4 - x_5 - 54 \to \max,$$

$$\mathbf{I}:\ \Omega_I : \begin{cases} -x_1 - 5x_2 - x_3 - x_4 + x_5 = -41, \\ 4x_2 + x_3 - x_5 = 28, \\ x_1 - x_4 + x_5 = 9, \end{cases}$$

$$x_j \ge 0, \quad j = 1, \mathbf{K}, 5,$$

should be taken for construction of a dual problem to it.

Solution. We have the canonical form of a primal problem recording. Let us construct Table 2 to transit to a dual problem.

Following the algorithm of transition to the dual problem in the case of the primal problem represented in the canonical form, we have the following dual problem:

$$W_{II} = -41y_1 + 28y_2 + 9y_3 \to \min,$$

$$\mathbf{II}:\ \Omega_{II} : \begin{cases} -y_1 + y_3 \ge 4, \\ -5y_1 + 4y_3 \ge 9, \\ -y_1 + y \ge 1, \\ -y_1 - y_3 \ge 2, \\ y_1 - y_3 + y_3 \ge -1, \end{cases}$$

$$y_i \ge 0, \quad i = 1, 2, 3.$$

TABLE 2. TRANSITION TO THE DUAL PROBLEM

Y\X	$x_1 \ge 0$	$x_2 \ge 0$	$x_3 \ge 0$	$x_4 \ge 0$	$x_5 \ge 0$	?	B
y_1	-1	-5	-1	-1	1	=	-41
y_2	0	4	1	0	-1	=	28
y_3	1	0	0	-1	1	=	9
?	≥	≥	≥	≥	≥		
C	4	9	1	2	-1		

No. 3 *The following primal linear optimization problem*

$$W_I = x_1 - 3x_2 + 5x_3 + 2x_4 \to \min,$$

$$\mathbf{I}:\ \Omega_I : \begin{cases} 4x_1 - 5x_2 + 3x_3 - x_4 \ge 3, \\ -x_1 + 2x_2 + 4x_3 - 7x_4 \ge 2, \\ 3x_1 - x_2 - x_3 - 7x_4 \ge 3, \\ x_1 + 8x_2 - 5x_3 - 9x_4 \ge 8. \end{cases}$$

should be taken for construction of a dual problem to it.

Solution. For a dual problem, we construct Table 3.

TABLE 3. TRANSITION TO THE DUAL PROBLEM

Y\X	x_1	x_2	x_3	x_4	?	B
$y_1 \ge 0$	4	-5	3	-1	≥	3
$y_2 \ge 0$	-1	2	4	-7	≥	2
$y_3 \ge 0$	3	-1	-1	-7	≥	3
$y_4 \ge 0$	1	8	-5	-9	≥	8
?	=	=	=	=		
C	1	-3	5	2		

Following the algorithm of transition to a dual problem, we obtain the following dual problem:

$$W_{II} = 3y_1 + 2y_2 + 3y_3 + 8y_4 \to \max,$$

$$\mathbf{II}: \Omega_{II}: \begin{cases} 4y_1 - y_2 + 3y_3 + y_4 = 1, \\ -5y_1 + 2y_2 - y_3 + 8y_4 = -3, \\ 3y_1 + 4y_2 - y_3 - 5y_4 = 5, \\ -y_1 - 7y_2 - 7y_3 - 9y_4 = 2, \end{cases}$$

$$y_i \ge 0, \quad i = 1, 2, 3, 4.$$

No. 4 *The following primal linear optimization problem*

$$W_I = 2x_1 + 8x_2 - x_3 + 5x_4 \to \max,$$

$$\mathbf{I}: \Omega_I: \begin{cases} 4x_1 - 5x_2 + 3x_3 - x_4 \le 3, \\ -x_1 + 2x_2 + 4x_3 - 7x_4 \le 2. \end{cases}$$

should be taken for construction of a dual problem to it.

Solution. For a dual problem, we construct Table 4.

TABLE 4. TRANSITION TO THE DUAL PROBLEM

$Y\backslash X$	x_1	x_2	x_3	x_4	?	B
$y_1 \ge 0$	4	-5	3	-1	$\le$	3
$y_2 \ge 0$	-1	2	4	-7	$\le$	2
?	=	=	=	=		
C	2	8	-1	5		

Following the algorithm of transition to a dual problem, we obtain the following dual problem:

$$W_{II} = 3y_1 + 2y_2 \to \min,$$

$$\mathbf{II}: \Omega_{II}: \begin{cases} 4y_1 - y_2 = 2, \\ -5y_1 + 2y_2 = 8, \\ 3y_1 + 4y_2 = -1, \\ -y_1 - 7y_2 = 5, \end{cases}$$

$$y_i \ge 0, \quad i = 1, 2.$$

The dual problem for the general form of a primal LO problem

The given dual problem pairs allow generalizing the definition of duality in linear optimization problems for the case of representing a primal problem in general form.

Let us have a general linear optimization problem

$$W_I = C X \to \max,$$

$$\mathbf{I}: \Omega_I: \left\{ \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \begin{bmatrix} \le \\ = \end{bmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \right\},$$

$$x_j \ge 0, \quad j = 1, 2, ..., l,$$

or in expanded recording form

$$W_I = \sum_{j=1}^{n} c_j x_j \to \max,$$

$$\mathbf{I}: \Omega_I: \begin{cases} \sum_{j=1}^{n} a_{ij} x_j \le b_i, & i = 1, 2, 3, ...k, \\ \sum_{j=1}^{n} a_{ij} x_j = b_i, & i = k+1, k+2, k+3, ...m, \end{cases}$$

$$x_j \ge 0, \quad j = 1, 2, ..., l,$$

a problem looking as follows shall be called dual to it

$$W_{II} = Y B \to \min,$$

$$\mathbf{II}: \Omega_{II}: \left\{ \begin{pmatrix} Y_1 & Y_2 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{bmatrix} \ge \\ = \end{bmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \right\},$$

$$y_i \ge 0, \quad i = 1, 2, ..., k,$$

or in different way,

$$W_{II} = \sum_{i=1}^{m} b_i y_i \to \min,$$

$$\mathbf{II}: \Omega_{II}: \begin{cases} \sum_{i=1}^{m} y_i a_{ij} \ge c_j, & j = 1, 2, 3, ...l, \\ \sum_{i=1}^{m} y_i a_{ij} = c_i, & i = l+1, l+2, l+3, ...n, \end{cases}$$

$$y_i \ge 0, \quad i = 1, 2, ..., k.$$

Model example No. 5.

The following primal linear optimization problem

$$W_I = 7x_1 - 4x_2 + 3x_3 - 2x_4 + x_5 \to \min,$$

$$\mathbf{I}: \Omega_I: \begin{cases} -3x_1 - 5x_2 + 9x_3 - x_4 + 8x_5 \ge 24, \\ x_1 + 2x_2 - x_3 + 3x_4 - 7x_5 \le 11, \\ x_1 + 4x_2 + x_3 - 2x_4 - x_5 = 8, \\ -x_1 + 3x_2 + 6x_3 - 5x_4 - 3x_5 = 21, \end{cases}$$

$$x_3 \ge 0, x_5 \ge 0,$$

should be taken for construction of a dual problem to it.

Solution. To begin the transition to a dual problem, let us prepare the primal problem constraint system – for a minimum problem, we need inequalities $\ge$ to be present only. The change the sign of the second inequality for the opposite one.

$$W_I = 7x_1 - 4x_2 + 3x_3 - 2x_4 + x_5 \to \min,$$

$$\mathbf{I}: \Omega_I: \begin{cases} -3x_1 - 5x_2 + 9x_3 - x_4 + 8x_5 \ge 24, \\ -x_1 - 2x_2 + x_3 - 3x_4 + 7x_5 \ge -11, \\ x_1 + 4x_2 + x_3 - 2x_4 - x_5 = 8, \\ -x_1 + 3x_2 + 6x_3 - 5x_4 - 3x_5 = 21, \end{cases}$$

$$x_3 \ge 0, x_5 \ge 0,$$

We make the transition to a dual problem in Table 5.

TABLE 5. TRANSITION TO THE DUAL PROBLEM

$Y\backslash X$	x_1	x_2	$x_3 \ge 0$	x_4	$x_5 \ge 0$	?	B
$y_1 \ge 0$	-3	-5	9	-1	8	$\ge$	24
$y_2 \ge 0$	-1	--2	1	-3	7	$\ge$	-11
y_3	1	4	1	-2	-1	=	8
y_4	-1	3	6	-5	-3	=	21
?	=	=	$\le$	=	$\le$		
C	7	-4	3	-2	1		

The dual problem is

$$W_{II} = 24y_1 - 11y_2 + 8y_3 + 21y_4 \to \max,$$

$$\mathbf{II}: \Omega_{II}: \begin{cases} -3y_1 - y_2 + y_3 - y_4 = 7, \\ -5y_1 - 2y_2 + 4y_3 + 3y_4 = -4, \\ 9y_1 + y_2 + y_3 + 5y_4 \le 7, \\ y_1 + 3y_2 + 2y_3 + 5y_4 = 2, \\ 8y_1 + 7y_2 - y_3 - 3y_4 \le 7 \end{cases}$$

$$y_1 \ge 0, \; y_2 \ge 0.$$

Let us consider the case of nonpositive unknowns present in a primal problem and of the inequality sign failing to correspond to the target function extremum type.

We prove that each nonpositive unknown $x \le 0$ of the primal problem is associated with a constraint – inequality, for max – $\ge$, and for min – $\le$.

The pair of LO problems is dual. It is true,

$$\begin{array}{c} W_I = CX - \max, \\ \mathbf{I}: \Omega_I : AX \ge B, \\ X \ge 0, \end{array} \xrightarrow{def\ \ Dual} \begin{array}{c} W_{II} = YB - \min, \\ \mathbf{II}: \Omega_{II} : YA \le C^T, \\ Y \le 0. \end{array}$$

$$\begin{array}{c} W_I = CX - \max, \\ \mathbf{I}: \Omega_I : AX \ge B, \\ X \ge 0, \end{array} \overset{\times(-1)}{\Leftrightarrow} \begin{array}{c} W_I = CX - \max, \\ \mathbf{I}: \Omega_I : -AX \le -B, \\ X \ge 0, \end{array} \xrightarrow{def\ \ Dual}$$

$$\xrightarrow{def\ \ Dual} \begin{array}{c} W_{II} = Y'(-B) - \min, \\ \mathbf{II}: \Omega_{II} : Y'(-A) \ge C^T, \\ Y' \ge 0, \end{array} \overset{Y'=-Y}{\Leftrightarrow} \begin{array}{c} W_{II} = YB - \min, \\ \mathbf{II}: \Omega_{II} : YA \le C^T, \\ Y \le 0. \end{array}$$

Thus we have proven that in case of the inequality sign failing to correspond to the target function optimum type, the corresponding dual unknowns have to be nonpositive $y \le 0$.

It is ascertained in a similar way that each nonpositive unknown $x \le 0$ of the primal problem is associated with the constraint – inequality in dual problem having a sign opposite to the main definition. Based on this, the given problem pairs are dual.

$$\begin{array}{c} W_I = CX - \max, \\ \mathbf{I}: \Omega_I : AX \le B, \\ X \le 0, \end{array} \xrightarrow{def\ \ Dual} \begin{array}{c} W_{II} = YB - \min, \\ \mathbf{II}: \Omega_{II} : YA \le C^T, \\ Y \ge 0, \end{array}$$

$$\begin{array}{c} W_I = CX - \max, \\ \mathbf{I}: \Omega_I : AX \ge B, \\ X \le 0, \end{array} \xrightarrow{def\ \ Dual} \begin{array}{c} W_{II} = YB - \min, \\ \mathbf{II}: \Omega_{II} : YA \le C^T, \\ Y \le 0, \end{array}$$

$$\begin{array}{c} W_I = CX - \min, \\ \mathbf{I}: \Omega_I : AX \ge B, \\ X \le 0, \end{array} \xrightarrow{def\ \ Dual} \begin{array}{c} W_{II} = YB - \max, \\ \mathbf{II}: \Omega_{II} : YA \ge C^T, \\ Y \ge 0. \end{array}$$

The dual problem for the arbitrary form of a primal LO problem

Generalizing the provings and the consequences given above, we can obtain the general algorithm of transition to the dual problem for the arbitrary form of recording a primal linear optimization problem.

Definition

For the arbitrary form of a primal linear optimization problem representation

$$W_I = CX \to \max,$$

$$\mathbf{I}: \Omega_I : \left\{ \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \begin{bmatrix} \le \\ \ge \\ = \end{bmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} \right.,$$

$$x_j \ge 0, \quad j = 1, 2, \ldots, k,$$
$$x_j \le 0, \quad j = k+1, k+2, \ldots, l,$$
$$x_j, \quad j = l+1, l+2, \ldots, n,$$

or in expanded algebraic recording form

$$W_I = \sum_{j=1}^{n} c_j x_j - \max,$$

$$\mathbf{I}: \Omega_I : \begin{cases} \sum_{j=1}^{n} a_{ij} x_j \le b_i, & i = 1, 2, 3, \ldots, s, \\ \sum_{j=1}^{n} a_{ij} x_j \ge b_i, & i = s+1, s+2, s+3, \ldots, t, \\ \sum_{j=1}^{n} a_{ij} x_j = b_i, & i = t+1, t+2, t+3, \ldots, m, \end{cases}$$

$$x_j \ge 0, \quad j = 1, 2, \ldots, k,$$
$$x_j \le 0, \quad j = k+1, k+2, \ldots, l,$$
$$x_j, \quad j = l+1, l+2, \ldots, n,$$

a problem looking as follows shall be called dual to it:

$$W_{II} = YB \to \min,$$

$$\mathbf{II}: \Omega_{II} : \left\{ \begin{pmatrix} Y_1 & Y_2 & Y_3 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{bmatrix} \ge \\ \le \\ = \end{bmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} \right.,$$

$$y_i \ge 0, \quad i = 1, 2, \ldots, s,$$
$$y_i \ge 0, \quad i = s+1, s+2, \ldots, t,$$
$$y_i, \quad i = t+1, t+2, \ldots, m,$$

or in expanded algebraic form

$$W_{II} = \sum_{i=1}^{m} b_i\, y_i - \min,$$

$$\mathbf{II} : \Omega_{II} : \begin{cases} \sum_{i=1}^{m} y_i\, a_{ij} \geq c_j, & j = 1,2,3,\ldots k, \\ \sum_{i=1}^{m} y_i\, a_{ij} \leq c_j, & j = k+1, k+2, k+3, \ldots l, \\ \sum_{i=1}^{m} y_i\, a_{ij} = c_i, & i = l+1, l+2, l+3, \ldots n, \end{cases}$$

$$y_i \geq 0, \quad i = 1, 2, \ldots, s,$$
$$y_i \leq 0, \quad i = s+1, s+2, \ldots, t,$$
$$y_i, \quad i = t+1, t+2, \ldots, m.$$

***Model example No.* 5.**

Construct a dual problem to the given primal problem

$$W_I = 7x_1 - 4x_2 + 9x_3 - 2x_4 - \max,$$

$$\begin{cases} -2x_1 - 3x_2 + 5x_3 + x_4 = 1, \\ 3x_1 + 4x_2 - 8x_3 - 2x_4 \geq 21, \\ x_1 - 2x_2 - 3x_3 + 4x_4 \leq 12, \end{cases}$$

$$x_2 \leq 0, x_4 \geq 0,$$

Solution.

We make the transition to a dual problem in Table 6.

TABLE 6. TRANSITION TO THE DUAL PROBLEM

Y\X	x_1	$x_2 \leq 0$	x_3	$x_4 \geq 0$	?	B
y_1	-2	-3	5	1	=	1
$y_2 \leq 0$	3	4	-8	-2	≥	21
$y_3 \geq 0$	1	-2	-3	4	≤	12
?	=	≤	=	≥		
C	7	-4	9	-2		

The dual problem is

$$W_{II} = y_1 + 21y_2 + 12y_3 \rightarrow \min,$$

$$\mathbf{II} : \Omega_{II} : \begin{cases} -2y_1 + 3y_2 + y_3 = 7, \\ -3y_1 + 4y_2 - 2y_3 \leq -4, \\ 5y_1 - 8y_2 - 3y_3 = -9, \\ y_1 - 2y_2 + 4y_3 \geq -2, \end{cases}$$

$$y_2 \leq 0, \; y_3 \geq 0.$$

CONCLUSIONS

Therefore, for preparing the project budget and avoiding groundless or minimizing unforeseeable expenses, it has been proposed to use the dual problem construction algorithm. For existing dual problem pairs, it has been strictly proven that they are conjugate and this is the main criterion for construction of duality pairs. Operations of transition to a dual problem from primal one represented in the general recording form have been strictly determined for the first time. In view of this, the transition to the dual problem does not provide for a complicated formal order.

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