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# METHOD FOR APPROXIMATE CONSTRUCTION OF THREE-DIMENSIONAL MASS DISTRIBUTION FUNCTION AND GRADIENT OF AN ELLIPSOIDAL PLANET BASED ON EXTERNAL GRAVITATIONAL FIELD PARAMETERS

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**Purpose.** To investigate the technique for constructing a three-dimensional distribution function for the masses of the interior of the Earth and its derivatives, coordinated with the parameters of the planet's gravitational field to fourth order inclusive. By using the mass distribution function constructed, to make an interpretation of the features of the internal structure of an ellipsoidal planet. **Methodology.** Based on the created initial approximation of the function, which includes a reference density model, further refinements are built. Using Stokes constants up to the second order inclusive, we give the following approximation, which we subsequently take as zero. In this case, the use of Stokes constants up to the fourth order inclusive leads to the solution of systems of equations. It is established that the addition of one identity leads to uniqueness of the solution. One system with Stokes constants $c_{40}, c_{42}, c_{44}$. is an exception. It is necessary to note that the computation process is controllable, since the power moments of the density derivatives are reduced to quantities that take into account the value of the density on the surface of the ellipsoid. **Results.** In contrast to the second-order model describing gross global inhomogeneities, the obtained distribution function gives a detailed picture of the location of the density anomalies (the deviation of the three-dimensional function from the averaged over the sphere is “isodense”) Analysis of maps at different depths 2891 km (core-mantle), 5150 km (internal-external core) allows us to draw preliminary conclusions about the global mass redistribution due to the rotating component of the force of gravity over the entire radius, as well as due to the horizontal components of the density gradient. On the contrary, the minimum of such a deviation is observed in the polar parts of the Earth, which also has its explanation: the magnitude of the rotational force decreases when approaching the pole. The mass distribution function is constructed using the proposed method to describe in more detail the picture of the mass distribution. Of particular interest are sketch maps of the components of the density anomaly function gradient, namely the component which coincides with the axis $Oz$ - for the upper part of the shell which is negative, and for the lower part it is positive. This means that the gradient vector is directed toward the centre of mass. The nature of the values for other two components is different both in sign and in magnitude and depends on the placement point. The cumulative consideration and consideration of all the quantities makes possible a more complete interpretation of the processes inside the Earth. **Originality.** In contrast to the traditional approach, the changes for the density derivatives of one variable (depth), obtained from the Adams-Williams equation, in this paper made an attempt to obtain derivatives with Cartesian coordinates. Used in the described method, the parameters of the gravitational field up to the fourth order inclusively increases the order of approximation of the mass distribution function of three variables from two to six, and its derivatives up to five. In this case, unlike the traditional method, the defining here is the construction of the derivatives, from which the mass distribution function and the use of geophysical information accumulated in the referential PREM model are reproduced. **Practical significance.** The resulting mass distribution function of the Earth can be used as the next approximation when using Stokes constants of higher orders in the presented algorithm. Its application makes it possible to interpret global anomalies of the gravitational field and to study geodynamic processes deeply inside the Earth.

*Key words*: potential, harmonic function, the mass distribution model, Stoke’s constants, density gradient.

### *Introduction*

Any steps in the direction of studying the three-dimensional structure of the internal structure of the Earth and celestial bodies provide a significant contribution to our notion of them. After all, it is the mass displacement inside the planet that largely gives the key to understanding the dynamics of tectonic plate movement and, as a consequence of this fact, the emergence of global natural disasters. The inhomogeneity of masses located in zones close to the Earth's core, in a certain way, clarifies the formation of the Earth's magnetic field.

Unlike the creation of a one-dimensional mass distribution functions, the three-dimensional case has only in recent decades been practically considered. For example, in the well-known monograph [Bullen, 1978], only a small paragraph is devoted to this problem. Today, a method of seismic tomography based on observational data of seismic waves and information on natural oscillations of the Earth has been developed [Anderson, 1984]. Based on this, deviations of seismic waves from the radial distribution in a number of regions of the Earth [Liu, 2016], [Martyshko, 2017] are constructed, which made it possible to find deviations from the one-dimensional distribution of mass distribution inhomogeneities by known dependences [Anderson, 1984]. An additional means in this direction is information on the external gravitational field [Mashimov, 1991], the anisotropy of which is largely a consequence of deviation from homogeneity. Therefore, the development of methods using parameters of the gravitational field is an important link in the study of internal structure [Tserklevich, 2012]. We note that practically all methods of constructing density models use Stokes constants only of zero and second order (mass and polar moment of inertia) [Bullen, 1978]. The creation of models using Stokes constants higher than the second order is possible only approximately, due to ambiguous determination of the potential (the problem of zero-potential bodies) using iterative methods. In this case, as a rule, the reference density model is taken to be consistent with the Stokes constants up to the second order inclusive [Meshcheryakov, 1986], and further refinements are performed under the condition of minimal deviation from the adopted model, taking into account the coefficients of the potential expansion to a certain degree [Meshcheryakov, 1991]. In the proposed study, an attempt is made to obtain an approximation based on several other conditions, which leads to a procedure for constructing the functions of the derivatives of the density (gradient), and then - to establish the form of the density itself.

The reduction of power density moments to controlled values (quantities determined on the ellipsoid surface) makes it possible to analyze and control the calculations. The gradient and the mass distribution function constructed using the proposed method, taking into account the parameters of the third and, partially, fourth-order gravity field, describe the mass distribution more detailed. In this case, the gradient of density anomalies at the points of ellipsoidal shells on the radius vectors considered (depths) is directed toward the center of mass of the planet. Thus, this allows predicting the redistribution of masses within the planet $\left(\frac{\partial d}{\partial x_3} < 0\right)$. After all, using the same data in the described method increases the order of approximation from two to four due to the possibility of restoring the mass distribution of the planet's interior to its derivatives. Therefore, unlike the second-order model, which describes gross global inhomogeneities, the obtained distribution function gives a detailed picture of the placement of density anomalies (the deviation of the three-dimensional function from the averaged over the sphere - “isodense”). Analysis of maps at different depths (2,891 km core-mantle, 5150 km inner-outer core) allows us to draw preliminary conclusions about the global mass redistribution due to the rotating component of gravity along the entire radius: its decrease along the axis of rotation and increase, moving away from axis. This is particularly evident in equatorial regions. On the contrary, in polar regions there is a minimum of such a deviation, which also has its own explanation (the magnitude of the rotational force decreases with distance from the equator).

The study of the internal structure of the Earth and other planets does not lose relevance at the present time [Moritz, 1994]. Data on the Earth's gravitational field are a powerful source in its study, including the internal structure. This, in turn, requires the creation of methods for their use. It is known [Dzewonski, 1981] that in the construction of spherically symmetric distributions of the Earth's masses, the mass and the Stokes second-order constant (the polar moment of inertia) are used. Other coefficients of the expansion of the potential by spherical functions are described in [Martinenc, 1987], [Shcherbakov, 1978], [Meshcheryakov, 1975, 1981]. In particular, in the article [Moritz, 1973], it is proposed to represent a three-dimensional distribution by a harmonic function consistent with the expansion coefficients up to a certain order. In [Meshcheryakov, 1975, 1981] an approximate method is proposed for constructing three-dimensional models of the Earth's density, taking into account the features of the internal structure and corresponding to Stokes constants of a given order. The degree of reliability of the found model distributions can not be estimated. In this connection, there is a need to develop a methodology that allows you to analyze the calculation process and objectively evaluate the reliability of density functions. An attempt of such a construction was made in [Fys, 2008], [Cherniaga, 2012], where the problem was reduced to controlled quantities (surface integrals). The mass density models obtained in this way reflect our ideas about them, and the relative values (the derivatives of the three-dimensional functions) can be close to the true ones.

### *Purpose*

The purpose of this paper is to construct and investigate the technique for creating a three-dimensional distribution function for the masses of the interior of the Earth and its derivatives, coordinated with the parameters of the planet's gravitational field to fourth order inclusive. By using the mass distribution function constructed, to make an interpretation of the features of the internal structure of an ellipsoidal planet..

### *Methodology*

The paper [Fys, 2016] gives general aspects of constructing three-dimensional mass distributions inside spherical planets. Further generalization is a transition to an ellipsoidal figure [Moritz, 1973], which does not significantly complicate the algorithm for determining the mass density distribution function. Therefore, we will consider just such a case.

We recall briefly the scheme of the algorithm for such a construction. By Stokes constants $C_{nk}$,

$S_{nk}$, $(n,k\le 2)$, dynamic compression $H$ and one of the most representative density models $d^0(r)$ (for example, the PREM model [Dzewonski, 1981]), we determine the three-dimensional mass distribution function inside the planetary ellipsoid

$$t\left\{\frac{x_1^2}{a_1^2}+\frac{x_2^2}{a_2^2}+\frac{x_3^2}{a_3^2}\le 1\right\}:$$

$$d_2(x_1,x_2,x_3)=d^0(r)+\sum_{m+n+k=0}^{2} b_{vnk}W_{mnk}(x_1,x_2,x_3).$$

The degree moments and coefficients of the decomposition are determined by the formulas taken from the monograph [Meshcheryakov, 1991].

In contrast to the commonly accepted methodology, we construct an alternative model of density $d_4^a$ and its derivatives based on the algorithm (described in [Fys, 2016]) and the same data (Stokes constant to second order including dynamic compression and PREM model):

$$d_4^a(x_1x_2,x_3)=d^0(r)+\sum_{m+n+k=0}^{3} a^1_{mnk}\int_0^{x_1}W_{mnk}(x_1,x_2,x_3)\,dx_1,+$$

$$+\sum_{m+n+k=0}^{3} a^2_{mnk}\int_0^{x_2}W_{mnk}(0,x_2,x_3)\,dx_2+\sum_{m+n+k=0}^{3} a^3_{mnk}\int_0^{x_3}W_{mnk}(0,0,x_3)\,dx_3,$$

$$\frac{\partial d}{\partial x_i}=\frac{1}{a_i}\sum_{m+n+k=0}^{N} a^i_{mnk}W_{mnk}(x_1,x_2,x_3),\quad i=1,2,3.$$

Comparison of numerical results (difference of density distribution values) presented in Fig. 2-4 [Fys, 2016], confirm the advisability of continuing such studies. Therefore, we proceed to construct an approximation that agrees with the Stokes constants of the third and, partially, fourth orders. Firstly, we note that this technique requires knowledge of third order and fourth-order density moments. They, unlike the previous one, are not determined by observational data, so we take their approximate values from the constructed model $d_4^a$ [Fys, 2016].

The fifth-order and sixth-order surface power moments are determined from the system of equations obtained in [Fys, 2016], which in the matrix form can be represented as:

$$C=A_i s_i, \qquad (1)$$

where $C$ - vector-column with coefficients

$$C_{nk}=c_{nk}-C^*_{nk},\; S_{nk}=s_{nk}-S^*_{nk},$$

$c_{nk}, s_{nk}$ – observation data and

$$C^*_{nk}=\frac{1}{Ma_e^n}\int_t \frac{\partial d_4^a}{\partial x_i}u^i_{nk}\,dt\;, \quad S^*_{nk}=\frac{1}{Ma_e^n}\int_t \frac{\partial d_4^a}{\partial x_i}v^i_{nk}\,dt\;.$$

The matrices $A_i$ and values $s_i$ are determined for each group taking into account the identities:

$$s_{p+2qs}+s_{pqs+2}+s_{pq+2s}=s_{pqs},$$

Elements of (1) for all blocks of equations will be written

$$C_{30}=\frac{1}{Ma_e^3}\int_t d\left(x_3^3-\frac{3}{2}x_3\left(x_1^2+x_2^2\right)\right)dt,$$

$$C_{32}=\frac{1}{Ma_e^3}\int_t d\left(\frac{1}{4}x_3\left(x_1^2-x_2^2\right)\right)dt, \qquad (2)$$

$$A_1=\frac{1}{24}\begin{pmatrix} 6 & -18 & -18 & 0 & 0 & 0\\ 0 & 24 & 0 & -12 & -36 & 0\\ 0 & 0 & 24 & 0 & -36 & -12\\ 0 & 3 & -3 & 0 & 0 & 0\\ 0 & 0 & 0 & 2 & -6 & 0\\ 24 & 48 & 48 & 24 & 48 & 24\\ 0 & 0 & 0 & 0 & 6 & -2 \end{pmatrix},$$

$$b=\begin{pmatrix} \left(C_{30}-C^*_{30}\right)\\ \left(C_{30}-C^*_{30}\right)\\ \left(C_{30}-C^*_{30}\right)\\ \left(C_{32}-C^*_{32}\right)\\ \left(C_{32}-C^*_{32}\right)\\ \left(C_{32}-C^*_{32}\right)\\ s_{001} \end{pmatrix}, \qquad x=\begin{pmatrix} s_{005}\\ s_{203}\\ s_{023}\\ s_{401}\\ s_{221}\\ s_{141} \end{pmatrix}.$$

Then

$$s_{005}+2s_{203}+2s_{023}+s_{401}+2s_{221}+s_{041}=s_{001},$$

$$C_{31}=\frac{1}{Ma_e^3}\int_t d\left(x_3^2x_1-\frac{1}{4}x_1\left(x_1^2+x_2^2\right)\right)dt,$$

$$C_{33}=\frac{1}{Ma_e^3}\int_t d\left(x_1\left(\frac{1}{3}x_1^2-x_2^2\right)\right)dt,$$

$$A_2=\frac{1}{48}\begin{pmatrix} 16 & -12 & -12 & 0 & 0 & 0\\ 0 & 24 & 0 & -3 & -6 & 0\\ 0 & 0 & 48 & 0 & -12 & -4\\ 0 & 16 & -48 & 0 & 0 & 0\\ 0 & 0 & 0 & 4 & -24 & 0\\ 0 & 0 & 0 & 0 & 16 & -16\\ 48 & 96 & 96 & 48 & 96 & 48 \end{pmatrix},$$

$$b=\begin{pmatrix} \left(C_{31}-C^*_{31}\right)\\ \left(C_{31}-C^*_{31}\right)\\ \left(C_{31}-C^*_{31}\right)\\ \left(C_{33}-C^*_{33}\right)\\ \left(C_{33}-C^*_{33}\right)\\ \left(C_{33}-C^*_{33}\right)\\ s_{010} \end{pmatrix}, \qquad x=\begin{pmatrix} s_{104}\\ s_{302}\\ s_{122}\\ s_{500}\\ s_{320}\\ s_{140} \end{pmatrix},$$

Also,

$$s_{500}+2s_{320}+2s_{302}+s_{140}+2s_{122}+s_{104}=s_{100},$$

$$S_{31}=\frac{1}{Ma_e^3}\int_t d\left(x_3^2x_2-\frac{1}{4}x_2\left(x_1^2+x_2^2\right)\right)dt,$$

$$S_{33}=\frac{1}{Ma_e^3}\int_t d\left(x_2\left(x_1^2-\frac{1}{3}x_2^2\right)\right)dt,$$

$$A_2=\frac{1}{48}\begin{pmatrix}16 & -12 & -12 & 0 & 0 & 0\\ 0 & 48 & 0 & -4 & -12 & 0\\ 0 & 0 & 24 & 0 & -6 & -3\\ 0 & 48 & -16 & 0 & 0 & 0\\ 0 & 0 & 0 & 16 & -16 & 0\\ 48 & 96 & 96 & 48 & 96 & 48\\ 0 & 0 & 0 & 0 & 24 & -4\end{pmatrix},$$

$$b=\begin{pmatrix}\left(S_{31}-S_{31}^*\right)\\ \left(S_{31}-S_{31}^*\right)\\ \left(S_{31}-S_{31}^*\right)\\ \left(S_{33}-S_{33}^*\right)\\ \left(S_{33}-S_{33}^*\right)\\ \left(S_{33}-S_{33}^*\right)\end{pmatrix},\qquad x=\begin{pmatrix}s_{014}\\ s_{212}\\ s_{032}\\ s_{410}\\ s_{230}\\ s_{050}\end{pmatrix},$$

where

$$S_{32}=\frac{1}{2Ma_e^3}\int_t d\,x_1x_2x_3\,dt.$$

The direct use of identity (2) from [Fys, 2016] in the last formula gives

$$s_{311}=2S_{32}-\frac{2}{7}a_{011}^1,\ s_{131}=2S_{32}-\frac{2}{7}a_{101}^2,\ s_{113}=2S_{32}-\frac{2}{7}a_{110}^3.$$

Continuing to evolve the formulas for the Stokes constant of fourth order we get:

$$C_{40}=\frac{1}{Ma_e^4}\int_t d\left(x_3^4-3x_3^2\left(x_1^2+x_2^2\right)+\frac{3}{8}\left(x_1^2+x_2^2\right)^2\right)dt,$$

$$C_{42}=\frac{1}{Ma_e^4}\int_t d\left(\frac{3}{2}x_3^2\left(x_1^2-x_2^2\right)+\frac{1}{4}\left(x_2^4-x_1^4\right)\right)dt,$$

$$C_{44}=\frac{1}{32Ma_e^4}\int_t d\left(x_1^4+x_2^4-6x_1^2x_2^2\right)dt,$$

where

$$A_5=\frac{1}{160}\begin{pmatrix}32 & -160 & -160 & 60 & 120 & 60 & 0 & 0 & 0 & 0\\ 0 & 160 & 0 & -160 & -480 & 0 & 12 & 40 & 60 & 0\\ 0 & 0 & 160 & 0 & -480 & -160 & 0 & 60 & 40 & 12\\ 0 & 80 & -80 & -40 & 0 & 40 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 80 & -240 & 0 & -8 & 0 & 40 & 0\\ 0 & 0 & 0 & 0 & 240 & -80 & 0 & -40 & 0 & 8\\ 0 & 0 & 0 & 5 & -30 & 5 & 0 & 0 & 0 & 0\\ 160 & 480 & 480 & 480 & 960 & 480 & 160 & 480 & 480 & 160\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -10 & 5 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -10 & 1\end{pmatrix},$$

and $s_5^1=\begin{pmatrix}s_{006}\\ s_{204}\\ s_{024}\\ s_{402}\\ s_{222}\\ s_{042}\\ s_{600}\\ s_{420}\\ s_{240}\\ s_{060}\end{pmatrix}$, $\qquad s_5^2=\begin{pmatrix}C_{40}-C_{40}^*\\ C_{40}-C_{40}^*\\ C_{40}-C_{40}^*\\ C_{42}-C_{42}^*\\ C_{42}-C_{42}^*\\ C_{42}-C_{42}^*\\ C_{44}-C_{44}^*\\ C_{44}-C_{44}^*\\ C_{44}-C_{44}^*\\ s_{000}\end{pmatrix}$.

Similarly,

$$s_{006}+3(s_{204}+s_{024}+s_{402}+s_{042}+s_{420}+s_{240})+6s_{222}s_{600}+s_{060}=s_{000}.$$

$$C_{41}=\frac{1}{Ma_e^4}\int_t d\left(x_3^3x_1-\frac{3}{4}x_3x_1\left(x_1^2+x_2^2\right)\right)dt,$$

$$C_{43}=\frac{1}{6Ma_e^4}\int_t d\left(x_1x_3\left(x_1^2-3x_2^2\right)\right)dt,$$

where

$$A_6=\frac{1}{48}\begin{pmatrix}12 & -18 & -18 & 0 & 0 & 0\\ 0 & 24 & 0 & -9 & -18 & 0\\ 0 & 0 & 48 & 0 & -36 & -12\\ 0 & 4 & -12 & 0 & 0 & 0\\ 0 & 0 & 0 & 2 & -12 & 0\\ 48 & 96 & 96 & 48 & 96 & 48\\ 0 & 0 & 0 & 0 & 8 & -8\end{pmatrix},$$

$$s_6^1=\begin{pmatrix}s_{105}\\ s_{303}\\ s_{123}\\ s_{501}\\ s_{321}\\ s_{141}\end{pmatrix},\qquad C_6=\begin{pmatrix}C_{41}-C_{41}^*\\ C_{41}-C_{41}^*\\ C_{41}-C_{41}^*\\ C_{43}-C_{43}^*\\ C_{43}-C_{43}^*\\ C_{43}-C_{43}^*\\ s_{101}\end{pmatrix}.$$

We have

$$2s_{303}+2s_{123}+s_{105}+s_{501}+2s_{321}+s_{141}=s_{101}$$

$$S_{41}=\frac{1}{Ma_e^4}\int_t d\left(x_3^3x_2-\frac{3}{4}x_3x_2\left(x_1^2+x_2^2\right)\right)dt,$$

$$S_{43}=\frac{1}{6Ma_e^4}\int_t d\left(x_2x_3\left(3x_1^2-x_2^2\right)\right)dt,$$

$$A_7=\frac{1}{48}\begin{pmatrix}12 & -18 & -18 & 0 & 0 & 0\\ 0 & 48 & 0 & -12 & -36 & 0\\ 0 & 0 & 24 & 0 & -18 & -9\\ 0 & 24 & -4 & 0 & 0 & 0\\ 0 & 0 & 0 & 8 & -8 & 0\\ 48 & 96 & 96 & 48 & 96 & 48\\ 0 & 0 & 0 & 0 & 12 & -2\end{pmatrix},$$

$$s_7^1 = \begin{pmatrix} s_{015} \\ s_{213} \\ s_{033} \\ s_{411} \\ s_{231} \\ s_{051} \end{pmatrix}, \qquad C_7 = \begin{pmatrix} S_{41} - S_{41}^* \\ S_{41} - S_{41}^* \\ S_{41} - S_{41}^* \\ S_{43} - S_{43}^* \\ S_{43} - S_{43}^* \\ S_{43} - S_{43}^* \\ s_{011} \end{pmatrix}.$$

Beside this,

$$2s_{213} + 2s_{033} + s_{015} + s_{411} + 2s_{231} + s_{051} = s_{011}$$

$$S_{42} = \frac{1}{Ma_e^4} \int_t d\left(3x_3^2 x_1 x_2 \left(x_1^2 - x_2^2\right) - \frac{1}{2} x_1 x_2 \left(x_2^2 + x_1^2\right)\right) dt,$$

$$S_{44} = \frac{1}{8Ma_e^4} \int_t d x_1 x_2 \left(x_1^2 - x_2^2\right) dt.$$

$$A_8 = \frac{1}{32} \begin{pmatrix} 32 & -16 & -16 & 0 & 0 & 0 \\ 0 & 54 & 0 & -4 & -8 & 0 \\ 0 & 0 & 54 & 0 & -8 & -4 \\ 0 & 4 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 32 & 64 & 64 & 32 & 64 & 32 \\ 0 & 0 & 0 & 0 & 2 & -1 \end{pmatrix},$$

$$s_8^1 = \begin{pmatrix} s_{114} \\ s_{312} \\ s_{132} \\ s_{510} \\ s_{330} \\ s_{150} \end{pmatrix}, \qquad C_8 = \begin{pmatrix} S_{42} - S_{42}^* \\ S_{42} - S_{42}^* \\ S_{42} - S_{42}^* \\ S_{44} - S_{44}^* \\ S_{44} - S_{44}^* \\ S_{44} - S_{44}^* \\ s_{110} \end{pmatrix}.$$

So

$$2s_{312} + 2s_{132} + s_{114} + s_{510} + 2s_{330} + s_{150} = s_{110}.$$

Calculations performed according to this technique show that the systems of equations are numbered, respectively, by the number *N*, have a unique solution for $N = 2, 3, 4, 6, 7, 8$. If $N = 5$, then the system is ambiguous, and for the uniqueness of the solution it is necessary to involve additional conditions. Therefore, this part of the contribution is not taken into account in further studies and the construction of the next approximation of density $d_m$ $(m = 5, 6)$ and its anomalies $\Delta d_m = d_m - d_m^s$, where $d_m^s$ - is the density averaged value over the "sphere". The results of the numerical experiments are reflected in the corresponding figures in the form of sketch maps of different depths. See informational Figures 1 to 6.

***Results***

The constructed three-dimensional density model preserves all the basic properties of the PREM reference model: the magnitude of the jumps and the depth of their occurrence, as well as the nature of the density variation along the radius. In this case, in contrast to the model $d_2$ (Figures 1a, 2a, 3a), the density anomaly maps $\Delta d_m$ (Figures 1b, 2b, 3b) are more structured, that is, they give a more detailed picture of the mass distribution within the Earth. Thus, mass redistribution is observed at different depths, however, the mass consolidation in the equatorial regions and the deficit when approaching the poles is characteristic for all cases. Such a movement of masses can be associated with the rotational motion of the planet. It should be noted that extreme values do not always correlate with peaks of geoid anomalies. On the map of the surface, despite the coincidence of some density anomalies and geoid, for example when $\left(q \approx 90^{\circ}, l \approx 300^{\circ}\right)$, but for others, the opposite dependence is inherent $(q \approx 90^{\circ},\ l \approx 250^{\circ})$. That applies to other depths (radii). Therefore, the use of these results should be treated with caution, given their approximate nature.

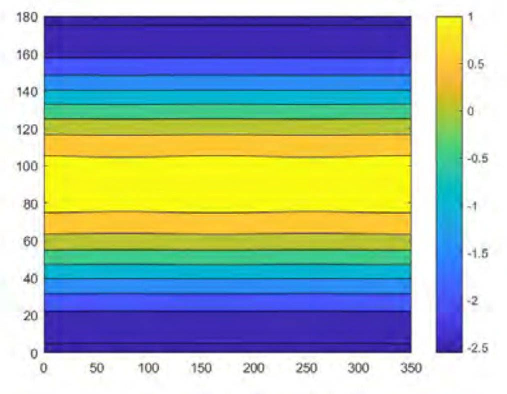


Fig. 1a

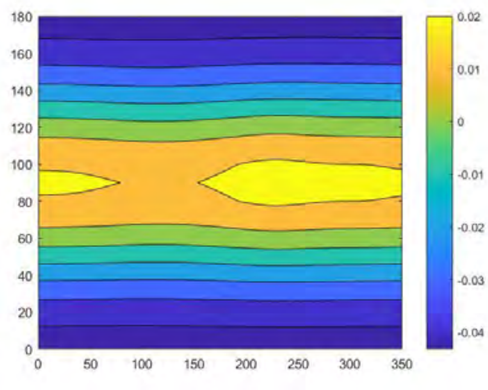


Fig. 1b

**Fig. 1(a-b).** Maps of the distribution of density anomalies of the Earth $\Delta d_6$ at a depth of 5150 km according to the model $d_2$ and the method described in the paper

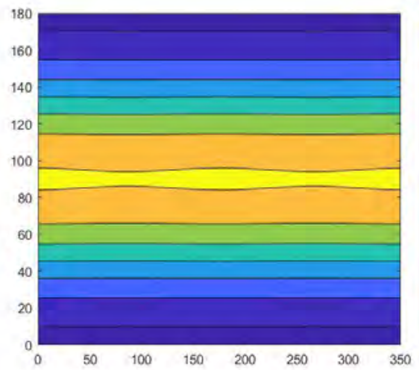


Fig. 2a

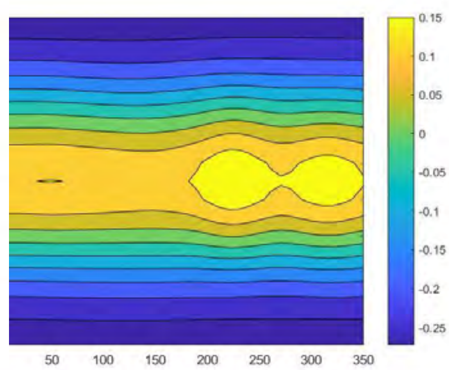


Fig. 2b

**Fig. 2(a-b).** Maps of the distribution of density anomalies of the Earth $\Delta d_6$ at a depth of 2891 km according to the model $d_2$ and the method described in the paper.

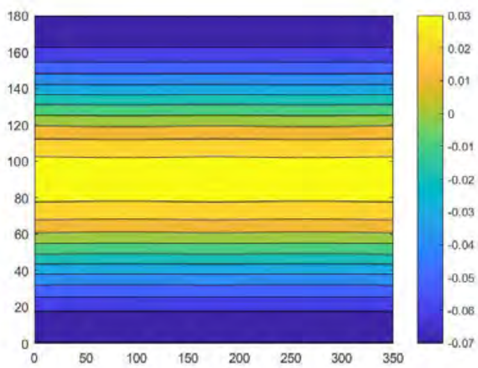


Fig. 3a

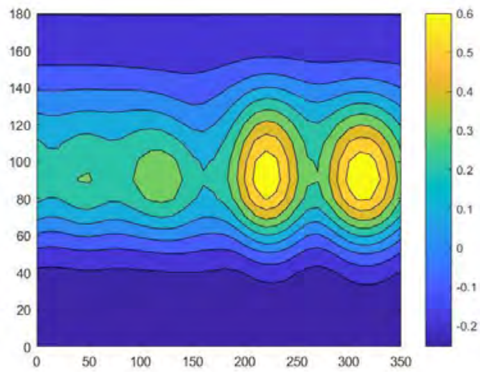


Fig. 3b

**Fig. 3(a-b).** Maps of the distribution of density anomalies of the Earth $\Delta d_6$ on its surface according to the model $d_2$ and the method described in the paper.

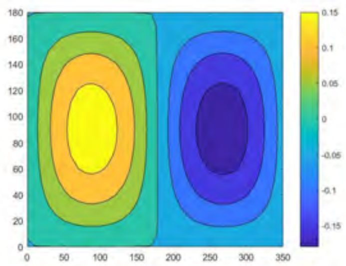

Fig. 4a

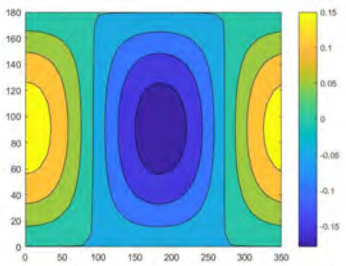

Fig. 4b

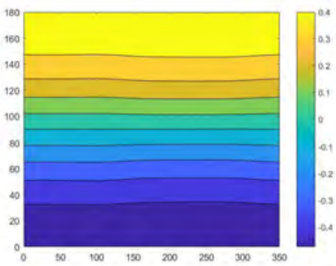

Fig. 4c

**Fig. 4(a-c).** Maps of the composite anomalies of the Earth interior density gradient $\Delta d_6$ at a depth of 5150 km.

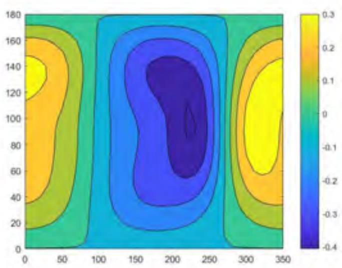

Fig. 5a

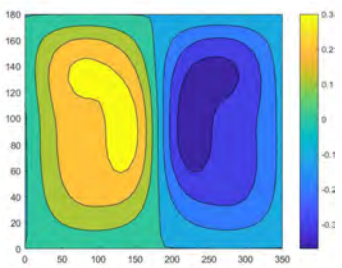

Fig. 5b

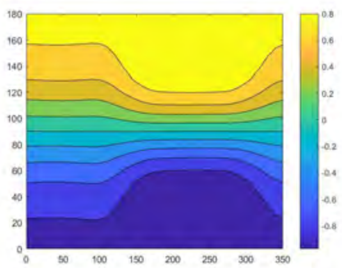

Fig. 5c

**Fig. 5(a-c).** Maps of the composite anomalies of the Earth interior density gradient $\Delta d_6$ at a depth of 2891 km

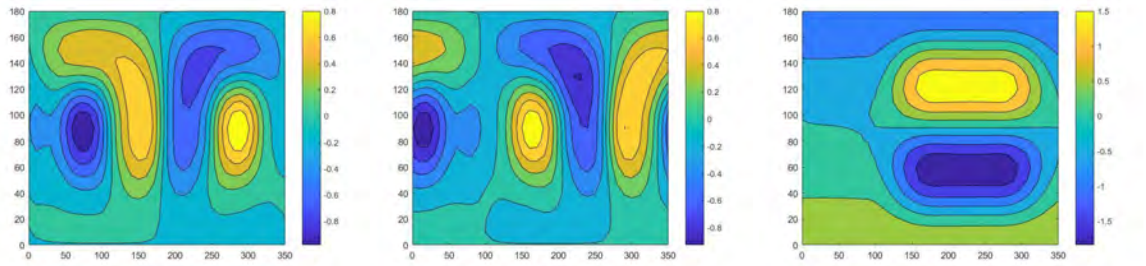

Fig. 6a Fig. 6b Fig. 6c

**Fig. 6(a-c).** Maps of the composite anomalies of the Earth interior density gradient $\Delta d_6$ on the surface

An important aspect in this technique is the generation and interpretation of the behavior of the components of the density anomaly gradient, and especially its third component $\left(\frac{\partial d}{\partial x_3}\right)$. As can be seen from Figures 4c, 5c, 6c, a characteristic value for them is a positive value if $0 \le q \le 90^{\mathbf{o}}$ and negative, when $0 \le q \le 90^{\mathbf{o}}$. This means that the gradient vector is directed toward the center of mass.

Consequently, based on this information, we obtain a density model that gives a more detailed picture of the distribution of masses inside the planet.

The horizontal components of the gradient $\frac{\partial d}{\partial x_1}, \frac{\partial d}{\partial x_2}$ at different depths behave differently and can be an additional tool in the redistribution of masses. In the ideal case, this leads to construction of volumetric maps showing the magnitude and direction of the density gradient vectors, which would greatly simplify the interpretation of the obtained results. Unfortunately, we can not do this yet.

### *Conclusions*

1. The proposed method for the approximate construction of the mass distribution of an ellipsoidal Earth does not substantially differ from that of a spherically symmetrical planet, but it makes it possible for fuller use of information about the planet's gravitational field.
2. To a certain order $(N < 5)$, an approximate method of determining the density and its derivatives allows to monitor the calculation process and evaluate the reliability of such a construction.
3. Constructed sketch maps of lines of constant density make it possible to analyse the locations of anomalous masses and their possible displacement due to the rotating component of the gravity force.
4. The presented maps of components of the density gradient vector give additional possibilities in interpreting possible redistribution of the masses of the Earth.
5. For all maps presented, the third component $\left(\frac{\partial d}{\partial x_3}\right)$ is directed toward the centre of mass, that is, the property of the radial distribution of the growth of the density function with depth.

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## МЕТОД НАБЛИЖЕНОЇ ПОБУДОВИ ГРАДІЄНТА ФУНКЦІЇ ТРИВИМІРНОГО РОЗПОДІЛУ МАС НАДР ЕЛІПСОЇДАЛЬНОЇ ПЛАНЕТИ НА ОСНОВІ ПАРАМЕТРІВ ЗОВНІШНЬОГО ГРАВІТАЦІЙНОГО ПОЛЯ

**Мета.** Дослідити методику побудови тривимірної функції розподілу мас надр усередині Землі та її похідних, узгоджену з параметрами гравітаційного поля планети до четвертого порядку включно. За побудованою таким способом функцією розподілу мас зробити інтерпретацію особливостей внутрішньої будови еліпсоїдальної планети. **Методика.** На основі створеного початкового наближення функції, яке включає референцну модель густини, вибудовуються подальші уточнення. Використовуючи стоксові постійні до другого порядку включно, подаємо наступне наближення, яке надалі приймаємо як нульове. При цьому використання стоксових постійних до четвертого порядку включно приводить до розв’язування систем рівнянь. Встановлено, що долучення однієї тотожності приводить до однозначності розв’язку. Винятком є одна система зі стоксовими постійними $c_{40}, c_{42}, c_{44}$. Зауважимо, що процес обчислень є контрольованим, оскільки степеневі моменти похідних густини зводяться до величин, що враховують значення густини на поверхні еліпсоїда. **Результати**. На відміну від моделі другого порядку, яка описує грубі глобальні неоднорідності, отримана функція розподілу дає детальнішу картину розміщення аномалій густини (відхилення тривимірної функції від усередненої по сфері – “ізоденс”). Аналіз карт на різних глибинах 2891 км (ядро-мантія) та 5150 км (внутрішнє-зовнішнє ядро) дозволяє зробити попередні висновки про глобальний перерозподіл мас за рахунок обертової складової сили

тяжіння по всьому радіусу, а також за рахунок горизонтальних компонент градієнта густини. Цей факт є особливо помітним для екваторіальних областей. Навпаки, в полярних частинах Землі спостерігається мінімум такого відхилення, що також має своє пояснення: величина сили обертання зменшується при зміщенні до полюса. Побудована за допомогою запропонованого методу функція розподілу мас більш детально описує картину розподілу мас. Особливий інтерес становлять картосхеми компонент градієнта аномалій густини, а саме компонента, що співпадає з віссю $Oz$ – для верхньої частини оболонки вона від'ємна, для нижньої – додатна. Це означає, що вектор градієнта напрямлений в сторону центра мас. Характер значень для двох інших компонент різний і за знаком так і за величиною та залежить від точки розміщення. Сукупний розгляд та врахування всіх величин дає можливість повнішої інтерпретації процесів усередині Землі. **Наукова новизна**. На відміну від традиційного підходу зміни для похідних густини однієї змінної (глибини), отриманих із рівняння Адамса-Вільямса, в цій роботі зроблено спробу одержати похідні за декартовими координатами. Використання в описаному методі параметрів гравітаційного поля до четвертого порядку включно збільшує порядок апроксимації функції розподілу мас трьох змінних з двох до шести, а її похідних – до п'яти. При цьому, на відміну від традиційної методики, визначаючим тут є побудова похідних, з яких відтворюється функція розподілу мас та використання геофізичної інформації, що акумульована в реферецній моделі PREM. **Практична значущість.** Отриману функцію розподілу мас Землі можна використати як наступне наближення при використанні стоксових постійних вищих порядків у поданому алгоритмі. Її застосування дає можливість інтерпретувати глобальні аномалії гравітаційного поля та вивчати глибинні геодинамічні процеси всередині Землі.

*Ключові слова*: потенціал, гармонічна функція, модель розподілу мас, стоксові постійні, градієнт густини.

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## МЕТОД ПРИБЛИЖЕННОГО ПОСТРОЕНИЯ ГРАДИЕНТА ФУНКЦИИ ТРЕХМЕРНОГО РАСПРЕДЕЛЕНИЯ МАС НЕДР ЭЛЛИПСОИДАЛЬНОЙ ПЛАНЕТЫ С ИСПОЛЬЗОВАНИЕМ ПАРАМЕТРОВ ВНЕШНЕГО ГРАВИТАЦИОННОГО ПОЛЯ

**Цель.** Исследовать методику построения трехмерной функции распределения масс недр внутри Земли и ее производных, согласованную с параметрами гравитационного поля планеты к четвертому порядку включительно. По построенной таким способом функцией распределения масс осуществить интерпретацию особенностей внутреннего строения эллипсоидальной планеты. **Методика.** На основе созданного начального приближения функции, включающего референцную модель плотности, выстраиваются дальнейшие уточнения. Используя Стокса постоянные до второго порядка включительно, даем следующее приближение, которое в дальнейшем принимаем как нулевое. При этом использование стоксовых постоянных до четвертого порядка включительно приводит к решению систем уравнений. Установлено, что присоединение одного тождества приводит к однозначности решения. Исключением является одна система с стоксовыми постоянными $c_{40}, c_{42}, c_{44}$. Заметим, что процесс вычислений является контролируемым, так как степенные моменты производных плотности сводятся к величинам, которые учитывают значение плотности на поверхности эллипсоида. **Результаты.** В отличие от модели второго порядка, описывающей грубые глобальные неоднородности, полученная функция распределения дает подробную картину размещения аномалий плотности (отклонение трехмерной функции от усредненной по сфере – "изоденс"). Анализ карт на разных глубинах 2891 км (ядро-мантия), 5150 км (внутреннее-внешнее ядро) позволяет сделать предварительные выводы о глобальном перераспределении масс за счет вращающейся составляющей силы тяжести по всему радиусу, а также за счет горизонтальных компонент градиента плотности. Этот факт особенно заметен для экваториальных областей. Напротив, в полярных частях Земли наблюдается минимум такого отклонения, что также имеет свое объяснение: величина силы вращения уменьшается при смещении к полюсу. Построена с помощью предложенного метода функция распределения масс более подробно описывает картину распределения масс. Особый интерес представляют картосхемы компонентов градиента функции аномалий плотности, а именно компонента, что совпадает с осью $Oz$ – для верхней части оболочки она отрицательная, для нижней положительная. Это значит, что вектор градиента направлен в сторону центра масс. Характер значений для двух других компонент разный как по знаку, так и по величине и зависит от точки размещения. Совокупное рассмотрение и учет всех величин дает возможность более полной интерпретации процессов внутри Земли. **Научная новизна.** В отличие от традиционного подхода изменения для производных плотности одной переменной (глубины), полученных из уравнения Адамса-Уильямса, в данной работе сделана попытка получить производные по декартовым координатам. Использование в описанном методе параметров гравитационного поля до четвертого порядка включительно увеличивает порядок аппроксимации функции распределения масс трех переменных с двух до шести, а ее производных – до пяти. При этом, в отличие от традиционной методики, определяющим здесь является построение производных, из которых воспроизводится функция распределения масс и использования геофизической информации, аккумулированная в реферецной модели PREM. **Практическая значимость.** Полученная функция распределения масс Земли может быть использована как следующее приближение при использовании стоксовых постоянных высших порядков в представленном алгоритме. Ее применение дает возможность интерпретировать глобальные аномалии гравитационного поля и изучать глубинные геодинамические процессы внутри Земли.

*Ключевые слова*: потенциал, гармоническая функция, модель распределения масс, стоксовые постоянные, градиент плотности.

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