

ЗАСОБИ ВИМІРЮВАНЬ ЕЛЕКТРИЧНИХ ТА МАГНІТНИХ ВЕЛИЧИН

НЕПЕВНОСТІ БАГАТОПАРАМЕТРОВИХ НЕПРЯМИХ ВИМІРЮВАНЬ ЗА ДОПОМОГОЮ ЕЛЕКТРИЧНИХ СХЕМ ПОСТІЙНОГО СТРУМУ UNCERTAINTIES OF MULTIVARIABLE INDIRECT MEASUREMENTS OF DC ELECTRICAL CIRCUITS

Варша 3.

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Abstract. In this paper three examples of processing uncertainties of a indirect multi-variable measurement system are considered. It was proposed to extend the vector method of estimating measurement uncertainties, given in Supplement 2 to GUM, on the statistical description of the accuracy of whole ranges of indirect multivariable measurement system. Formula for the covariance matrix of relative uncertainties of the vector measurand is given. The covariance matrixes of uncertainties of few DC electrical measurement circuits are presented, i.e.: for indirect measurement of three resistances with using them in three variants of balanced Wheatstone bridge or without disconnection this circuit but with apply unconventional current supplies; the measurement of three internal resistances of the star circuit from its terminals and estimation of uncertainty of powers of two currents if two other currents are measured and their uncertainties are known. Formulas for absolute and relative uncertainties and their correlation coefficients are given. The general conclusion is that in the description accuracy of multivariable measurement systems the relative uncertainties are sometimes preferable than the absolute ones, and uncertainties of their main measurement functions have been also considered.

Key words: Uncertainty, Electrical circuit, Multiplicative measurement equations.

Анотація. Висвітлено особливості опрацювання результатів вимірювання за допомогою непрямой багатопараметрової вимірювальної системи. Розглянуто три випадки вивчення непевностей вимірювання. Запропоновано поширити векторний метод оцінювання непевностей вимірювання, поданий у додатку 2 до GUM, на статистичне описання параметрів точності непрямой багатопараметрової вимірювальної системи. Подано формули для коваріантної матриці відносних непевностей векторного межеранда. Матриця непевностей для випадку декількох електричних систем вимірювання на постійному струмі підлягає аналізу. Інакше кажучи, аналізують непрямі вимірювання електричного опору за допомогою моста Вінстона, причому вимірювання внутрішніх опорів зіркової конфігурації з оцінюванням непевності струму напруги живлення підлягають розгляду. Наведені формули для абсолютної та відносної непевностей, а також їх коефіцієнти кореляції. Основний висновок такий: описання параметрів точності багатопараметрової вимірювальної системи відносними непевностями іноді краще порівняно із подібним описанням на основі абсолютних непевностей, що й повинно враховуватись під час розгляду.

Ключові слова: непевність, електричне коло, багатопараметрові вимірювальні рівняння.

Introduction

In indirect methods of measurements, results the tested quantities (observables) and their accuracy is determined from direct measurements of the set of jointed other quantities, named the multivariate or vector measurand. In general case of multivariable measurements, the relation between values X of quantities measured in input and data of Y quantities

obtained on output after processing is a functional $F(X, Y) = 0$. Usually it can be formulated as the following multivariable function

$$Y = F(X). \quad (1)$$

The multivariable vector dependence (1) can be linear or non-linear. The general flow chart of multivariable measurement system is shown in Fig. 1.

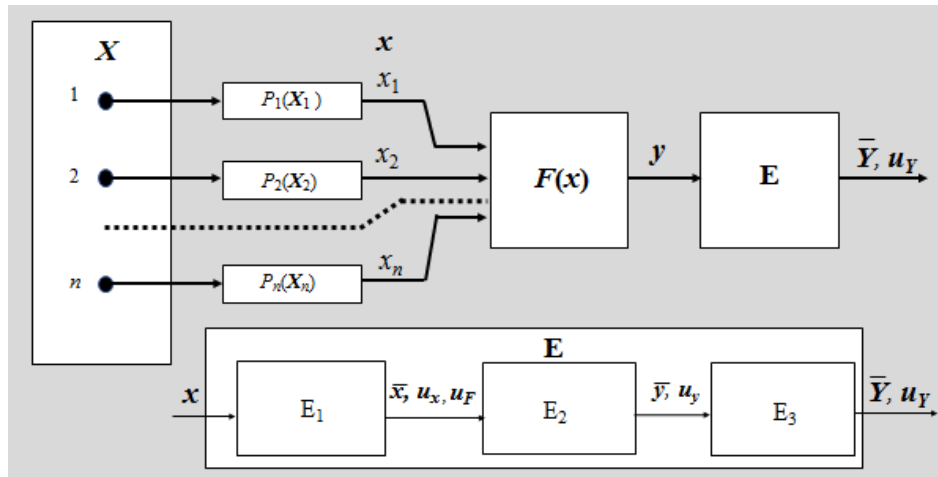


Fig. 1. Signal processing in multi-variable indirect measurement system

In Fig. 1 and in the text below the next symbols are used: X, Y – vectors of values of input and output variables; $x=[x_1, x_2, \dots, x_n]$, $y=[y_1, y_2, \dots, y_m]$ – vectors of input x_i and output y_i signals; $\bar{x}, \bar{y}, u_x, u_{\delta x}, u_y, u_{\delta y}$ – vectors of estimators of values of input x and output y signals and their absolute and relative standard uncertainties; $U_x, U_y, U_{\delta x}, U_{\delta y}$ – covariance matrixes; $F(x), u_F, u_{\delta F}, U_F, U_{\delta F}$ – function of processing signals x to y , their absolute and relative uncertainties and its covariance matrixes, E – processing unit of y to obtain vectors of $\bar{Y}, u_y, u_{\delta y}$ and covariance matrixes $U_y, U_{\delta y}$.

The method of estimation uncertainties in indirect multivariable measurements are given in GUM Supplement 2 [1]. A collection of n individual quantities X_i of the input vector measurand X are measured directly, and from their signals x_i the output vector Y of m estimators \bar{Y}_i (named observables [2–3]) and covariance matrixes U_x, U_y of absolute standard uncertainties $u_x \equiv \sigma_x, u_y \equiv \sigma_y$ are calculated. The vector Y depends on whether the functional $F(X)$ is linear ($m \leq n$) or nonlinear ($m \leq \text{sum of linear and nonlinear equations}$). The relation between output and input covariance matrixes is

$$U_Y = S U_X S^T \quad (2)$$

where:

$$U_X = \begin{bmatrix} \sigma_{x1}^2 & \dots & \rho_{x1n} \sigma_{x1} \sigma_{xn} \\ \dots & \dots & \dots \\ \rho_{xn1} \sigma_{xn} \sigma_{x1} & \dots & \sigma_{xn}^2 \end{bmatrix},$$

$$S = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \dots & \frac{\partial y_1}{\partial x_n} \\ \dots & \dots & \dots \\ \frac{\partial y_m}{\partial x_1} & \dots & \frac{\partial y_m}{\partial x_n} \end{bmatrix},$$

$$U_Y = \begin{bmatrix} \sigma_{y1}^2 & \dots & \rho_{y1m} \sigma_{y1} \sigma_{ym} \\ \dots & \dots & \dots \\ \rho_{y1m} \sigma_{ym} \sigma_{y1} & \dots & \sigma_{ym}^2 \end{bmatrix} \quad (2a, b, c)$$

Covariance matrixes U_x, U_y are symmetric, i.e. correlation coefficients $\rho_{xij} = \rho_{xji}$, and $\rho_{yij} = \rho_{yji}$.

S is the sensitivity matrix obtained after linearization of function F for small changes of X elements.

All, or some results of components of the measurand Y can be used further separately or jointly. In the latter case it is necessary to take also in considerations the correlation coefficients between the uncertainties of its components y_i , which are in non-diagonal elements of the covariance matrix U_y .

For calculations made *of line*, i.e. after collection all data, the uncertainties of function F can be obtained negligible. Recommendation for estimation uncertainties of this case is in the GUM Supplement 2. The realization of indirect measurements of m – components of the multivariate measurand Y can be made also *on line* by automatic instrumental measurement systems. In this case the uncertainties of F should be also considered.

The metrological description of the multivariable instrumental measurement system needs the accuracy for the whole values of measurement ranges of input and output signals. The accuracy of each range is up to now described by the maximal value (worth case) of limited absolute error $|\Delta_{y_i}|_{\max}$. The absolute error of any output signal y_i of Y may be presented in the similar two component form as for digital voltmeter, i.e. $\Delta_{y_i} = \Delta_{y_{i0}} + \Delta_{(y_i - y_{i0})}$ and the limited absolute error is

$$|\Delta y_i| \leq |\Delta y_{i0}| + y_i |\varepsilon_{s_i}(y_i)| \quad \text{for } i=1, \dots, m \quad (3)$$

where: Δy_{i0} – absolute error of initial value y_{i0} of the range, $\varepsilon_{s_i} \equiv \Delta(y_i - y_{i0})/y_i$ – relative error of the difference $(y_i - y_{i0})$ of output signal or reading.

If $|\Delta y_{i0}| \ll |\Delta(y_i - y_{i0})|$, then the relative limited error (worth case) of the component y_i is

$$|\Delta y_i / y_i|_{\max} \cong |\varepsilon_{s_i}|_{\max} \quad (3a)$$

The probability of existence the maximal limited error in each range is very low. Then the randomized description as in GUM [1] by uncertainties type B can be more valuable. It may be made in the similar two component form as for limited errors in the equation (3). Then it should contain the expanded absolute uncertainty U_0 of the initial value y_{i0} of each range and expanded relative uncertainty U_r of its increase $(y_i - y_{i0})$ for all values of the range. Both these uncertainties should be given for defined P probability of the confidence level, e.g. for $P=0.95$ is $U_0 \approx 0.95u$ and $U_r \approx 0.95u_r$ (u, u_r – standard uncertainties marked as in GUM). In the most cases these components of uncertainty are non-correlated, and very often relative uncertainty U_r is constant for the whole range or its function or maximal its value can be used. Moreover, the type B standard uncertainty is significantly smaller then maximal limited error because it is estimated as the square root of possible values of components, and not as their sum. The standard uncertainty of the single output value y_i is

$$u_{y_i} = \sqrt{u_{y_{i0}}^2 + (y_i - y_{i0})^2 u_{r_i}^2} \quad (4)$$

If $u_{y_{i0}}^2 \ll (y_i - y_{i0})^2 u_{r_i}^2$, then the accuracy of y_i is described only by the single value of relative uncertainty $u_{y_i} \cong u_{r_i} y_i \equiv \delta_{y_i}$, unchanged in almost whole measuring range, or as $\delta_{y_i} \leq u_{r_i, \max} y_i$.

Up to date there are no internationally accepted regulations how to describe statistically by the uncertainties the accuracy of different kind of instrumental systems for indirect multivariable measurements. For the multiplicative type of measurement equations we found that is possible to use given below the new vector formula (5) between covariance matrixes $U_{\delta X}$ and $U_{\delta Y}$ of relative standard uncertainties $u_{x_i} \equiv \delta_{x_i}$, $u_{y_i} \equiv \delta_{y_i}$. Their correlation coefficients are the same as in (2)

$$U_{\delta Y} = S_B U_{\delta X} S_B^T \quad (5)$$

where:

$$U_{\delta X} = \begin{bmatrix} \delta_{x1}^2 & \dots & \rho_{x1n} \delta_{x1} \delta_{xn} \\ \dots & \dots & \dots \\ \rho_{xn1} \delta_{xn} \delta_{x1} & \dots & \delta_{xn}^2 \end{bmatrix},$$

$$S_B = \begin{bmatrix} \frac{x_1}{y_1} \frac{\partial y_1}{\partial x_1} & \dots & \frac{x_n}{y_1} \frac{\partial y_1}{\partial x_n} \\ \dots & \dots & \dots \\ \frac{x_1}{y_m} \frac{\partial y_m}{\partial x_1} & \dots & \frac{x_n}{y_m} \frac{\partial y_m}{\partial x_n} \end{bmatrix},$$

$$U_{\delta Y} = \begin{bmatrix} \delta_{y1}^2 & \dots & \rho_{y1m} \delta_{y1} \delta_{ym} \\ \dots & \dots & \dots \\ \rho_{y1m} \delta_{ym} \delta_{y1} & \dots & \delta_{ym}^2 \end{bmatrix} \quad (5a-c)$$

All that should be clearer on analysis of few examples of indirect multivariate measurements with the multiplicative and additive type of functional F given below, i.e. indirect measurements of three arm resistances of the Wheatstone bridge and measurement of star circuit internal resistances from its terminals. The description of the uncertainty of active power measurements will be also discussed. Some general conclusions are given in the end.

1. Case of multiplicative measurement equations

The unknown values R_2, R_3, R_4 of three resistors can be determined without use the high accuracy digital ohmmeter. Two cases of indirect measurements are possible. The first, when these resistors and the regulated multi-decade resistor R_1 are connected as the Wheatstone bridge circuit but three times in three different orders in its loop, i.e. R_2, R_3, R_4 (Fig. 2a), or R_2, R_4, R_3 and R_3, R_2, R_4 . Three bridge' balances $U_{CD} = 0$ give three settings R_{x1}, R_{x2}, R_{x3} of R_1 . The settings R_{x2}, R_{x3} can be obtained also without disconnection this bridge, by unconventional supply by current sources $J_1=J_3$ connected parallelly to opposite arms 1, 3 and balancing outputs AB or DC on diagonals (Fig. 2b).

In both circuits the same settings R_{x1}, R_{x2}, R_{x3} on multi decade resistor R_1 satisfied circuit balances, i.e.:

$$R_{x1} = R_2 \frac{R_4}{R_3}, R_{x2} = R_2 \frac{R_3}{R_4}, R_{x3} = R_3 \frac{R_4}{R_2} \quad (6a, b, c)$$

From above relations the unknown resistances as elements of the output vector Y can be calculated

$$R_2 = \sqrt{R_{x1} R_{x2}}, \quad R_3 = \sqrt{R_{x2} R_{x3}},$$

$$R_4 = \sqrt{R_{x1} R_{x3}} \quad (7a, b, c)$$

As solutions (26) are of the multiplicative type the equation (3) for direct calculating the relative uncertainties can be used. The measurement sensitivity function S_B for relative uncertainties is

$$S_B = \begin{bmatrix} \frac{R_{x1}}{R_2} \frac{\partial R_2}{\partial R_{x1}} & \frac{R_{x2}}{R_2} \frac{\partial R_2}{\partial R_{x2}} & \frac{R_{x3}}{R_2} \frac{\partial R_2}{\partial R_{x3}} \\ \frac{R_{x1}}{R_3} \frac{\partial R_3}{\partial R_{x1}} & \frac{R_{x2}}{R_3} \frac{\partial R_3}{\partial R_{x2}} & \frac{R_{x3}}{R_3} \frac{\partial R_3}{\partial R_{x3}} \\ \frac{R_{x1}}{R_4} \frac{\partial R_4}{\partial R_{x1}} & \frac{R_{x2}}{R_4} \frac{\partial R_4}{\partial R_{x2}} & \frac{R_{x3}}{R_4} \frac{\partial R_4}{\partial R_{x3}} \end{bmatrix} =$$

$$= \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad (8)$$

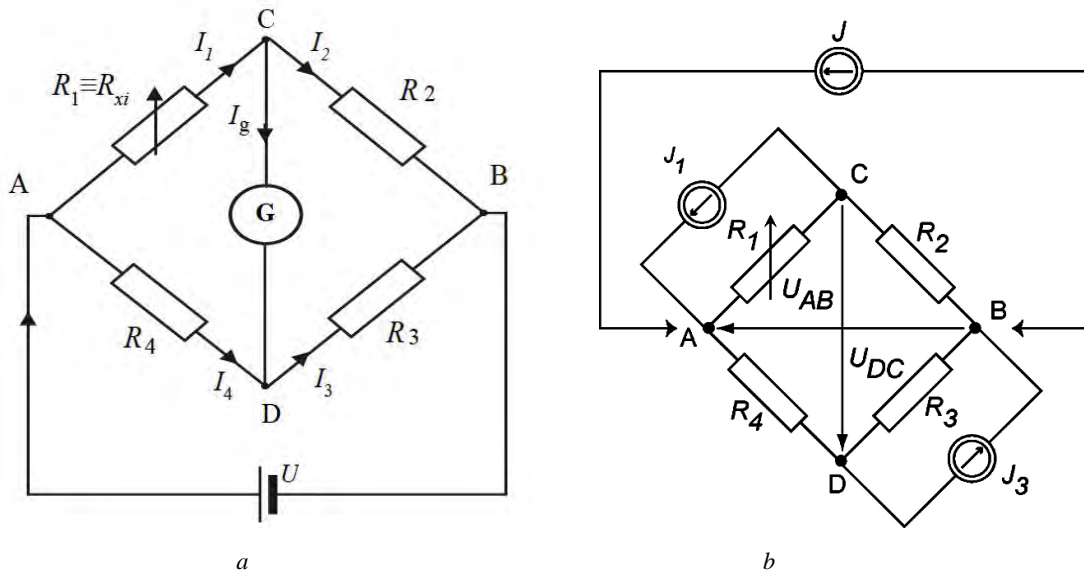


Fig. 2. The structures of DC bridges for measurement three resistances R_2, R_3, R_4 : a) the first of arm resistances of three variants of the balanced Wheatstone bridge; b) two different the bridge loop circuit supplies: classic one as in a) from single source J (or U) and in balance $U_{DC} = 0$ is $R_1 R_3 = R_2 R_4$; unconventional double current supply $J_1 = J_3$ in parallel to opposite arms 1 and 3 (or $J_3 = 0$ and J_1 switched between these arms) and then eq. of outputs DC and AB balances are used for measurement, i.e. if $U_{AB} = 0, R_1 R_4 = R_2 R_3$ or if $U_{DC} = 0, R_1 R_2 = R_3 R_4$

For estimation uncertainty of measured resistances, we assumed firstly that input variables R_{x1}, R_{x2}, R_{x3} are not correlated. and using formula (3) relative uncertainties, i.e. $U_{BY} = S_B U_{BX} S_B^T$ is:

$$U_{BY} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta_{R_{x1}}^2 & 0 & 0 \\ 0 & \delta_{R_{x2}}^2 & 0 \\ 0 & 0 & \delta_{R_{x3}}^2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} \delta_{R_{x1}}^2 + \delta_{R_{x2}}^2 & \delta_{R_{x2}}^2 & \delta_{R_{x1}}^2 \\ \delta_{R_{x2}}^2 & \delta_{R_{x2}}^2 + \delta_{R_{x3}}^2 & \delta_{R_{x3}}^2 \\ \delta_{R_{x1}}^2 & \delta_{R_{x3}}^2 & \delta_{R_{x1}}^2 + \delta_{R_{x3}}^2 \end{bmatrix} \quad (9)$$

So, the standard relative uncertainties of output quantities are defined:

$$\delta_{R_2} = \frac{1}{2} \sqrt{\delta_{R_{x1}}^2 + \delta_{R_{x2}}^2}, \quad \delta_{R_3} = \frac{1}{2} \sqrt{\delta_{R_{x2}}^2 + \delta_{R_{x3}}^2}, \quad \delta_{R_4} = \frac{1}{2} \sqrt{\delta_{R_{x1}}^2 + \delta_{R_{x3}}^2} \quad (10a, b, c)$$

and correlations coefficients:

$$\rho_{R_2 R_3} = \frac{\frac{1}{4} \delta_{R_{x2}}^2}{\frac{1}{2} \sqrt{\delta_{R_{x1}}^2 + \delta_{R_{x2}}^2} \frac{1}{2} \sqrt{\delta_{R_{x2}}^2 + \delta_{R_{x3}}^2}} = \frac{1}{\sqrt{1 + \left(\frac{\delta_{R_{x2}}}{\delta_{R_{x1}}}\right)^2} \sqrt{1 + \left(\frac{\delta_{R_{x3}}}{\delta_{R_{x2}}}\right)^2}} > 0$$

$$\rho_{R_2 R_4} = \frac{\frac{1}{4} \delta_{R_{x1}}^2}{\frac{1}{2} \sqrt{\delta_{R_{x1}}^2 + \delta_{R_{x2}}^2} \frac{1}{2} \sqrt{\delta_{R_{x1}}^2 + \delta_{R_{x3}}^2}} =$$

$$= \frac{1}{\sqrt{1 + \left(\frac{\delta_{R_{x2}}}{\delta_{R_{x1}}}\right)^2} \sqrt{1 + \left(\frac{\delta_{R_{x3}}}{\delta_{R_{x1}}}\right)^2}} > 0 \quad (11a, b, c)$$

$$\rho_{R_3 R_4} = \frac{\frac{1}{4} \delta_{R_{x3}}^2}{\frac{1}{2} \sqrt{\delta_{R_{x2}}^2 + \delta_{R_{x3}}^2} \frac{1}{2} \sqrt{\delta_{R_{x1}}^2 + \delta_{R_{x3}}^2}} = \frac{1}{\sqrt{1 + \left(\frac{\delta_{R_{x2}}}{\delta_{R_{x3}}}\right)^2} \sqrt{1 + \left(\frac{\delta_{R_{x1}}}{\delta_{R_{x3}}}\right)^2}} > 0$$

All above correlations coefficients are positive.

If $\delta_{R_{x1}} = \delta_{R_{x2}} = \delta_{R_{x3}} = \delta$, $\delta_{R_2} = \delta_{R_3} = \delta_{R_4} = \frac{\sqrt{2}}{2} \delta$; $\rho_{R_2 R_3} = \rho_{R_2 R_4} = \rho_{R_3 R_4} = \frac{1}{2}$, then the coverage region is ellipsoid of parameter $w = 1 - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} + 2 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2} > 0$

In this case the ellipsoidal coverage region for relative uncertainties of Y with probability 0.95 determines the ellipsoid with half axis $a = 2,8\delta$, $b = 1,4\delta$, $c = 1,4\delta$. In formulas it is used that the coverage factor/extension/coefficient for 95% of coverage region in 3D (three-dimensional) Gauss distribution is equal $k_p = 2,8$. This ellipsoid is contiguous in six points to the cube with edges $d = 2 \cdot 2,8 \cdot \frac{\sqrt{2}}{2} \delta = 3,96 \delta$. The relations between capacity of ellipsoid and cube is $4\pi abc / (3d^3) = 37\%$.

1.1. Example of the additive type of multi-variate measurement equations

In many practical situations the star circuit of resistances connection is applied and there is no possibility of disconnection them from the common point 0 and even this point is not available, or star structure is the equivalent circuit only. So, three values of star resistances must be determined indirectly from measurements of three input resistances between terminals A, B, C (Fig. 3). If changes of star resistances must be remote monitored from a distance, then the special measurement circuit E is used for these indirect measurements. Let us as the first step assume that the values of resistances of star are determined precisely without any disturbances and modifications by A/D converters and arithmetical modules located in E.

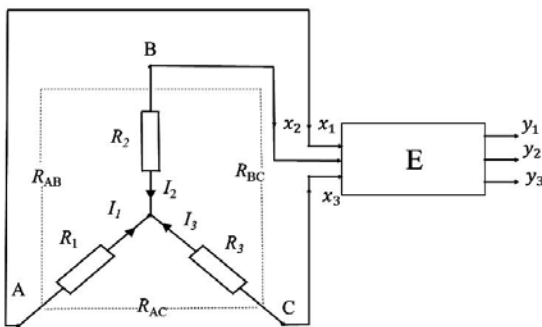


Fig. 3. The diagram of the star circuits with module of performance measurements

The main measurement equations are:

$$\begin{aligned} R_{AB} &= R_1 + R_2, & R_{BC} &= R_2 + R_3, \\ R_{AC} &= R_1 + R_3 \end{aligned} \quad (12)$$

or in the matrix form:

$$\begin{bmatrix} R_{AB} \\ R_{BC} \\ R_{AC} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \quad (12a)$$

To obtain solutions, the both sides of eq.(6a) are multiplied by the inverse to above matrix and the main formula (1) has here the matrix form

$$Y = F \cdot X \quad (13)$$

Where: $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} R_{AB} \\ R_{BC} \\ R_{AC} \end{bmatrix}$, $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}$,

$$F = \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \quad (13a, b, c)$$

The star circuit resistances are

$$\begin{aligned} R_1 &= \frac{R_{AB}}{2} - \frac{R_{BC}}{2} + \frac{R_{AC}}{2}, \\ R_2 &= \frac{R_{AB}}{2} + \frac{R_{BC}}{2} - \frac{R_{AC}}{2}, \end{aligned}$$

$$R_3 = -\frac{R_{AB}}{2} + \frac{R_{BC}}{2} + \frac{R_{AC}}{2} \quad (14a, b, c)$$

Then corrections are implemented for known systematic errors. Unknown systematic errors are randomized and estimated as components of the type B uncertainty u_B . Next the results of absolute standard uncertainties σ_{AB} , σ_{BC} , σ_{AC} are find as a square of quadratic values of uncertainties u_A and u_B (type A and B), and relative uncertainties δ_{AB} , δ_{BC} , δ_{AC} should be calculated.

To find the absolute uncertainties and correlation coefficients of star resistances as output quantities, the vector method given in Supplement 2 to GUM is used [1]. Covariance matrices are related by formula (2), i.e. $U_Y = S \cdot U_X \cdot S^T$. In which: U_Y , U_X – covariance matrixes of output vector Y and input vector X , S - the Jacobian matrix sensitivity coefficients of absolute uncertainties.

For the resistances of stair circuit

$$\begin{aligned} S = F &= \begin{bmatrix} \partial R_1 / \partial R_{AB} & \partial R_1 / \partial R_{BC} & \partial R_1 / \partial R_{AC} \\ \partial R_2 / \partial R_{AB} & \partial R_2 / \partial R_{BC} & \partial R_2 / \partial R_{AC} \\ \partial R_3 / \partial R_{AB} & \partial R_3 / \partial R_{BC} & \partial R_3 / \partial R_{AC} \end{bmatrix} = \\ &= \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \end{aligned} \quad (15)$$

0 correlated variables in the input

Let us consider the general case when absolute uncertainties of input quantities σ_{AB} , σ_{BC} , σ_{AC} are correlated. Then in the covariance matrix U_X of input quantities in such case the non-zero elements in non-diagonal positions are appearing. They are defined with correlation coefficients ρ_{AB} , ρ_{BC} , ρ_{AC}

$$U_X = \begin{bmatrix} \sigma_{AB}^2 & \rho_{AB} \sigma_{AB} \sigma_{BC} & \rho_{BC} \sigma_{AB} \sigma_{AC} \\ \rho_{AB} \sigma_{AB} \sigma_{BC} & \sigma_{BC}^2 & \rho_{AC} \sigma_{BC} \sigma_{AC} \\ \rho_{BC} \sigma_{AB} \sigma_{AC} & \rho_{AC} \sigma_{BC} \sigma_{AC} & \sigma_{AC}^2 \end{bmatrix} \quad (16)$$

If the relative uncertainties of measured

resistances on stair circuit terminals are the same, i.e.:

$$\delta_{AB} = \delta_{BC} = \delta_{AC} = \delta,$$

then absolute uncertainties of input quantities are: $\sigma_{AB} = \delta \cdot R_{AB}$, $\sigma_{AC} = \delta \cdot R_{AC}$ and $\sigma_{BC} = \delta \cdot R_{BC}$. Then output absolute uncertainties are:

$$\begin{aligned} \sigma_{y1} &= \frac{\delta}{2} \sqrt{R_{AB}^2 + R_{BC}^2 + R_{AC}^2 + 2(\rho_{BC} R_{AB} R_{AC} - \rho_{AB} R_{AB} R_{BC} - \rho_{AC} R_{BC} R_{AC})} \\ \sigma_{y2} &= \frac{\delta}{2} \sqrt{R_{AB}^2 + R_{BC}^2 + R_{AC}^2 + 2(\rho_{AB} R_{AB} R_{BC} - \rho_{BC} R_{AB} R_{AC} - \rho_{AC} R_{BC} R_{AC})} \\ \sigma_{y3} &= \frac{\delta}{2} \sqrt{R_{AB}^2 + R_{BC}^2 + R_{AC}^2 + 2(\rho_{AC} R_{BC} R_{AC} - \rho_{BC} R_{AB} R_{AC} - \rho_{AB} R_{AB} R_{BC})} \end{aligned} \quad (17a, b, c)$$

The output relative uncertainties:

$$\begin{aligned} \delta_{y1} &= \frac{\delta \sqrt{1 + \beta^2 + \gamma^2 + 2(\rho_{BC} \gamma - \rho_{AB} \beta - \rho_{AC} \beta \gamma)}}{1 - \beta + \gamma} \\ \delta_{y2} &= \frac{\delta \sqrt{1 + \beta^2 + \gamma^2 + 2(\rho_{AB} \beta - \rho_{BC} \gamma - \rho_{AC} \beta \gamma)}}{1 + \beta - \gamma} \\ \delta_{y3} &= \frac{\delta \sqrt{1 + \beta^2 + \gamma^2 + 2(\rho_{AC} \beta \gamma - \rho_{BC} \gamma - \rho_{AB} \beta)}}{\beta + \gamma - 1} \end{aligned} \quad (18a, b, c)$$

where:

$$\beta = \frac{R_{BC}}{R_{AB}} \text{ and } \gamma = \frac{R_{AC}}{R_{AB}}.$$

The correlations coefficients of output quantities are defined as follows

$$\begin{aligned} \rho_{y_1 y_2} &= \frac{\delta^2}{4} \frac{R_{AB}^2 - R_{BC}^2 - R_{AC}^2 + 2\rho_{AC} R_{BC} R_{AC}}{\sigma_1 \sigma_2}, \\ \rho_{y_1 y_3} &= \frac{\delta^2}{4} \frac{R_{AC}^2 - R_{BC}^2 - R_{AB}^2 + 2\rho_{AB} R_{BC} R_{AC}}{\sigma_1 \sigma_3}, \\ \rho_{y_2 y_3} &= \frac{\delta^2}{4} \frac{R_{BC}^2 - R_{AC}^2 - R_{AB}^2 + 2\rho_{BC} R_{AB} R_{AC}}{\sigma_2 \sigma_3} \end{aligned} \quad (19a, b, c)$$

If $R_{AB} = R_{BC} = R_{AC} = R$, the uncertainties are

$$\begin{aligned} \sigma_{y_1} &= \frac{\delta R}{2} \sqrt{3 + 2(\rho_{BC} - \rho_{AB} - \rho_{AC})}, \\ \sigma_{y_2} &= \frac{\delta R}{2} \sqrt{3 + 2(\rho_{AB} - \rho_{BC} - \rho_{AC})}, \\ \sigma_{y_3} &= \frac{\delta R}{2} \sqrt{3 + 2(\rho_{AC} - \rho_{AB} - \rho_{BC})} \end{aligned} \quad (20a, b, c)$$

Ø non-correlated variables in the input

For non-correlated variables $\rho_{AB} = \rho_{BC} = \rho_{AC} = 0$, and from (20) the absolute uncertainties are

$$\sigma_{y_1} = \sigma_{y_2} = \sigma_{y_3} = \frac{1}{2} \sqrt{\sigma_{R_{AB}}^2 + \sigma_{R_{AC}}^2 + \sigma_{R_{BC}}^2} \quad (21)$$

And from (13) correlation coefficients

$$\begin{aligned} \rho_{y_1 y_2} &= \frac{\sigma_{AB}^2 - \sigma_{BC}^2 - \sigma_{AC}^2}{\sigma_{AB}^2 + \sigma_{BC}^2 + \sigma_{AC}^2}, \\ \rho_{y_1 y_3} &= \frac{-\sigma_{AB}^2 - \sigma_{BC}^2 + \sigma_{AC}^2}{\sigma_{AB}^2 + \sigma_{BC}^2 + \sigma_{AC}^2}, \\ \rho_{y_2 y_3} &= \frac{-\sigma_{AB}^2 + \sigma_{BC}^2 - \sigma_{AC}^2}{\sigma_{AB}^2 + \sigma_{BC}^2 + \sigma_{AC}^2} \end{aligned} \quad (22a, b, c)$$

If $\sigma_{AB} = \sigma_{BC} = \sigma_{AC} = \sigma$ then

$$\begin{aligned} \sigma_{y_1} = \sigma_{y_2} = \sigma_{y_3} &= \frac{\sqrt{3}}{2} \sigma \\ \rho_{y_1 y_2} = \rho_{y_1 y_3} = \rho_{y_2 y_3} &= -\frac{1}{3}. \end{aligned}$$

It can be show that determinant of matrix U_y by the sign of parameter w

$$\begin{aligned} w &= 1 - \rho_{y_1 y_2}^2 - \rho_{y_1 y_3}^2 - \rho_{y_2 y_3}^2 + \\ &+ 2 \cdot \rho_{y_1 y_2} \rho_{y_1 y_3} \rho_{y_2 y_3} > 0 \end{aligned} \quad (23)$$

is always positive.

Defining $x = \frac{\sigma_{BC}^2 + \sigma_{AC}^2}{\sigma_{AB}^2} > 0$, $v = \frac{\sigma_{BC}^2 - \sigma_{AC}^2}{\sigma_{AB}^2}$ we express correlation coefficients by

$$\rho_{y_1 y_2} = \frac{1-x}{1+x}; \quad \rho_{y_1 y_3} = -\frac{1+v}{1+x}; \quad \rho_{y_2 y_3} = \frac{v-1}{1+x} \quad (24a, b, c)$$

so, the parameter $w = 4 \frac{x^2 - v^2}{(1+x)^2}$ must be $x^2 > v^2$, what is always fulfilled.

That is why the characteristic equation of inverse matrix has three positive roots.

The border of cover region for values of results with given probability $P \leq 0.95$ is ellipsoid, closed in solid cubic, and contiguous in six points the wall of cubic with edge distance

$$k_p \sqrt{\sigma_{AB}^2 + \sigma_{BC}^2 + \sigma_{AC}^2} \quad (25)$$

$k_p=2,8$ – cover factor/ extension coefficients.

Summary of solutions of some cases / $P \leq 0,95$

- Ø if $\sigma_{AB} = \sigma_{BC} = \sigma_{AC} = \sigma_{in}$,
 $\rho_{AB} = \rho_{BC} = \rho_{AC} = \rho_{in}$;
- Ø $\sigma_{out} = \sigma_{y_1} = \sigma_{y_2} = \sigma_{y_3} = \frac{1}{2} \sigma_{in} \sqrt{3 - 2\rho_{in}}$;
- Ø $\rho_{y_1 y_2} = \rho_{y_1 y_3} = \rho_{y_2 y_3} = \frac{2\rho_{in} - 1}{2 - 2\rho_{in}}$;
 $\rho_{y_1 y_2} \cdot \rho_{y_2 y_3} \cdot \rho_{y_1 y_3} \leq 0$ for $\rho_{in} \leq \frac{1}{2}$.
- Ø if $\rho_{in} = 0$: $\sigma_{out} = \frac{\sqrt{3}}{2} \sigma_{in}$,
 $\rho_{y_1 y_2} = \rho_{y_1 y_3} = \rho_{y_2 y_3} = -\frac{1}{3}$; half axes:
 $1,4 \sigma_{in}, 2,8 \sigma_{in}, 2,8 \sigma_{in}$
- Ø if $\rho_{in} = \frac{1}{2}$: $\min \sigma_{out} = \frac{1}{2} \sigma_{in}$,
 $\rho_{y_1 y_2} = \rho_{y_1 y_3} = \rho_{y_2 y_3} = 0$; radius $1,4 \sigma_{in}$
- Ø if $\rho_{in} = -1$: $\max \sigma_{out} = \frac{\sqrt{3}}{2} \sigma_{in}$,
 $\rho_{y_1 y_2} = \rho_{y_1 y_3} = \rho_{y_2 y_3} = -\frac{2}{3}$; $w < 0$.

1.2. Influence of uncertainties u_F of matrix F in stair circuit measurements

In the instrumental system for measurements the stair circuit resistances processing of output values and their uncertainties is made in digital unit E . The main matrix equation $Y = F \cdot X$ was given in (13) and (13a, b, c). Solution of vector Y elements is in (14). Let us now consider uncertainties of amplification/attenuation of signals in measurement channels. The realization of signals processing has linear disturbances in channels changing levels of signals e.g.:

$$X_S = \begin{bmatrix} k_1 x_1 \\ k_2 x_2 \\ k_3 x_3 \end{bmatrix} \quad (26)$$

where k_1, k_2, k_3 is amplifying coefficients.

The analog/digital processing input signals has uncertainty u_F . Therefore, the functional matrix F is must be modified, and a new matrix is defined as follows:

$$F_S = \frac{1}{2} \begin{bmatrix} k_1(1 + \delta_1) & -k_2(1 + \delta_1) & k_3(1 + \delta_1) \\ k_1(1 + \delta_2) & k_2(1 + \delta_2) & k_3(1 + \delta_2) \\ -k_1(1 + \delta_3) & k_2(1 + \delta_3) & k_3(1 + \delta_3) \end{bmatrix} \quad (27)$$

where: $\delta_1, \delta_2, \delta_3$ — coefficients dedicated to the components of output quantities.

The vector function of output quantities is additionally perturbed by uncertainties associated with zero set errors $(\frac{\Delta_{10}}{1+\delta_1}; \frac{\Delta_{20}}{1+\delta_2}; \frac{\Delta_{30}}{1+\delta_3})$ and results the output components now are:

$$y_1 = (1 + \delta_1) \left(\frac{k_1 x_1 - k_2 x_2 + k_3 x_3}{2} + \frac{\Delta_{10}}{1 + \delta_1} \right)$$

$$\sigma_{y_1} = (1 + \delta_1) \sqrt{k_1^2 \sigma_{x_1}^2 + k_2^2 \sigma_{x_2}^2 + k_3^2 \sigma_{x_3}^2 + 2(k_1 k_2 \rho_{x_1 x_2} \sigma_{x_1} \sigma_{x_2} - k_1 k_2 \rho_{x_1 x_2} \sigma_{x_1} \sigma_{x_2} - k_2 k_3 \rho_{x_2 x_3} \sigma_{x_2} \sigma_{x_3}) + \sigma^2 \left(\frac{\Delta_{10}}{1 + \delta_1} \right)}$$

$$\sigma_{y_2} = (1 + \delta_2) \sqrt{k_1^2 \sigma_{x_1}^2 + k_2^2 \sigma_{x_2}^2 + k_3^2 \sigma_{x_3}^2 + 2(k_1 k_2 \rho_{x_1 x_2} \sigma_{x_1} \sigma_{x_2} - k_1 k_2 \rho_{x_1 x_2} \sigma_{x_1} \sigma_{x_2} - k_2 k_3 \rho_{x_2 x_3} \sigma_{x_2} \sigma_{x_3}) + \sigma^2 \left(\frac{\Delta_{20}}{1 + \delta_2} \right)}$$

$$\sigma_{y_3} = (1 + \delta_3) \sqrt{k_1^2 \sigma_{x_1}^2 + k_2^2 \sigma_{x_2}^2 + k_3^2 \sigma_{x_3}^2 + 2(k_2 k_3 \rho_{x_2 x_3} \sigma_{x_2} \sigma_{x_3} - k_1 k_2 \rho_{x_1 x_2} \sigma_{x_1} \sigma_{x_2} - k_1 k_3 \rho_{x_1 x_3} \sigma_{x_1} \sigma_{x_3}) + \sigma^2 \left(\frac{\Delta_{30}}{1 + \delta_3} \right)} \quad (29a, b, c)$$

1.3. Power indirect measurements

Now the considerations of the case of directly measurements, where two of values of currents I_1 and I_2 ($I_2 > I_1$) and non-correlated values of its uncertainties σ_{I1}, σ_{I2} conducted from measurement are done. Assuming the linear dependence of estimated current $I(k)$ as a function of both measured currents in the form:

$$I(k) = kI_1 + (1 - k)I_2, \quad (30)$$

where: $0 \leq k \leq 1$.

The outside range of above defined values of k leads on the linear dependence too, however the limits of the interval of k for extrapolation of values of current in the range $I_2 \leq I(k) \leq I_1$ must be explicit evaluated e.g. $\frac{I_1 - I_2}{I_1 - I_2} \leq k \leq \frac{I_2 - I_2}{I_1 - I_2}$.

Active power $P(k)$ in the case of flow of current $I(k)$ through resistance R is as follow as:

$$P(k) = (kI_1 + (1 - k)I_2)^2 R = [k^2 I_1^2 + (1 - k)^2 I_2^2 + 2k(1 - k)I_2 I_1] R \quad (31)$$

The goal of this part is estimation of power uncertainties of σ_{P1}, σ_{P2} both of current $I(k_1), I(k_2)$ and its correlation coefficient.

The covariance matrix of non-correlated input quantities e.g. measured currents I_1 and I_2 has following form:

$$y_2 = (1 + \delta_2) \left(\frac{k_1 x_1 + k_2 x_2 - k_3 x_3}{2} + \frac{\Delta_{20}}{1 + \delta_2} \right) \quad (28a, b, c)$$

$$y_3 = (1 + \delta_3) \left(\frac{-k_1 x_1 + k_2 x_2 + k_3 x_3}{2} + \frac{\Delta_{30}}{1 + \delta_3} \right)$$

Using formulas which are derived for absolute uncertainties of star circuit and modifying it, the absolute uncertainties after modification with zero set errors are more complicated as:

$$U_X = \begin{bmatrix} \sigma_{I1}^2 & 0 \\ 0 & \sigma_{I2}^2 \end{bmatrix} \quad (32)$$

The matrix of linearized functional of both active powers $P(k_1)$ i $P(k_2)$ emitted on the resistor R as the output quantities is the following:

$$S = \begin{bmatrix} 2k_1[I_1 k_1 + (1 - k_1)I_2] & 2(1 - k_1)[I_2(1 - k_1) + k_2 I_1] \\ 2k_2[I_1 k_2 + (1 - k_2)I_2] & 2(1 - k_2)[I_2(1 - k_2) + k_2 I_1] \end{bmatrix} \quad (33a, b)$$

$$S^T = \begin{bmatrix} 2k_1[I_1 k_1 + (1 - k_1)I_2] & 2k_2[I_1 k_2 + (1 - k_2)I_2] \\ 2(1 - k_1)[I_2(1 - k_1) + k_2 I_1] & 2(1 - k_2)[I_2(1 - k_2) + k_2 I_1] \end{bmatrix}$$

The covariance matrix of output quantities is described by formulae:

$$U_Y \equiv \begin{bmatrix} \sigma_{P1}^2 & \rho_P \sigma_{P1} \sigma_{P2} \\ \rho_P \sigma_{P1} \sigma_{P2} & \sigma_{P2}^2 \end{bmatrix} = S U_X S^T \quad (34)$$

After simply transformations the variance of both uncertainties of active powers is given:

$$\sigma_{P1}^2 = 4[I_1 k_1 + (1 - k_1)I_2]^2 (k_1^2 \sigma_{I1}^2 + (1 - k_1)^2 \sigma_{I2}^2)$$

$$\sigma_{P2}^2 = 4[I_1 k_2 + (1 - k_2)I_2]^2 (k_2^2 \sigma_{I1}^2 + (1 - k_2)^2 \sigma_{I2}^2) \quad (35)$$

and its correlation coefficient:

$$\rho_P = \frac{k_1 k_2 \sigma_{I1}^2 + (1 - k_1)(1 - k_2) \sigma_{I2}^2}{\sqrt{k_1^2 \sigma_{I1}^2 + (1 - k_1)^2 \sigma_{I2}^2} \sqrt{k_2^2 \sigma_{I1}^2 + (1 - k_2)^2 \sigma_{I2}^2}} \quad (36)$$

Above formulae is identical as correlation coefficient for intensity currents of $I(k_1)$ and $I(k_2)$.

Conclusion

Few examples of determining the uncertainties in case of multi-parameter linear and nonlinear formulas

have been presented, for example indirect measurements of resistance by Wheatstone bridge, a star circuit and module of difference of magnetic field induction vector. Covariance matrix of relative uncertainties is also applied. We proposed of using the covariance matrix for relative uncertainties and corresponding measured function dedicated for relative uncertainties as well as relative errors were applied in the classical approach.

It is shown that in the case when two or more parameters /for example element of electronic circuits/ are measured together, the uncertainties of above parameters are correlated. So, if the above correlated elements will be used without disconnection in the next circuits of device then in the determination of uncertainties of such new device we should consider the corresponding correlations coefficients obtained from first measurements.

Information about uncertainty calculations of multivariable AC measurements is in [3]. Supplement 2 of GUM [1] does not cover situations existing in instrumental systems, when realization of functional $F(\mathbf{x})$ is not accurate. Such inaccuracy can be due to approximation of transfer functions and limited their frequency ranges, using in signal processing an A/D converters, analogue multipliers, and other functional

elements, necessary in indirect measurements. Therefore $F(\mathbf{x})$ is also saddled with own uncertainties u_F .¹ Even in the most precise measurements the rounding of results also becomes essential, including one resulting from the precision of digital circuits [2–3].

In the last days authors have developed the vector method for the description of the accuracy of multivariate measurements systems with considering uncertainties u_F , $u_{\delta F}$ of the functional F parameters. This method is wider then recommendations given in GUM Supplement 2 [1], which do not consider inaccuracy of F . Details and new formulas will be provided in the next authors' work.

References

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- [4] L. Finkelstein, "Fundamental concepts of measurement", *ACTA IMEKO*, vol. 3, no. 1, p. 10–15 May 2014.

¹Such problems of measurement technology are included in the *measurement science*, discipline wider then metrology. The concept and term of this discipline were proposed by prof. L. Finkelstein from the City University of London in 1970-s years during his IMEKO activity [4].