Analysis of Steady-States of Continuous-Time Impulse K-Winners-Take-All Neural Network

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INTRODUCTION

K-winners-take-all (KWTA) neural networks (NNs) define the *K* maximal out of *N* inputs, where $1 \leq K < N$ is a positive integer [1] – [3]. In the particular case when $K = 1$, the KWTA network becomes a winner-takes-all (WTA) network [4], [5]. WTA and KWTA neural networks are related with simultaneous recurrent NNs which require fulfilling iterations to converge to steady-states [6].

KWTA NNs are used in different applications, especially, for pattern recognition, in processing data and signals, in making decisions, in competitive learning, and in parallel sorting and filtering. The KWTA networks are used in telecommunications and vision systems, for image processing, classifying, clustering, decoding, mobile robot navigation, and feature extraction. KWTA tools are applied for networking cognitive phenomena and impulse neural networks [3].

Various NNs have been proposed for obtaining solutions the WTA and KWTA problems. For instance, a design, implementation and verification of WTA network in a number of CMOS integrated circuits is presented in [7]. Continuous-time KWTA NNs realized in analog hardware are faster, have smaller size and larger power-efficiency comparatively to digital counterparts [8], [9].

Neurons can be modelled based on impulse networks. An impulse can be described by Dirac delta function or by exponential decay. Trains of impulses and exponential decays are employed to model the potential of a neuron membrane. The sum of exponential decays between spikes and delta functions are used to describe a level of activity that averages in time the influences of postsynaptic spikes on the neuron. Impulse neurons are applied to model psychophysical data with a laminar cortical network. Impulse-based computation, caused by networks of computation in the central nervous system, are capable to provide essential performance advantages comparatively to traditional methods for specific kinds of large-scale problems [10].

In this paper, a train of impulses defined by the sum of Dirac delta functions is applied in continuous-time KWTA NN. For this reason, in contrast to other competitors for which a trajectory of the state variable of the network to the KWTA operation has continuous nonlinear, continuous piecewise-linear or continuous linear shape, a state variable trajectory of the network to the KWTA operation has stepping form. Therefore, the theoretical speed of convergence of state variable trajectories of the network to the KWTA operation goes to infinity if the period of firing impulse approximates zero. This provides a capability of the NN to choose instantaneously without transient modes, the *K* maximal out of N inputs. This is the principal advantage of the network. There is analyzed an existence and uniqueness of the network steady-states. Computer simulations illustrating and confirming theoretical results are given.

PROBLEM FORMULATION

Let us consider a vector of inputs $\boldsymbol{a} = \left(a_{n_1}, a_{n_2}, \dots a_{n_N}\right)^T \in \Re^n$, $1 < N < \infty$ elements of which are unknown but have finite values. The inputs have different values and can be disposed in a descending order of quantity meeting the inequalities

$$\infty > a_{n_1} > a_{n_2} > \dots > a_{n_N} > -\infty , \tag{1}$$

where $n_1, n_2, \dots n_N$ are the unknown numbers of the first maximal input, the second maximal input and so on up to the *N*th maximal input. It is required to investigate existence and uniqueness of steady-states of the network that can define instantaneously without transient modes, the *K* maximal of these inputs, which are referred to as the winners. The designed network should process the vector of inputs $\boldsymbol{a}$ to obtain a vector of outputs $\boldsymbol{b} = \left(b_{n_1}, b_{n_2}, \dots b_{n_N}\right)^T$ such that the following KWTA property holds [1], [3]:

$$b_{n_i} > 0, i = 1,2,\dots,K;\ b_{n_j} < 0, j = K+1, K+2,\dots,N. \tag{2}$$

It should be also possible to determine the KWTA operation as follows [11]:

$$d_{n_i} = 1, i = 1,2,...,K;\ d_{n_j} = 0, j = K+1, K+2,...,N. \tag{3}$$

Note that in the case of using outputs (3) of the network only *K* winners out of *N* inputs will be determined. No information is obtained on the ordering of inputs by value which can be employed further, for example, for obtaining solutions of such problems as clustering, classification, etc. [3].

IMPULSE CONTINUOUS-TIME KWTA NEURAL NETWORK

Consider the continuous-time KWTA NN described in [3] that is governed by the following state equation:

$$\frac{dx}{dt} = rD(x)\sum_{l=0}^{m}\delta(t-t_l) \qquad (4)$$

and by output equation

$$b_{n_k} = a_{n_k} - x, k = 1,2,\ldots,N, \qquad (5)$$

where

$$D(x) = \sum_{k=1}^{N} S_k(x) - K \qquad (6)$$

is a function of difference between obtained and required quantity of positive outputs,

$$S_k(x) = \begin{cases} 1, & if \quad a_{n_k} - x > 0; \\ 0, & otherwise \end{cases} \qquad (7)$$

is a step activation function,

$$\delta(t-t_l) = \begin{cases} +\infty, & if \quad t = t_l; \\ 0, & if \quad t \neq t_l \end{cases} \qquad (8)$$

is an impulse shaped as the Dirac delta function, $\sum_{l=1}^{m}\delta(t-t_l)$ is a train of impulses, t_l is a lth time instant of firing impulses, m is a number of impulses necessary to reach a convergence of search process to the KWTA operation, and r is a resolution of the network.

For the continuous-time KWTA NN described by state equation (4) and by output equation (5), the KWTA operation (3) can be defined as follows:

$$d_k(x) = S_k(x), \qquad (9)$$

where $k=1,2,\ldots,N$.

In contrast to other competitors with continuous nonlinear trajectory of the state variable x, the network described by state equation (4) and by output equation (5) presents stepping state trajectory of x. Therefore, it can reach theoretically any finite speed of processing inputs defined by the period of firing impulses. If this period tends to zero, a processing time of inputs of the network also goes to zero. This is main advantage of the network. A practical speed of processing input vector of the network is limited by limitations of its software or hardware realization. In particular, a software realization of the network has limited computational accuracy. On the other hand, a hardware realization of the network, the restrictions are finite speed of comparator, non-idealities of integrator, mismatch, etc.

EXISTENCE AND UNIQUENESS OF STEADY-STATES OF THE NETWORK

We analyze the existence and uniqueness of the KWTA steady-states of the state equation (4) solutions. Integrating both sides of this equation gives

$$x(t)=\int_{t_0}^{t} rD(x)\sum_{l=0}^{m}\delta(\lambda-t_l)d\lambda = x_0 + r\sum_{l=0}^{m} D(x_l)H(t-t_l), \qquad (10)$$

where

$$H(t-t_l)=\begin{cases} 1, & if \quad t-t_l>0; \\ [0,1], & if \quad t-t_l=0; \\ 0, & if \quad t-t_l<0 \end{cases}$$

is a Heaviside step activation function. As the terms x_0, r, $D(x_l)$ and $H(t-t_l)$ of the solution (10) are finite, $x(t)$ is finite as well. In the KWTA steady-states the solution (10) should meet the inequalities

$$a_{K+1} \le x_0 + r\sum_{l=0}^{m} D(x_l)H(t-t_l) < a_K . \qquad (11)$$

Let us rewrite the inequalities (11) in the following shape:

$$a_{K+1} \le x_0 + rI < a_K , \qquad (12)$$

where $I=\sum_{l=0}^{m} D(x_l)H(t-t_l)$ is an integer number. There exists an integer I such that inequalities (12) hold under the bounds $0<r\le min\left|a_{n_i}-a_{n_j}\right|$, $i\ne j=1,2,\ldots,N$ for arbitrary distinct inputs (1) and for any finite initial condition x_0. This means that a constant number $x^*\in\Re$ exists, such that $a_{K+1}\le x^* < a_K$. For x^*, equality $D(x^*)=\sum_{k=1}^{N} S_k(x^*)-K=0$ means that x^* is a steady-state solution of the equation (4). Taking into account (2) and (5), it follows that steady-state KWTA outputs $b_{n_k}=a_{n_k}-x^*$,

$k=1,2,3,\ldots,N$ of the network described by state equation (4) and by output equation (5) exist. This implies that x^* is a KWTA steady-state solution of the state equation (4).

It is necessary to note that the steady-state solution of the state equation (4) can accept arbitrary finite value in the region $a_{K+1} \le x^* < a_K$ meeting the equality $D(x^*) = \sum_{k=1}^{N} S_k(x^*) - K = 0$. The solution x^* is not unique as this equality is satisfied for each $x^* \in [a_{K+1}, a_K]$. It follows from this fact that the outputs $b_{n_k} = a_{n_k} - x^*$, $k=1,2,\ldots,N$ are also not unique. However, the KWTA property (2) of the network described by state equation (4) band output equation (5) is defined by signs of outputs b_{n_K} rather than by their values. These signs are unique for any $a_{K+1} \le x^* < a_K$. Therefore, the network described by state equation (4) and by output equation (5) has a unique KWTA property (2).

COMPUTER SIMULATIONS

Let us consider example with corresponding computer simulations which illustrate the performance of the continuous-time impulse KWTA neural network described by state equation (4) and by output equation (5). We set the input vector $\boldsymbol{a}$= $(1.4,3.1,2.3,9.2,10,7.6,\ 5.7,4.8,6.9,8.5)^T$, i. e. $N=10$, $K=7$, $r = 0.5 < \min\left|a_{n_i} - a_{n_j}\right|$, $i \neq j = 1,2,\ldots,10$ and the initial value of the state variable $x_0 = 0$. A 1.81 GHz desktop PC and Euler solver of non-stiff ordinary differential equations in the Simulink environment of Matlab program (ODE1) of the fixed-step size 1×10^{-11} are employed. Impulse source is realized by sequential connection of pulse generator, differentiator and absolute value blocks.

Fig. 1 shows, in normalized units, the train of $m=4$ impulses fired with the period τ=0.15 *ms*. In Fig. 2, the dynamics of elements of output vector $\boldsymbol{b}$ of the KWTA NN with impulse train for $\tau = 0.5\ ns$ are presented. As one can see from this figure, the trajectories of the network outputs converge to unique KWTA steady-states starting from their initial values by using four impulses after 2 ns.

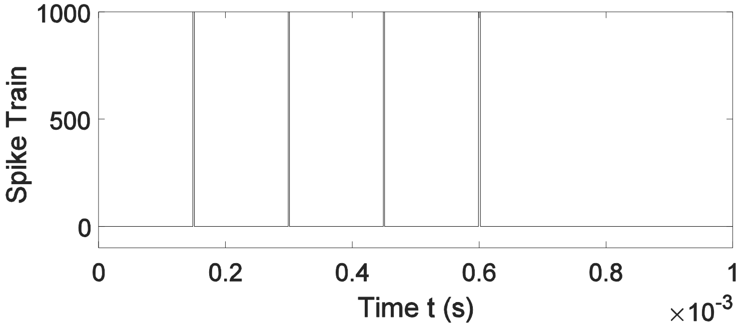

Fig. 1. Train of $m=4$ impulses (8) fired with the period τ=0.15 *ms* of the KWTA NN described by state equation (4) and by output equation (5).

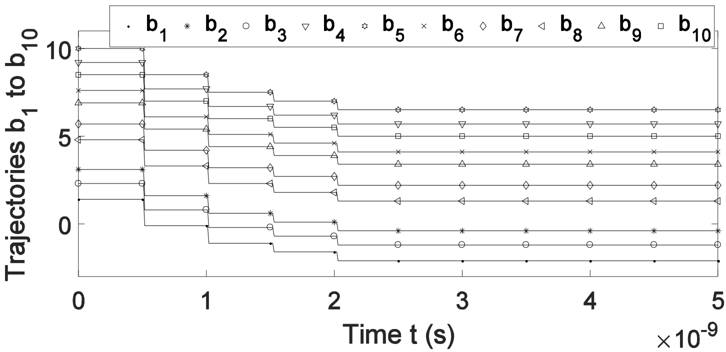

Fig. 2. Dynamics of elements of output vector ***b*** of the network for $\tau = 0.5\ ns$.

CONCLUSION

This paper describes a continuous-time KWTA NN that contains impulse train. The network is capable of defining K maximal among arbitrary finite value N unknown different inputs positioned in an unknown region, where $1 \le K < N$. There is analyzed existence and uniqueness of steady-states of the network. In contrast to other comparable competitors, state variable trajectories of the network have piecewise-constant, i. e. stepping shape. Therefore, steady-state KWTA operation of the network can be reached theoretically instantaneously as the period of firing impulses approximates zero. Thus, the network can define the K maximal among N inputs without transient dynamics, i.e. instantaneously.

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