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MODELING OF THE EARTH'S GRAVITATIONAL FIELD USING SPHERICAL FUNCTIONS

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Aim. There are many methods for modeling a regional gravitational field in which the Legendre spherical functions of integer degree of the real order are used. They relate, however, mainly to the region which form represents a segment of the sphere. In addition, for their use, the input data must be transformed into a sphere segment with its center at the north pole. The aim of this work is to find a system of functions that would have orthogonal properties on an arbitrary spherical trapezium, as well as researching the properties of such a system. **Method.** Based on the Legendre spherical functions on the spherical segment, an orthogonal system of functions to an arbitrary spherical trapezoid was developed. Such functions can not be explicitly stated, nor do they have recurring relationships. **Results.** This article examines the associated Legandre spherical functions on the spherical trapezium where the functions are orthogonal and provide the formulas for defining the norms of these functions. The obtained functions can be used to build regional models of the gravitational fields on the arbitrary spherical trapezium. The orthogonality of the functions ensures a sustainable solution when determining the unknown model coefficients. To model the regional gravitational field with high accuracy, it is necessary to grid the input data (define the transformants of the geopotential), and then use the partial discrete orthogonality of these functions in longitudial direction or full discrete orthogonality similar to the second Neumann's method. This allows significant reduction of computing time without any loss of accuracy, as the functions under study do not have any recursive relations and it is required to use the decomposition into the hypergeometric series to define these functions. ***The scientific novelty and practical significance.*** In this paper we first obtained a system of functions that were orthogonally consistent to an arbitrary spherical trapezium. It can be used to construct a regional gravitational field, a regional magnetic field, and also for high-precision interpolation or approximation tasks, for example the construction of a regional ionosphere model.

Key words: spherical functions, spherical trapezium, orthogonality.

Introduction

It is common to use the associated Legendre spherical harmonics for building the gravitational field of the Earth, as the harmonics are orthogonal overall the sphere and obey the Laplace's equation similar to the potential function [Hobson 1931]. In addition, the associated Legendre spherical harmonics can be decomposed into the finite hypergeometric series that allow using simple recursive relations for their calculation. However, if the measurements are conducted not on the whole surface of the Earth but rather on a certain part of it such as in a regional model, then it is impractical to use the associated Legendre spherical harmonics which lose their orthogonality and the solution becomes unstable.

To solve this problem, in 1985, it was suggested to use the spherical cap harmonic analysis that presupposes transformation of the input data into the spherical cap and usage of the integer-order Legendre spherical harmonics as the basic function system but of the real degree [Haines 1985]. Such functions obey the Laplace's equation and form two orthogonal systems of functions on the spherical cap. However, in general these functions are not orthogonal. They do not have recurrent relations and they can be calculated by their decomposition into the infinite hypergeometric series [Haines 1988]. In addition, it was suggested to use the adjusted spherical harmonic analysis [De Santis 1992] that presupposes the transformation of the coordinate system from the spherical cap into the half sphere. In this case, it was proposed to use the adjusted spherical harmonic analysis that presupposes the transformation of the coordinate system from the spherical cap into the hemisphere. In this case, the eigenvalues of spherical functions become integers. However, in spite of this, such functions are not orthogonal but they form two orthogonal systems of functions similar to the functions on the spherical cap. Additionally, such methods as translated origin spherical harmonic analysis [De Santis 1991], revised spherical cap harmonic analysis [Thebault *et al.* 2006], and other methods were suggested.

Aim

Computation of high-accuracy gravitational fields require models of high order. For example, the global models EGM 2008 and EIGEN-6C4 are built to 2190 order [Pavlis *et al.* 2012]. Despite the fact that the models of the regional fields of the same accuracy will have a much lower maximum order, it is recommended to use the orthogonal system of functions as the basic system to obtain a stable solution [De Santis & Torta 1997]. This article aims at finding and investigating such systems of functions on the spherical trapezium.

Method

Generally for modeling the gravitational fields in the spherical coordinate system (r, θ, λ), functions each of which depends only on one coordinate

$$V = f(r) \cdot g(q) \cdot h(l) \quad (1)$$

and which obey the Laplace's equation are used. Laplace's equation in the latitudial direction is represented as a differential equation of the second order.

$$\sin q \cdot g''(q) + \cos q \cdot g'(q) + \left[n(n+1)\sin q - \frac{m^2}{\sin q} \right] \cdot g(q) = 0 . \quad (2)$$

The Legendre spherical functions of the first and second kind provide a solution for it. Since the Legendre spherical functions of the second kind diverge at the poles, the Legendre spherical functions of the first kind are used and can be represented with the hypergeometrical series as follows [Hobson 1931; Hwang & Chen 1997]:

$$P_{nm}(\cos q) = \sin^m q \cdot F\left(m-n, n+m+1, 1+m, \frac{1-\cos q}{2} \right), \quad (3)$$

where n and m depend on the imposed boundary conditions and $g(q) \equiv P_{nm}(\cos q)$.

If to find such values of n on some segment of the sphere $q \le q_0$, for which the following equation will be true:

$$\frac{dP_{nm}(\cos q_0)}{d(\cos q)} = 0 , \quad (4)$$

as well as

$$P_{nm}(\cos q_0) = 0, \quad (5)$$

then the corresponding functions will form two orthogonal systems of functions on the segment of the sphere under study [Haines 1985; Haines 1988].

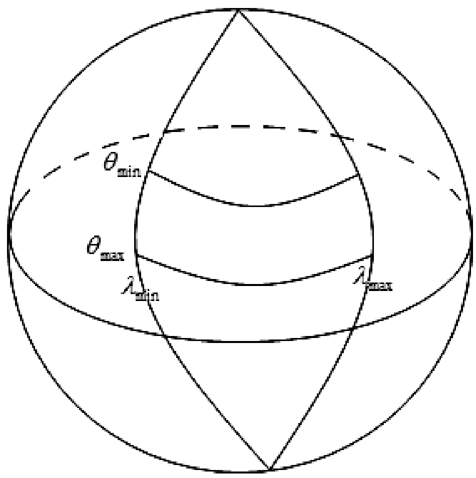

Fig. 1. Spherical trapezium

The following functions are examined on the spherical trapezium (fig.1) limited by the coordinates $q_{\min}, q_{\max}, l_{\min}, l_{\max}$

$$\left. \begin{array}{l} P_{km} = \sin^m(q - q_{\min}) \cdot F\left(m-n_k, n_k+m+1, 1+m, \frac{1-\cos(q-q_{\min})}{2} \right), \ if\ q_{\min} \le q \le q_{mean} \\ P_{km} = (-1)^{k+m} \sin^m(q_{\max} - q) \cdot F\left(m-n_k, n_k+m+1, 1+m, \frac{1-\cos(q_{\max}-q)}{2} \right), \ if\ q_{mean} \le q \le q_{\max} \end{array} \right\}, \quad (6)$$

where k and m are integer values, and q_{mean} is the average value, namely $q_{mean} = (q_{\min} + q_{\max})/2$. In its turn, the values of n_k will depend on k and m. They can be calculated [Haines 1985; Hwang & Chen 1997] using the following equation if k–m is an odd value:

$$\tilde{F}(n_k, m, \cos q_0) = 0 , \quad (7)$$

or using the equation:

$$n_k \cos q_{mean} \tilde{F}(n_k, m, \cos q_0) - (n_k - m)\tilde{F}(n_k - 1, m, \cos q_0) = 0 , \quad (8)$$

if k–m is an even value, where $q_0 = (q_{\max} - q_{\min})/2$ and $\tilde{F}$ represents conditional hypergeometric series [Hwang & Chen 1997]:

$$\tilde{F}(n, m, m) = F(m-n, m+n+1, m+1, \frac{1-m}{2}). \quad (9)$$

From the equations (7) and (8), it becomes obvious that the functions (6) are continuous as they have the same value in the point $q = q_{mean}$. When *k–m* is even, then $(-1)^{k+m} = 1$, and when *k–m* is odd, then $P_{km}(q_{mean}) = 0$.

Results

Functions (6) form two orthogonal systems of functions with respect to the weight function $\sqrt{\sin(q - q_{\min})}$ over the segment $[q_{\min}; q_{mean}]$ [Haines 1985; Smythe 1950]:

$$\int_{q_{\min}}^{q_{mean}} P_{km}(q) P_{sm}(q) \sin(q - q_{\min}) dq = 0, \qquad (10)$$

where $k \neq s$ and *k-m* and s-*m* are both either even or odd. A similar equation can be used for $[q_{mean}; q_{\max}]$

$$\int_{q_{mean}}^{q_{\max}} P_{km}(q) P_{sm}(q) \sin(q_{\max} - q) dq = 0. \qquad (11)$$

According to the functions (6), if *k-m* is even, then the functions P_{km} will be even on the segment $[q_{\min}; q_{\max}]$ relatively to q_{mean} and if s-*m* is odd, then the functions P_{km} will be odd on this segment relatively to q_{mean} . Integral of the product of an even function and an odd one is equal to zero. In conclusion, functions (6) are orthogonal on the segment $[q_{\min}; q_{\max}]$:

$$\int_{q_{\min}}^{q_{\max}} P_{km}(q) P_{sm}(q) \sin(q_0 - |q - q_{mean}|) dq = 0. \qquad (12)$$

Let us build the graph of the functions (6), for example, on the segment [2\0º; \7\0º]. For this, the value of n_k when $q_0 = 25°$ was calculated using the formulas (7)–(9). These values are provided in the Table 1.

Table 1

Values of $n_k(m)$ when $q_0 = 25°$

k/m	0	1	2	3
0	0.000			
1	5.004	3.806		
2	8.296	8.296	6.632	
3	12.148	11.743	11.324	9.318

Fig. 2 shows the graph of the functions (6) when m=0 and $k = \overline{0,3}$.

Let us examine the following functions on the segment $[l_1; l_2]$:

$$\left.\begin{aligned} h_m^c &= \cos\left(2p\, m \frac{l - l_1}{l_2 - l_1}\right) \\ h_m^s &= \sin\left(2p\, m \frac{l - l_1}{l_2 - l_1}\right) \end{aligned}\right\}, \qquad (13)$$

where *m* is integer value. It is easy to illustrate that such functions build an orthogonal system of functions on the segment $[l_1; l_2]$:

$$\left.\begin{aligned} &\int_{l_1}^{l_2} \sin\left(2p\, m \frac{l - l_1}{l_2 - l_1}\right) \cdot \sin\left(2p\, l \frac{l - l_1}{l_2 - l_1}\right) d\,l = 0 \\ &\int_{l_1}^{l_2} \sin\left(2p\, m \frac{l - l_1}{l_2 - l_1}\right) \cdot \cos\left(2p\, l \frac{l - l_1}{l_2 - l_1}\right) d\,l = 0 \\ &\int_{l_1}^{l_2} \cos\left(2p\, m \frac{l - l_1}{l_2 - l_1}\right) \cdot \cos\left(2p\, l \frac{l - l_1}{l_2 - l_1}\right) d\,l = 0 \end{aligned}\right\} \qquad (14)$$

where m $\neq$ 1.

Fig. 3 shows a graph of functions (13) when $m = \overline{0,3}$ on the segment [3\0º; \5\0º].

To summarize the above-mentioned, the following functions are obtained on the spherical trapezium that is limited by the coordinates $q_{\min}, q_{\max}, l_{\min}, l_{\max}$:

$$\begin{aligned} R_{km}(q, l) &= P_{km}(\cos q) \cos\left(2p\, m \frac{l - l_1}{l_2 - l_1}\right), \\ S_{km}(q, l) &= P_{km}(\cos q) \sin\left(2p\, m \frac{l - l_1}{l_2 - l_1}\right). \end{aligned} \qquad (15)$$

Equations (12) and (14) show that functions (15) are orthogonal on the spherical trapezium, namely:

$$\begin{aligned} &\left.\begin{aligned} &\iint_s R_{nm}(q, l) R_{sr}(q, l) ds = 0 \\ &\iint_s S_{nm}(q, l) S_{sr}(q, l) ds = 0 \end{aligned}\right\} \; if \;\; s \neq n \;\; or \;\; r \neq m; \\ &\iint_s R_{nm}(q, l) S_{sr}(q, l) ds = 0 \quad in \;\; any \;\; case, \end{aligned} \qquad (16)$$

while an element of a sphere and integration is carried out over the spherical trapezium.

Fig. 4 shows a graph of a function $R_{63}(q, l)$.

Functions (6) change their sign for *k-m* times in the interval $q_{\min} \leq q \leq q_{\max}$ whereas functions (13) have $2m$ zeroes in the interval $l_{\min} \leq l \leq l_{\max}$. Therefore, functions (15) divide the spherical trapezium into parts where they are alternately positive and negative similar to a chessboard. Fig. 5 shows the geometrical representation of the function $R_{63}(q, l)$.

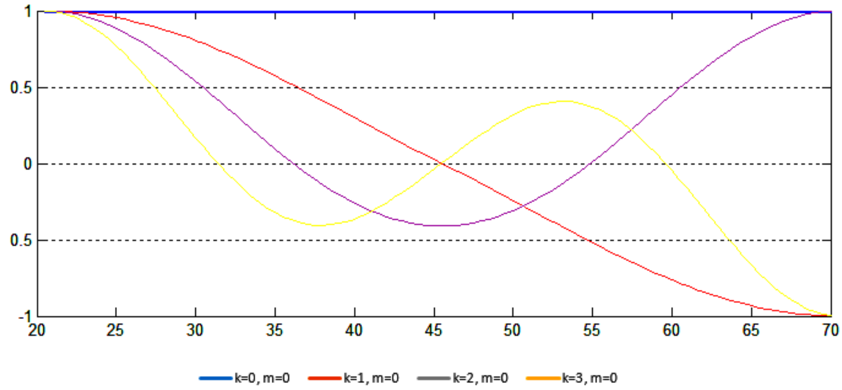

Fig. 2. Legendre spherical functions (6) on the segment [2\0º; \7\0º]

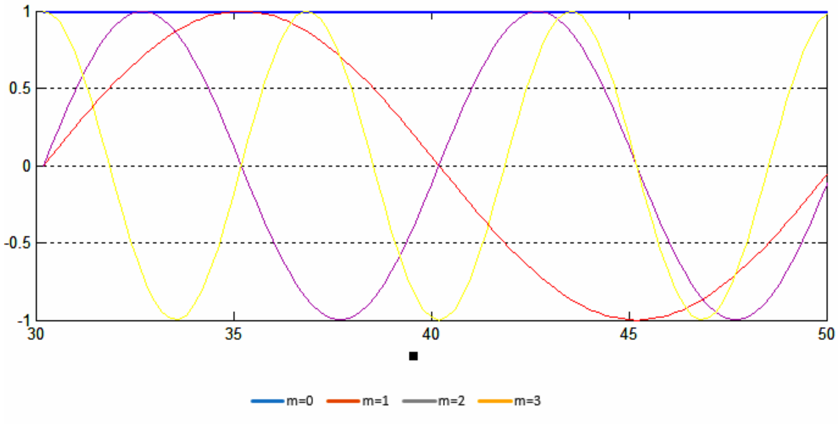

Fig. 3. Functions that are orthogonal on the segment [3\0º; \5\0º]

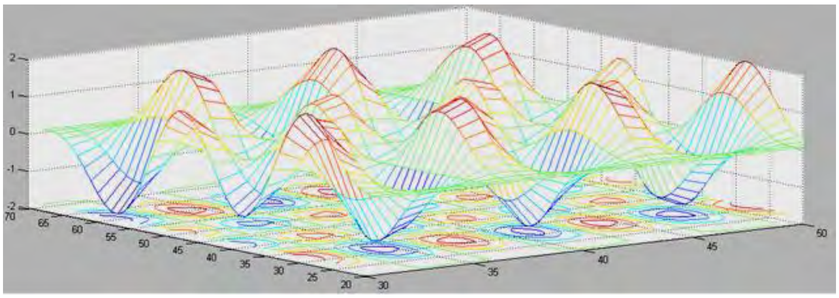

Fig. 4. Graph of a function $R_{63}(q,l)\cdot 10^3$

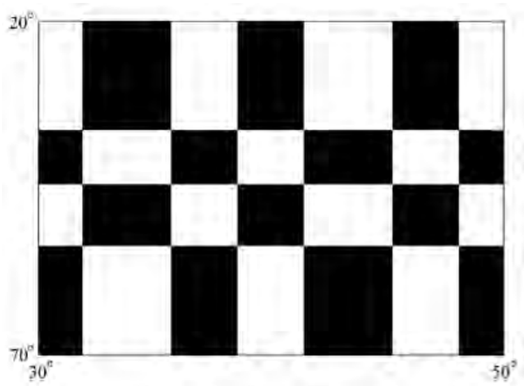

Fig. 5. Zeroes of the function $R_{63}(q,l)$

Norm of the functions (6) can be calculated using the following formula [Haines 1985; Hwang 1993]:

$$\left.\begin{aligned} N_{km}^2 &= 2\int_0^{q_0} P_{n_k(m)m}^2(\cos q)\sin q\,dq = 2\frac{(\cos^2 q_0 - 1)}{2n_k+1}\frac{\partial}{\partial n}\left[P_{n_k(m)m}(\cos q_0)\right]\frac{dP_{n_k(m)m}(\cos q_0)}{d(\cos q)}, \ \textit{if} \ \ k-m=odd \\ N_{km}^2 &= 2\int_0^{q_0} P_{n_k(m)m}^2(\cos q)\sin q\,dq = -2\frac{(1-\cos^2 q_0)}{2n_k+1}P_{n_k(m)m}(\cos q_0)\frac{\partial}{\partial n}\left[\frac{dP_{n_k(m)m}(\cos q_0)}{d(\cos q_0)}\right], \ \textit{if} \ \ k-m=even \end{aligned}\right\}$$

$$\left.\begin{aligned} N_{km}^2 &= 2\int_0^{q_0} P_{n_k(m)m}^2(\cos q)\sin q\,dq = 2\frac{(\cos^2 q_0 - 1)}{2n_k+1}\frac{\partial}{\partial n}\left[P_{n_k(m)m}(\cos q_0)\right]\frac{dP_{n_k(m)m}(\cos q_0)}{d(\cos q)}, \ \textit{if} \ \ k-m=odd \\ N_{km}^2 &= 2\int_0^{q_0} P_{n_k(m)m}^2(\cos q)\sin q\,dq = -2\frac{(1-\cos^2 q_0)}{2n_k+1}P_{n_k(m)m}(\cos q_0)\frac{\partial}{\partial n}\left[\frac{dP_{n_k(m)m}(\cos q_0)}{d(\cos q_0)}\right], \ \textit{if} \ \ k-m=even \end{aligned}\right\} \quad (17)$$

Expressions for calculating $\frac{\partial}{\partial n}\left[P_{n_k(m)m}(\cos q_0)\right]$ and $\frac{\partial}{\partial n}\left[\frac{dP_{n_k(m)m}(\cos q_0)}{d(\cos q)}\right]$ are presented in [Hwang & Chen 1997; Macdonald 1900].

On the other hand, it is easy to represent the norm of functions (13) as follows:

$$Q^2 = \int_{l_1}^{l_2}\cos^2\left(2p\,m\frac{l-l_1}{l_2-l_1}\right)d\,l =.$$

$$= \int_{l_1}^{l_2}\sin^2\left(2p\,m\frac{l-l_1}{l_2-l_1}\right)d\,l = \frac{l_2-l_1}{2}. \qquad (18)$$

The expression for calculating the normalized functions (15) is the following:

$$\begin{aligned} \bar{R}_{km}(q,l) &= \frac{R_{km}(q,l)}{N_{km}Q}, \\ \bar{S}_{km}(q,l) &= \frac{S_{km}(q,l)}{N_{km}Q}. \end{aligned} \qquad (19)$$

Functions $\bar{R}_{km}(q,l)$ and $\bar{S}_{km}(q,l)$ form an orthonormalized system of functions on the spherical trapezium. Practically any function *V* defined on the spherical trapezium can be decomposed into series using the functions (19).

The function

$$V = \sum_{k=0}^{k_{\max}}\sum_{m=0}^{k}\bar{a}_{km}\bar{R}_{km}(q,l) + \bar{b}_{km}\bar{S}_{km}(q,l), \text{ is used} \qquad (20)$$

where $\bar{a}_{km}$ and $\bar{b}_{km}$ are unknown coefficients.

The scientific novelty and practical significance

An orthogonal system of functions on an arbitrary spherical trapezium is proposed. It can be used to construct a model of the regional gravitational field having high resolution.

Conclusions

This article suggests using the orthogonal functions on the spherical trapezium for modeling the regional gravitational field. The algorithm for this method is the following:

1. Calculate coordinates of vertices of the spherical trapezium $q_{\min}, q_{\max}, l_{\min}, l_{\max}$ where the input data is known.
2. Define the midpoint of the trapezium and find the ***n*** eigenvalues.
3. Calculate a norm of functions under study.
4. Determine the unknown coefficients of the model using the method of least squares.

It is worth mentioning that for building the models of high order, it is recommended to place the input data on a certain grid [Sneeuw 1994]. This not only reduces the time required for accurate calculations, but also uses the discrete orthogonality of the functions during calculation and rotation of a matrix of normal equations [Marchenko & Dzhuman 2015].

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МОДЕЛЮВАННЯ ГРАВІТАЦІЙНОГО ПОЛЯ ЗЕМЛІ З ВИКОРИСТАННЯМ СФЕРИЧНИХ ФУНКЦІЙ

Мета. Існує багато методів моделювання регіонального гравітаційного поля, в яких використовують сферичні функції Лежандра цілого ступеня, проте дійсного порядку. Проте вони стосуються переважно регіону, який за формою становить сегмент сфери. Крім того, для їх використання потрібно вхідні дані трансформувати на сегмент сфери з центром на північному полюсі. **Метою** цієї роботи є знаходження системи функцій, яка б мала ортогональні властивості на довільній сферичній трапеції, а також дослідження властивостей такої системи. **Методика.** Взявши за основу сферичні функції Лежандра на сферичному сегменті, розроблено систему функцій, ортогональну на довільній сферичній трапеції. Такі функції не можна задати в явному вигляді, а також вони не мають рекурентних співвідношень. **Результати.** Розглянуто приєднані сферичні функції Лежандра на сферичній трапеції, які мають властивість ортогональності у цьому регіоні. Наведено формули для знаходження норми таких функцій. Отримані функції можна використовувати для побудови регіональних моделей гравітаційних полів на довільній сферичній трапеції. Ортогональна властивість досліджуваних функцій забезпечує стійкий розв’язок під час знаходження невідомих коефіцієнтів моделі. Для високоточного моделювання регіонального гравітаційного поля необхідно перегрідувати вхідні дані (виміряні трансформанти геопотенціалу) на певний грід, і після цього можна використати часткову дискретну ортогональність даних функцій по довготі або повну дискретну ортогональність подібно до другого методу Неймана. Це дає змогу суттєво скоротити час обчислень без втрати точності, адже досліджувані функції не мають рекурсивних співвідношень і для їх знаходження необхідно використовувати розклад у гіпергеометричний ряд. **Наукова новизна і практична значущість.** У цій роботі вперше отримано систему функцій, ортогональну на довільній сферичній трапеції. Її можна використовувати для побудови регіонального гравітаційного поля, регіонального магнітного поля, а також для задач високоточної інтерполяції або апроксимації, наприклад побудови регіональної моделі іоносфери.

Ключові слова: сферичні функції, сферична трапеція, ортогональність.

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