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BOUNDARY-VALUE PROBLEM FOR SECOND-ORDER DIFFERENTIAL-OPERATOR EQUATION WITH INVOLUTION

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We study a nonlocal problem for differential-operator equations of order 2 with involution. The spectral properties of the operator of this problem are analyzed and the conditions for the existence and uniqueness of its solution are established. It is also proved that the system of eigenfunctions of the analyzed problem forms a Riesz basis.

Key words: differential equation, differential-operator equation, root function, operator of involution, essentially a nonself-adjoint operator, Riesz basis, nonlocal problem.

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Introduction

Suppose that H is a separable Hilbert space, A is a positive self-adjoint operator on H with point spectrum $\sigma_p(A) = \{z_k \in \mathbf{R} : z_k \sim \beta k^\alpha, \alpha, \beta > 0, k = 1, 2, ...\}$, i.e. $\lim\limits_{k\to\infty} \frac{\beta k^\alpha}{z_k} = 1$, $V(A) \equiv \{v_k(A) \in H : k = 1, 2, ...\}$ is a system of eigenfunctions that form an orthonormal basis in the space H, $H_1 \equiv L_2((0, 1), H) = \{u(t) : (0, 1) \to H, \|u(t); H\| \in L_2(0, 1)\}$, D_x is a strong derivative in the space H_1, i.e. $\left\|\frac{u(x+\Delta x)-u(x)}{\Delta x} - D_x u; H\right\| \to 0$, $\Delta x \to 0$, D is a differentiation operator in the space $L_2(0, 1)$, $I : L_2(0, 1) \to L_2(0, 1)$ is a operator of involution: $Iu(x) \equiv u(1-x)$, $u(x) \in L_2(0, 1)$, $p_0 \equiv \frac{1}{2}(E + I)$, $p_1 \equiv \frac{1}{2}(E - I)$, $W_2^2(0, 1) \equiv \{y \in L_2(0, 1) : Dy \in AC[0, 1], D^2 y \in L_2(0, 1)\}$, $(y, u; W_2^2(0, 1)) \equiv \sum\limits_{k=0}^{2}(D^k y, D^k u; L_2(0, 1))$, $\|y; W_2^2(0, 1)\|^2 \equiv (y, y; W_2^2(0, 1))$, $H(A^s) = \{v \in H : A^s v \in H\}$, $s \geq 0$, $(u, v; H(A^s)) \equiv (A^s u, A^s v; H)$, $\|y; H(A^s)\|^2 \equiv \|A^s y; H\|^2$, $L_{2,j}(0, 1) \equiv \{y \in L_2(0, 1) : y = p_j y\}$, $E_H f \equiv f$, $f \in H$, $p_{1,j} : H_1 \to H_1$, $p_{1,j} \equiv p_j \otimes E_H$, $H_{1,j} \equiv \{y(x) \in H_1 : y(x) \equiv p_{1,j} y(x)\}$, $j = 0, 1$, $H_2 \equiv \{y(x) \in H_1 : D_x^2 y \in H_1, A^2 y \in H_1\}$, $\|y; H_2\|^2 \equiv \|D_x^2 y; H\|^2 + \|A^2 y; H\|^2$, $L(H(A^m); H(A^q))$ is a set of bounded linear operators $S : H(A^m) \to H(A^q)$, $m, q \geq 0$, $L(H(A^m)) \equiv L(H(A^m); H(A^m))$, $H(A^0) = H$, $H^1 \equiv H(A^{\frac{3}{2}})$, $H^2 \equiv H(A^{\frac{1}{2}})$, $B_r \in L(H)$, $B_0 \in L(H_1)$, $B_r v_k = b_{r,k} v_k$, $r = 1, 2, k = 1, 2, \ldots$.

Consider the following problem:

$$L(D_x, A)y \equiv -D_x^2 y(x) + A^2 y + B_0(2x - 1)p_{1,0} y(x) = f(x), \tag{1}$$

$$\begin{aligned} l_1 y &\equiv B_1 y(0) + B_2 y(1) = h_1, \\ l_2 y &\equiv D_x y(0) - D_x y(1) = h_2, \end{aligned} \tag{2}$$

$$f(x) \in H_1, h_1 \in H^1, h_2 \in H^2.$$

We interpret the solution [1], [2] of problem (1), (2) as a function $y(x) \in H_2$satisfying the equalities

$$\begin{aligned} \|L(D_x, A)y - f; H_1\| = 0, \\ \left\|l_1 y - h_1; H^1\right\| = \left\|l_2 y - h_2; H^2\right\| = 0. \end{aligned} \tag{3}$$

Differential equation (1) includes operator of involution I. The first time study properties of the operator involution started C. Babbage [3]. In the paper [4] T. Carleman introduced the concept of operator shift – a generalization of the concept of involution I. Exploration partial differential equations with involution are devoted [4]–[10].

Properties spectral problems for ordinary differential and functional-differential equations with involution investigated in the works [2], [11]–[23] and [3, 24, 25, 26] respectively.

V. A. Il'in in [14]–[16] introduced the concept of essentially nonself-adjoint operator and proved the existence of present criterion of root functions of spectral problems for differential equations of arbitrary order.

In 1979 A. A. Shkalikov [27] it was proved that Riesz basis of subspaces (block basis) to $L_2(0, 1)$ root system forms a subspace that meet multiple or asymptotically close eigenvalues of differential operator A.

In works [28]–[30] studied the spectral properties of problem with linear potential with Dirichlet conditions in the interval $(0, 1)$.

In works [31]–[33] studied the properties of operators with increasing the multiplicity of the spectrum.

I. Auxiliary spectral problems

We now consider the operator L of problem (1), (2):

$$Ly \equiv L(D_x, A)y, y \in D(L),$$

$$D(L) \equiv \{y \in H_2 : l_1 y = 0, l_2 y = 0\},$$

Solutions of spectral problem

$$L(D_x, A)y = \lambda y(x), \lambda \in \mathbb{C}. \tag{4}$$

$$\begin{aligned} l_1 y &\equiv B_1 y(0) + B_2 y(1) = 0, \\ l_2 y &\equiv D_x y(0) - D_x y(1) = 0 \end{aligned} \tag{5}$$

consider as a product $y(x) = u(x)v_k(A)$, $u(x) \in W_2^2(0,1)$, $k = 1, 2, ...$

To determine the functions $u(x)$ obtain spectral problem

$$L_k(D,I)u \equiv -D^2u(x) + z_k^2u(x)+ \\ +b_{0,k}(2x-1)p_0u(x) = \lambda u(x), \tag{6}$$

$$l_{1,k}u \equiv b_{1,k}u(0) + b_{2,k}u(1) = 0, \\ l_{2,k}u \equiv Du(0) - Du(1) = 0. \tag{7}$$

Consider the particular case the problem (6), (7), if the specified conditions $B_1 = -B_2 = E$, $B_0 = 0$.

$$L_k(D)u \equiv -D^2u(x) + z_k^2u = \lambda u(x), \tag{8}$$

$$u(0) - u(1) = 0, Du(0) - Du(1) = 0. \tag{9}$$

It is easy to verify that the problem (8), (9) have point spectrum $\sigma_k \equiv \{\lambda_{n,k} \in \mathrm{R} : \lambda_{n,k} = (2\pi n)^2 + z_k^2,$ $n = 0, 1, ...\}$, and system of eigenfunctions

$$T \equiv \Big\{t_n^s \in L_2(0,1) : t_0^0(x) = 1,\ t_n^0(x) = \sqrt{2}\cos 2\pi nx,$$

$$t_n^1(x) = \sqrt{2}\sin 2\pi nx,\ n \in \mathrm{N}\Big\}.$$

We now consider the operator $L_{0,k}$, of the problem (8), (7)

$$L_{0,k}u \equiv L_k(D)u, u \in D(L_{0,k}),$$

$$D(L_{0,k}) \equiv \left\{u \in W_2^2(0,1) : l_{1,k}u = 0, l_{2,k}u = 0\right\}.$$

Let,

$$v_{0,k}^0(x) \equiv 1 + \beta_k(2x-1), \\ v_{n,k}^0(x) \equiv \sqrt{2}\sin 2\pi nx, n = 1, 2, ..., \tag{10}$$

$$v_{n,k}^1(x) \equiv \sqrt{2}(1 + \beta_k(2x-1))\cos 2\pi nx, \\ n = 1, 2, ..., \tag{11}$$

$$\beta_k \equiv (b_{1,k} - b_{2,k})^{-1}(b_{1,k} + b_{2,k}). \tag{12}$$

You can check that $v_{k,n}^s(x) \in D(L_{0,k})$, $s \in M_n$, $M_n \equiv \{0, \operatorname{sign} n\}$, $n = 0, 1, ...,$

$$L_{0,k}v_{n,k}^1 = \lambda_{n,k}v_{n,k}^1 + \xi_{n,k}v_{n,k}^0, \\ L_{0,k}v_{n,k}^0 = \lambda_{n,k}v_{n,k}^0, \xi_{k,n} = 8\pi n\beta_k, n = 1, 2, ... \tag{13}$$

Hence, $V(L_{0,k}) \equiv \left\{v_{n,k}^s(x) : s \in M_n, n = 0, 1, ...\right\}$ is a system of root functions of the operator $L_{0,k}$ in the sense of equality (13).

Theorem 1. *Let $b_{1,k} \neq b_{2,k}$. Then the operator $L_{0,k}$ of problem* (8)*, (7) have point spectrum σ_k, and system $V(L_{0,k})$ complete and minimal in $L_2(0,1)$.*

Proof. Let $y_s(x,\rho) = \left(e^{\rho x} + (-1)^s e^{\rho(1-x)}\right)$ – fundamental system of solutions of the differential equation (8), $u(x) = c_0y_0(x,\rho) + c_1y_1(x,\rho)$ – general solution of this differential equation (8),Re$\rho \le 0$ $\lambda = \rho^2$, $s = 0, 1$.

Substituting this general solution into boundary conditions (7), we obtain an equation for determining the eigenvalues

$$\Delta(\rho) \equiv \det\begin{pmatrix} 2\beta_k(1+e^\rho) & 2(1-e^\rho) \\ 2\rho(1-e^\rho) & 0 \end{pmatrix} = 0.$$

Hence, operator $L_{0,k}$ of problem (8), (7) have point spectrum σ_k.

Consider the adjoint problem

$$L_k(D)w \equiv -D^2w(x) + z_k^2w(x) = \lambda w(x),$$

$$l_{3,k}w \equiv w(0) - w(1) = 0,$$

$$l_{4,k}w \equiv b_{2,k}Dw(0) + b_{1,k}Dw(1) = 0.$$

Consider the operator $L_{0,k}^*$on $L_2(0,1)$, of the adjoint problem

$$L_{0,k}^*w \equiv L_k(D)w, w \in D\left(L_{0,k}^*\right),$$

$$D\left(L_{0,k}^*\right) \equiv \left\{w \in W_2^2(0,1) : l_{3,k}w = 0,\ l_{4,k}w = 0\right\}.$$

Let, $w_{0,k}^0(x) \equiv 1w_{q,k}^0(x) \equiv \sqrt{2}\cos 2\pi qx$, $w_{q,k}^1(x) \equiv \sqrt{2}(1 - \beta_{q,k}(2x-1))\sin 2\pi qx$, $q = 1, 2, ...$

You can check that $w_{n,k}^s(x) \in D\left(L_{0,k}^*\right)$, $s \in M_n$, $n = 0, 1, ...$

$$L_{0,k}^*w_{n,k}^1 = \lambda_{n,k}w_{n,k}^1 - \xi_{n,k}^0w_{n,k}^0, \\ \xi_{n,k}^0 = 8\pi n\beta_{n,k}, n = 1, 2, ... \tag{14}$$

Hence, the operator $L_{0,k}^*$ have point spectrum σ_k, and system of root functions $W(L_{0,k}) \equiv \left\{w_{q,k}^s \in L_2(0,1),\ s \in M_q,\ q = 0, 1, ...\right\}$. So, the system $V(L_{0,k})$ possesses a unique biorthogonal system $W(L_{0,k})$ in the sense of equality

$$\left(v_{n,k}^r, w_{q,k}^s; L_2(0,1)\right) = \delta_{r,s}\delta_{n,q},$$

$$r \in M_n, s \in M_q, q, k = 0, 1, ...$$

The theorem is proved.

Consider the operators $R_{0,k}, S_{0,k} : L_2(0,1) \to L_2(0,1)$, $Ey \equiv y$, $y \in L_2(0,1)$, $R_{0,k}t_n^p \equiv v_{n,k}^p$, $R_{0,k} \equiv E + S_{0,k}$, $p \in M_n$, $n = 0, 1, ...$

From the definition of the operator $R_{0,k}$ and the completeness of system $V(L_{0,k})$ in space $L_2(0,1)$ we get $S_{0,k} : L_{2,1}(0,1) \to 0$, $S_{0,k} : L_{2,0}(0,1) \to L_{2,1}(0,1)$. Then $S_{0,k}S_{0,k} : L_{2,0}(0,1) \to 0$, $S_{0,k}S_{0,k} : L_{2,1}(0,1) \to 0$, i.e., $S_{0,k}S_{0,k} = O$, where O is the zero operator in the space $L_2(0,1)$. Thus, $R_{0,k}^{-1} = E - S_{0,k}$.

Theorem 2. *Let $b_{1,k} \neq b_{2,k}$. Then system $V(L_{0,k})$ forms a Riesz basis in $L_2(0,1)$.*

To prove that the system $V(L_{0,k})$ forms a Riesz basis [34] in $L_2(0,1)$, it is sufficient, according to formula $R_{0,k} = E + S_{0,k}$, to show that the operator $S_{0,k} \in L(L_2(0,1))$.

Let ω be an arbitrary element from the space $L_2(0,1)$). We represent ω as a Fourier series in the system T.

$$\omega = \omega_0^0t_0^0 + \sum_{m=1}^{\infty}\omega_m^0t_m^0 + \omega_m^1t_m^1.$$

According to the definition of the operator $S_{0,k}$, we find

$$S_{0,k}\omega = \beta_k(2x-1)\left(\omega_0^0v_{0,k}^0 + \sum_{m=1}^{\infty}\omega_m^0\sqrt{2}\cos 2\pi mx\right)$$

Using the ratio

$$\|S_{0,k}\omega; L_2(0,1)\|^2 =$$

$$= \left\| \beta_k(2x-1), \omega_0^0 v_{0,k}^0 + \sum_{m=1}^{\infty} \omega_m^0 \sqrt{2}\cos 2\pi m x; L_2(0,1) \right\|^2$$

we estimate it

$$\|S_{0,k}\omega; L_2(0,1)\|^2 \le |\beta_k|^2 \|\omega; L_2(0,1)\|^2$$

Hence, the operator $R_{0,k} = E + S_{0,k}$ is bounded $L_2(0,1) \to L_2(0,1)$ and $\left(R_{0,k}^{-1}\right)^* = E - S_{0,k}^* \in L\left(L_2(0,1)\right)$. So using theorem N.K. Bary (see theorem 6. 2.1[33]) we obtain the statement of the theorem.

Further, we introduce operator $L_{1,k}$ of the problem (6), (7).

$$L_{1,k}u \equiv L_k(D,I)\,u, u \in D(L_{1,k}),$$
$$D(L_{1,k}) \equiv \left\{u \in W_2^2(0,1) : l_{1,k}u = 0, l_{2,k}u = 0\right\}.$$

By the direct substitution we can show that the

$$v_{0,0,k}(x) = 1 + \left(\beta_k + \tfrac{1}{12}b_{0,k}\right)(2x-1) - \\ -\tfrac{1}{12}b_{0,k}(2x-1)^3, v_{0,n,k}(x) = \sqrt{2}\sin 2\pi n x, \quad (15)$$

is eigenfunctions of operator $L_{1,k}$: $v_{s,n,k}(x) \in D(L_{1,k})$, $L_{1,k}v_{s,n,k}(x) = \lambda_{n,k}v_{s,n,k}(x)$, $s \in M_n$, $n = 0,1,\ldots$. Hence, the operator $L_{1,k}$ have point spectrum σ_k. Root function of the operator $L_{1,k}$ defined by relation

$$v_{1,n,k}(x) \equiv \sqrt{2}\left(1 + \beta_k(2x-1)\right)\cos 2\pi n x + \\ \xi_{n,k}\sqrt{2}(2x-1)^2 \sin 2\pi n x \quad (16)$$

$$\xi_{n,k} = b_{0,k}(16\pi n)^{-1}, n = 1,2,\ldots. \quad (17)$$

Hence, $V(L_{1,k}) \equiv \{v_{s,n,k}(x) : s \in M_n,\ n = 0,1,\ldots\}$ is a system of root functions of the operator $L_{1,k}$, in the sense of equality

$$L_k(D,I)v_{1,n,k}(x) = \lambda_{n,k}v_{1,n,k}(x) + \\ \xi_{1,n,k}v_{0,k,n}(x), \quad (18)$$

$$\xi_{1,n,k} = -8\xi_{n,k} + \rho_{n,k}, \rho_{n,k} = 8\pi n\beta_k, n = 1,2,\ldots. \quad (19)$$

Show that the system $V(L_{1,k})$ of root functions of the operator $L_{1,k}$ possesses a unique biorthogonal system $W(L_{1,k})$.

Further, we introduce operators $R_{1,k}, S_{1,k} : L_2(0,1) \to L_2(0,1)$,

$$R_{1,k} = E + S_{1,k}, \quad R_{1,k}t_k^p(x) \equiv v_{p,n,k}(x), \\ p \in M_n, n = 0,1,\ldots. \quad (20)$$

With formulas (15), (16) that have, that $(S_{1,k})^2 = 0$. Show that $S_{1,k} \in L\left(L_2(0,1)\right)$. To have any function $\omega \in L_2(0,1)$

$$\omega = \omega_0^0 + \sqrt{2}\sum_{m=1}^{\infty} \omega_m^0 \cos 2\pi m x + \omega_m^1 \sin 2\pi m x,$$

$$S_{1,k}\omega = \omega_0^0 + \sqrt{2}\sum_{n=1}^{\infty}\omega_{n,k}^0\left(\beta_k(2x-1)\cos 2\pi n x + \right.$$
$$\left.\xi_{n,k}(2x-1)^2 \sin 2\pi n x\right)$$

$$\|S_{1,k}\omega; L_2(0,1)\|^2 \le \max_n \left(1 + |\beta_k|^2 + \right. \\ \left. + |\xi_{n,k}|^2\right)\|\omega; L_2(0,1)\|^2. \quad (21)$$

So there operator $\left(R_{1,k}^*\right)^{-1} = E - S_{1,k}^*$ such that $\left(R_{1,k}^*\right)^{-1} : T \to W(L_{1,k})$. Hence, the system of $V(L_{1,k})$ of the operator $L_{1,k}$ complete and minimal in forms a in $L_2(0,1)$.

With formulas (12), (13), (15), (16) that have system $V(L_{1,k})$ and $V(L_{0,k})$ is squarely close in the sense of inequality

$$\sum_{n=0}^{\infty}\sum_{s\in M_n}\left\|v_{s,n,k}(x) - v_{n,k}^s(x); L_2(0,1)\right\|^2 \le$$

$$\le K\,|b_{0,k}|^2 < \infty, K = \frac{1}{72} + \sum_{n=1}^{\infty}\frac{1}{64\pi^2 n^2}.$$

So using theorem N.K. Bary (see theorem 6. 2.3[34]) we obtain the following statement.

Theorem 3. *Let $b_{1,k} \ne b_{2,k}$. Then the operator $L_{1,k}$ have point spectrum σ_k, and system $V(L_k)$ of root functions forms a Riesz basis in $L_2(0,1)$.*

II. The spectral problem (1), (2)

Let $b_{1,k} \ne b_{2,k}$, $k = 1,2,\ldots$. By the direct substitution we can show that the

$$v_{s,n,k}(L) \equiv v_{s,n,k}(x)v_k(A) \in D(L), \\ s \in M_n,\ k = 1,2,\ldots,\ n = 0,1,\ldots, \quad (22)$$

$$Lv_{0,n,k}(L) = \left(4\pi^2k^2 + z_k^2\right)v_{0,n,k}(L),$$
$$k = 1,2,\ldots,\ n = 0,1,\ldots,$$
$$Lv_{1,n,k}(L) = \left(4\pi^2k^2 + z_k^2\right)v_{1,n,k}(L) + \xi_{1,n,k}v_{0,n,k}(L),$$
$$k = 1,2,\ldots, n = 1,2,\ldots.$$

Therefore, we define the eigenvalues and the system of root functions of the operator L by the relations

$$\lambda_{n,k}(L) = 4\pi^2n^2 + z_k^2, k = 1,2,\ldots,\ n = 0,1,\ldots,$$

$$V(L) \equiv \{v_{s,n,k}(L) \in H_1 : v_{s,n,k}(L) \equiv v_{s,n,k}(x)v_k(A),$$
$$s \in M_n,\ k = 1,2,\ldots,\ n = 0,1,\ldots\}.$$

Let $\pi_k : H \to H$ – orthogonal projector, $\pi_k h \equiv (h, v_k(A); H)\,v_k(A)$ $k = 1,2,\ldots$. Consider the operators $R, S : H_1 \to H_1$, $R \equiv \sum_{k=1}^{\infty} R_{1,k}\pi_k$, $S = E_{H_1} - R$, and T_1 – the orthonormal basis in the space H_1 $T_1 \equiv \left\{t_{n,k}^s \in H_1 : t_{n,k}^s \equiv t_n^s(x)v_k(A),\ t_n^s(x) \in T,\right.$ $v_k(A) \in V(A), s \in M_n,\ n = 0,1,\ldots, k = 1,2,\ldots\}$.

With formulas (22) that have

$$v_{s,n,k}(L) = Rt_{n,k}^s, s \in M_n, \\ n = 0,1,\ldots, k = 1,2,\ldots. \quad (23)$$

From the definition of the operator R and the completeness of system $V(L)$ in space $L_2(0,1)$ we get $w_{r,p,q}(x)v_q(A) = R^*t_{p,q}^r$, $r \in M_p$, $p = 0,1,\ldots$, $q = 1,2,\ldots$.

Hence, system $V(L)$ of root functions of the operator L possesses a unique biorthogonal system $W(L) \equiv \{w_{r,p,q}(L) \in H_1 : w_{r,p,q}(L) \equiv w_{r,p,q}(x)v_q(A), r \in M_p,$

$p=0,1,\ldots,\ q=1,2,\ldots\}$ in the sense of equality $(v_{j,n,k},w_{r,p,q};H_1)=\delta_{j,r}\delta_{n,p}\delta_{k,q},\ j\in M_n,\ r\in M_p,\ n,p=0,1,\ldots,\ q,k=1,2,\ldots$.

Hence, we obtain the following statement.

Theorem 4. *Let $b_{1,k}\neq b_{2,k},\ k=1,2,\ldots$. Then the operator L have complete and minimal in H_1 system of root functions $V(L)$.*

Further, we introduce operator $B\equiv(B_1+B_2)(B_1-B_2)^{-1}\in L(H^1)$

Then

$$\left\|Bv_k(A);H^1\right\|=\left\|A^{\frac{3}{2}}Bv_k(A);H\right\|=$$
$$=\left\|\beta_k(z_k)^{\frac{3}{2}}v_k(A);H\right\|=|\beta_k|\left\|v_k(A);H^1\right\|,$$
$$\left\|Bv_k(A);H^1\right\|\leq\left\|B;L(H^1)\right\|\left\|v_k(A);H^1\right\|.$$

Therefore,

$$|\beta_k|\leq\left\|B;L(H^1)\right\|,k=1,2,\ldots, \tag{24}$$
$$\left\|B_0t^s_{n,k}(x);H_2\right\|^2=(b_{0,k})^2\left((2n\pi)^4+z_k^2\right),$$
$$\left\|t^s_{n,k}(x);H_2\right\|^2=(2n\pi)^4+z_k^2,$$
$$s\in M_n,\ n=0,1,\ldots,k=1,2,\ldots.$$

Therefore, if $B_0\in L(H_2)$, then

$$\left\|B_0t^s_{m,k}(x);H_2\right\|^2=(b_{0,k})^2\left((2m\pi)^4+z_k^2\right)\leq$$
$$\leq\|B_0;L(H_2)\|^2\left\|t^s_{m,k}(x);H_2\right\|^2,r\in M_n,$$
$$n=0,1,\ldots,k=1,2,\ldots,$$
$$|b_{0,k}|\leq\|B_0;L(H_2)\|,k=1,2,\ldots. \tag{25}$$

Show that $S\in L(H_1)$.

Let $g(x)$ be an arbitrary element from the space H_1. We represent $g(x)$ as a Fourier series in the system T_1: $g=\sum\limits_{s,m,k}g_{s,k,m}t^s_{m,k}$.

According to the definition of the operator S, we find

$$Sg=\sum_{k=1}^{\infty}\Big(g_{0,0,k}v_{0,0,k}(x)+\sum_{m=1}^{\infty}g_{0,m,k}v_{0,m,k}(x)+$$
$$+g_{1,m,k}v_{1,m,k}(x)\Big)v_k(A),$$

$$\begin{gathered}Sg=\sum_{k=1}^{\infty}\Big(g_{0,0,k}\big(1+(\beta_k+\tfrac{1}{12}b_{0,k})(2x-1)-\\ -\tfrac{1}{12}b_{0,k}(2x-1)^3\big)\Big)v_k(A)+\\ +\sum_{k=1}^{\infty}\Big(\sum_{m=1}^{\infty}\big(g_{0,m,k}+\\ +g_{1,m,k}\xi_{m,k}(2x-1)^2\big)\sqrt{2}\sin2\pi mx+\\ +g_{1,m,k}\beta_{m,k}(2x-1)\sqrt{2}\cos2\pi mx\Big)v_k(A).\end{gathered} \tag{26}$$

With formulas (17), (24), (25) that have

$$\begin{gathered}\|Sg,H_2\|\leq C\big(\|B;L(H^1)\|+\\ +\|B_0;L(H_2)\|\big)\|g,H_2\|,C>0.\end{gathered} \tag{27}$$

Hence, $S\in L(H_1)$.

So using theorem N. K. Bary (see theorem 6.2.1 [34]) we obtain the following statement.

Theorem 5. *Let $B\in L(H^1)$. Then the operator L have system of root functions $V(L)$ forms a Riesz basis in H_1.*

III. Property problem (1), (2)

Replaced condition (2) on equivalent terms

$$\begin{gathered}l_3y\equiv y(0)-y(1)+B(y(0)+y(1))=h_3,\\ l_2y\equiv D_xy(0)-D_xy(1)=h_2.\end{gathered} \tag{28}$$

Here $h_3\equiv(B_1-B_2)^{-1}h_1\in H^1$, $h_2\in H^2$.

Consider the particular case the problem (1), (28) if the specified conditions $B=0$, $B_0=0$

$$-D_x^2y(x)+A^2y=g(x), \tag{29}$$

$$\begin{gathered}y(0)-y(1)=g_1,D_xy(0)-D_xy(1)=g_2,\\ g_j\in H^j,j=1,2.\end{gathered} \tag{30}$$

Theorem 6. *For any $g\in H_1$, $g_1\in H^1$, $g_2\in H^2$ there exists a unique solution of problem* (29), (30).

Proof. We seek the solution of this problem in the form $y=u+v$, there u is a solution of the problem

$$\begin{gathered}-D_x^2u(x)+A^2u=g(x),u(0)-u(1)=0,\\ D_xu(0)-D_xu(1)=0,\end{gathered} \tag{31}$$

and v – solution of the problem

$$\begin{gathered}-D_x^2v(x)+A^2v(x)=0,v(0)-v(1)=g_1,\\ D_xv(0)-D_xv(1)=g_2.\end{gathered} \tag{32}$$

Consider the problem (31). We expand the functions $u(x)$, $g(x)$ in a series in the orthonormal basis T_1 in the space H_1:

$$u=\sum_{s,n,k}u_{s,n,k}t^s_{n,k},g=\sum_{s,n,k}g_{s,n,k}t^s_{n,k}.$$

Substituting into the (31) we get

$$u=\sum_{s,n,k}\left((2\pi n)^2+z_k^2\right)^{-1}g_{s,n,k}t^s_{n,k}.$$

We estimate a numbers

$$-D_x^2u=\sum_{s,n,k}(2\pi n)^2\left((2\pi n)^2+z_k^2\right)^{-1}g_{s,n,k}t^s_{n,k},$$

Therefore,

$$\left\|D_x^2u;H_1\right\|\leq\|g;H_1\|,$$

$$A^2u=\sum_{s,k,m}z_k^2\left((2\pi m)^2+z_k^2\right)^{-1}g^s_{k,m}t^s_{k,m},$$

$$\left\|A^2u;H_1\right\|\leq\|g;H_1\|,$$

Hence,

$$\|u;H_2\|\leq\sqrt{2}\,\|g;H_1\| \tag{33}$$

Consider the problem (32). Further, we introduce operators, $Y_j(x, A) \equiv e^{-Ax} + (-1)^j e^{-A(1-x)}$, $j = 0, 1$. So using lemma 4.1.2 (see [2]) we obtain

$$Y_j(x, A) \in L\left(H^2; H_2\right). \qquad (34)$$

The solution of the differential equation (32) has the form

$$v(x) = Y_0(x, A)\varphi_0 + Y_1(x, A)\varphi_1, \qquad (35)$$

where $\varphi_0, \varphi_1 \in H^1$ are unknown.

To determine the, $\varphi_0, \varphi_1 \in H^1$ we substitute expression (35) in the condition (32) and obtain

$$\phi_1 = \frac{1}{2} Y_1(0, A)^{-1} g_1, \phi_0 = -\frac{1}{2} Y_1(0, A)^{-1} A^{-1} g_2$$

Hence,

$$\begin{array}{l} v = \frac{1}{2} Y_1(x, A) Y_1(0, A)^{-1} g_1 - \\ -\frac{1}{2} Y_0(x, A) Y_1(0, A)^{-1} A^{-1} g_2. \end{array} \qquad (36)$$

With formulas (34) that have

$$\|v; H_2\|^2 \leq C\left(\left\|g_1; H^1\right\|^2 + \left\|g_2; H^2\right\|^2\right) \qquad (37)$$

Therefore follows from inequalities (33), (37) inequality

$$\|y; H_2\|^2 \leq C_1\left(\|g; H_1\|^2 + \left\|g_1; H^1\right\|^2 + \left\|g_2; H^2\right\|^2\right)$$

We now return to the original problem (1), (2). Consider in connection problem as the sum $y = y_0 + y_1$, $y_j \in H_{1,j}$, $j = 0, 1$.

To determine the unknowns $y_j \in H_{1,j}$, $j = 0, 1$, get the problem

$$-D_x^2 y_0(x) + A^2 y_0 = f_0(x), f_0(x) \in H_{1,0},$$

$$l_3 y_0 \equiv y_0(0) - y_0(1) = 0, l_2 y_0 \equiv D_x y_0(0) - D_x y_0(1) = h_2,$$

$$-D_x^2 y_1(x) + A^2 y_1 = -2B_0(2x-1)(y_0(x) + y_0(1-x)) +$$

$$+ f_1(x), f_1(x) \in H_{1,1},$$

$$y_1(0) - y_1(1) = -B(y_0(0) + y_0(1)) + h_3,$$

$$D_x y_1(0) - D_x y_1(1) = 0.$$

For unknown functions $y_j \in H_{1,j}$ get that problem is a particular case of the problem (29), (30).

Hence the statement is correct

Theorem 7. *Let* $B \in L(H^1)$, $B_0 \in L(H_2)$. *Then for any* $f \in H_1$, $h_1 \in H^1$, $h_2 \in H^2$, *there exists a unique solution of problem* (1)*,* (2)

$$\|y; H_2\|^2 \leq C\left(\|f; H_1\|^2 + \left\|h_1; H^1\right\|^2 + \left\|h_2; H^2\right\|^2\right),$$

$$C > 0.$$

Conclusion

We have investigated the properties of nonlocal problem with generalized conditions Ionkin's for the Sturm-Liouville equation with polynomial potential which contains an involution operator. Defined point spectrum and built a system of root functions of the spectral problem. It is proved that under certain conditions the system of root functions spectral problem forms a Riesz basis. It is proved that under certain conditions the solution of the problem exists and only one.

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КРАЙОВА ЗАДАЧА ДЛЯ ДИФЕРЕНЦІАЛЬНО-ОПЕРАТОРНОГО РІВНЯННЯ ДРУГОГО ПОРЯДКУ З ІНВОЛЮЦІЄЮ

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Вивчається нелокальна двоточкова задача для диференціально-операторних рівнянь з інволюцією. Встановлено спектральні властивості та умови існування і єдиності розв'язку. Наведено достатні умови, за яких система кореневих функцій задачі утворює базис Рісса.

Ключові слова: диференціальне рівняння, диференціально-операторне рівняння, коренева функція, оператор інволюції, несамоспряжений оператор, базис Рісса, нелокальна задача.

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