

## Pipeline pressure distribution finding methods

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The method of solving problems of mathematical physics, in particular for calculating a non-stationary gas flow in pipelines, is proposed in this article on the basis of the biorthogonal polynomial constructed by the authors. The method of solving the problem by means of the separation of variables in the base of biorthogonal polynomials is investigated. The analytical-approximate and approximate solutions of the problem as the sum of some biorthogonal and quasi-spectral polynomials are found. The comparative analysis between the obtained analytical-approximate and approximate solutions is conducted. The influence of parameters of methods, including the order of the partial sum, a bit grid, and an accuracy error of calculations on the obtained solution are studied. The results of calculation are presented in the form of tables.

**Keywords:** *spectral methods, mathematical model, non-stationary gas flow, linearization, biorthogonal and quasi-orthogonal polynomials.*

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### 1. Introduction

Spectral methods are used both in theoretical studies and for solving a wide class of problems of mathematics and mechanics. Their essence is that the functions included into the model are presented in the form of orthogonal series in accordance to the selected basis. Finding the solution is reduced to calculating the coefficients of orthogonal series of the desired solution. It is shown [6, 7] that the choice of orthogonal basis one should coordinate with the domain of definition of the desired solution. The positive sides are those that many orthogonal bases are studied well enough, they are easy to use, and the solving algorithms constructed on their basis are easy for automation. The negative side is that the summation of corresponding series is, as a rule, an incorrect problem. Therefore, not all the criteria to the solutions of problems one can satisfy by means of using of one orthogonal basis. Due to the fact, we are modifying the existing bases or constructing a new one to meet the broader criteria. One of the methods of consideration of these comments is the usage of biorthogonal bases. Nowadays, there are a few papers devoted to their research and practical application. This is mainly due to the fact that formation of these biorthogonal bases is connected with significant difficulties of calculation and they are not studied enough.

### 2. Formulation of the problem

The system of the interconnected differential equations in partial derivatives is the common mathematical model of gas flow in the pipeline in the isothermal case

$$\begin{cases} \frac{\partial p(y, t)}{\partial y} + \alpha \rho \frac{\partial}{\partial y} \left( \frac{v^2(y, t)}{2} \right) + \rho g \frac{\partial h(y, t)}{\partial y} + \frac{\lambda \rho v^2(y, t)}{2D} + \frac{\partial (\rho v(y, t))}{\partial t} = 0, \\ \frac{\partial (\rho v(y, t))}{\partial y} + \frac{1}{c^2} \frac{\partial p(y, t)}{\partial t} = 0, \end{cases} \quad (1)$$

where  $\omega = \rho v$  is the mass gas consumption (gas velocity);  $A$  is the sound velocity in gas;  $\alpha$  is the Coriolis coefficient;  $\rho$ ,  $v$ ,  $p$  are the density, the gas velocity and the gas pressure accordingly;  $\lambda$  is the hydraulic resistance coefficient;  $D$  is the diameter of the pipeline;  $g$  is gravity acceleration;  $h$  is the relative height of the pipeline location;  $t > 0$  is time;  $y \in [0, l]$  is the line coordinate,  $l$  is the length of the pipeline.

In practice, the limiting conditions are formulated on the input and output of compressor stations. As usual, the stationary pressure distribution is the initial state at the beginning of the non-stationary process. Therefore, the problem of mathematical physics in this case is as follows.

One can find the solution of the system (1) in accordance to the initial stationary pressure distribution

$$p(y, 0) = \sqrt{p_0^2 - \frac{\lambda z R T}{D} \left( \frac{\rho_s q_s}{s} \right)^2 y} \quad (2)$$

and limiting conditions on volumetric gas consumption

$$q_0(t) = q_{0n} + (q_0 - q_{0n}) e^{-\gamma_0 t}; \quad (3)$$

$$q_l(t) = q_{ln} + (q_l - q_{ln}) e^{-\gamma_l t}, \quad (4)$$

stand on inputs and outputs of the pipeline correspondingly.

In this case  $p_0$  is the value of pressure on the pipeline start;  $\rho_s$ ,  $q_s$  are the values of density and volumetric consumption in standard conditions,  $s = \pi D^2/4$ ;  $q_0$ ,  $q_{0n}$  are the volumetric gas consumptions at the initial and new states of gas flow and the parameter  $\gamma_0$ , which characterizes velocity of transition from one state into other on the pipeline start;  $q_l$ ,  $q_{ln}$ ,  $\gamma_l$  are the similar parameters at the pipeline end. In the case of transition to mass velocity the limiting conditions are the following [9]:

$$\omega_0 = \omega(0, t) = \frac{\rho_s}{s} q_0(t), \quad \omega_l = \omega(l, t) = \frac{\rho_s}{s} q_l(t). \quad (5)$$

### 3. Linearization of the output nonlinear system

The linearised variant of the system (1) [9] has the form

$$\begin{cases} A_0 \frac{\partial p(y, t)}{\partial y} + A_1 \frac{\partial \omega(y, t)}{\partial y} + \frac{\partial \omega(y, t)}{\partial t} + A_2 \omega(y, t) + A_3 p(y, t) = -A_4, \\ \frac{\partial \omega(y, t)}{\partial y} + \frac{1}{c^2} \frac{\partial p(y, t)}{\partial t} = 0. \end{cases} \quad (6)$$

where

$$\begin{aligned} a_p &= p_1(1 + fp_1) - b_p p_1, \quad c_0 = 1 - \alpha v_c^2 \rho_0 T_0 b_p / (p_0 T), \quad c_1 = \alpha v_c, \quad a_v = v_1 + v_2, \\ b_v &= -v_1 v_2 - \frac{1}{8} (v_2 - v_1)^2, \quad c_2 = \alpha b_v / (2D), \quad c_3 = \frac{\rho_0 T_0}{p_0 T} b_p \left( g \frac{\delta h}{\delta y} + \frac{\lambda a_v}{2D} \right), \\ c_4 &= \frac{\rho_0 T_0}{p_0 T} a_p \left( g \frac{\delta h}{\delta y} + \frac{\lambda a_v}{2D} \right), \quad p \in [p_1, p_2], \end{aligned}$$

$p_1$  and  $p_2$  stand for the limits of pressure change,  $v_A$  is the average volume of gas flow in the pipeline, which is accepted as known,  $v_1$  and  $v_2$  are the limits of change of gas flow velocity. For calculation of the coefficient of pressure  $z$ , which describes difference of the real gas from the ideal gas, one can use the empirical formula  $z = 1 / (1 + fp)$ , where  $p$  is measured in atmospheres, and  $f = (24 - 0.21t^\circ\text{C}) \cdot 10^{-4}$ ,  $t^\circ\text{C}$  is the temperature of gas Celsius;  $R$  is the gas constant.

We can note that initial distribution of pressure (2) is obtained in the stationary case from the nonlinear system of differential equations. Therefore, it is necessary to have its parametrical image in

the stationary case for finding the correct solution of the system (6). Thus, the problem of mathematical physics is as follows: to find the solution of the system (6) according to the initial (2) and the boundary (5) conditions.

#### 4. Analytical-approximate solution

We will consider the horizontal pipelines without taking into account the Coriolis force at the constant value of the pressure coefficient, that is  $\alpha = 0$  and  $c_0 = 1$ ,  $c_1 = 0$ ,  $c_2 = 0$  correspondingly. Then we write the system (6) in the form

$$\begin{cases} \frac{\partial p(y,t)}{\partial y} + \frac{\partial \omega(y,t)}{\partial t} + A_3 p(y,t) = -A_4, \\ \frac{\partial \omega(y,t)}{\partial y} + \frac{1}{c^2} \frac{\partial p(y,t)}{\partial t} = 0. \end{cases} \tag{7}$$

Since the length of the pipeline is  $l$ , that is  $0 \leq y \leq l$ , and functions which help us to solve this problem are considered on an interval  $[-1, 1]$ , then we change

$$y = \frac{l(x+1)}{2}, \quad w(y,t) = w\left(\frac{l(x+1)}{2}, t\right) = W(x,t), \quad p(y,t) = p\left(\frac{l(x+1)}{2}, t\right) = P(x,t). \tag{8}$$

Using the formulas (8), we write the system (7) and the conditions (2), (5) in the form

$$\begin{cases} \frac{2}{l} A_0 \frac{\partial P(x,t)}{\partial x} + \frac{\partial W(x,t)}{\partial t} + A_3 P(x,t) = -A_4, \\ \frac{2}{l} \frac{\partial W(x,t)}{\partial x} + \frac{1}{c^2} \frac{\partial P(x,t)}{\partial t} = 0, \end{cases} \tag{9}$$

$$P(x,0) = \sqrt{p_0^2 - \frac{\lambda z R T}{D} \left(\frac{\rho_s q_s}{s}\right)^2 \frac{l(x+1)}{2}}, \tag{10}$$

$$W_0 = W(-1,t) = \frac{\rho_s}{s} q_0(t), \quad W_l = W(1,t) = \frac{\rho_s}{s} q_l(t). \tag{11}$$

We can obtain the following from the system (9) and conditions (10), (11)

$$\frac{4c^2}{l^2} \frac{\partial W^2(x,t)}{\partial x^2} - \frac{\partial^2 W(x,t)}{\partial t^2} + \frac{2c^2 A_3}{l} \frac{\partial W(x,t)}{\partial x} = 0. \tag{12}$$

Even if

$$W(x,t) = V(x) e^{\beta x} G(t), \tag{13}$$

where  $\beta$  is the unknown parameter, which will be determined in the result of solving;  $V(x)$  are basis functions, which have the right ratios

$$\begin{aligned} \frac{d^2 V_i^{n+i}(x)}{dx^2} &= -\frac{V_i^{n+i}(x)}{\lambda_{i+(-1)^{i+1}}^n} + \frac{\tau_{i+(-1)^{i+1}}^n}{\lambda_{i+(-1)^{i+1}}^n} T'_{n+1+i}(x), \quad i = 1, \dots, n, \\ \frac{d^2 V_{n+1}^{n+1}(x)}{dx^2} &= \frac{(n+2)^2 \pi}{8} \sum_{k=1}^{\frac{n}{2}} \frac{c_2^{2k}}{\lambda_{2k}^n} \frac{V_{2k-1}^{n+1}(x)}{N_{2k-1}^n} + \frac{1}{15} n(n+1)(n+3)(n+4) T'_{n+2}(x), \\ \frac{d^2 V_{n+2}^{n+2}(x)}{dx^2} &= \frac{(n+1)^2 \pi}{2} \sum_{k=1}^{\frac{n}{2}} \frac{c_1^{2k-1}}{\lambda_{2k-1}^n} \frac{V_{2k}^n(x)}{N_{2k}^n} + \frac{1}{15} (n-1)n(n+2)(n+3) T'_{n+1}(x), \end{aligned}$$

where  $V_{n+1}^{n+1}(x) = T'_{n+2}(x)$ ,  $V_{n+2}^n(x) = T'_{n+1}(x)$ ,  $\bar{i} = 0$  for even values  $i$ , and  $\bar{i} = 1$  for uneven values  $i$ .  $T_{n+1} = T_{n+1}(x)$  and  $T_{n+2} = T_{n+2}(x)$  are the Chebyshev polynomials of degree  $n+1$  and  $n+2$ ,  $\lambda_i^n$ ,  $i = 1, \dots, n$  are the eigenvalues of the integral operator  $\pi_1^\infty L = \pi_1^\infty \int_{-1}^x \int_{-1}^{x_1} U_{2i}^{2s}(x) = \sum_{j=1}^s c_{2j}^{2i} \tilde{T}_{2j}(x)$ ,  $U_{2i-1}^{2s-1}(x) = \sum_{j=1}^s c_{2j-1}^{2i-1} \tilde{T}_{2j-1}(x)$  are the eigenvalues of this operator,  $\tilde{T}_j(x)$  is modified Chebyshev polynomials,  $\bar{U}_{2i}^{2s}(x) = \sum_{j=1}^s \bar{c}_{2j}^{2i} T_{2j}(x)$ ,  $\bar{U}_{2i-1}^{2s-1}(x) = \sum_{j=1}^s \bar{c}_{2j-1}^{2i-1} T_{2j-1}(x)$  are the eigenvalues of this turned operator,

$$\begin{aligned} \tau_{2i}^{2s}(x) &= \frac{c_{2s}^{2i}}{4(2s+1)(2s+2)}, & \tau_{2i-1}^{2s}(x) &= \frac{c_{2s-1}^{2i-1}}{4(2s)(2s+1)}, \\ \bar{\tau}_{2i}^{2s}(x) &= \frac{\bar{c}_{2s}^{2i}}{4(2s)(2s+1)}, & \bar{\tau}_{2i-1}^{2s}(x) &= \frac{\bar{c}_{2s-1}^{2i-1}}{4(2s-1)(2s)}, \end{aligned}$$

the biorthogonal functions

$$\begin{aligned} V_{2i-\bar{i}}^{n+\bar{i}}(x) &= \int_{-1}^x U_{2i-1+\bar{i}}^{n-\bar{i}-(-1)^{\bar{i}}}(x_1) dx_1, & \bar{V}_{2i-\bar{i}}^{n+\bar{i}}(x) &= -\sqrt{1-x^2} \int_{-1}^x \frac{\bar{U}_{2i-1+\bar{i}}^{n-\bar{i}-(-1)^{\bar{i}}}(x_1)}{\sqrt{1-x_1^2}} dx_1, & i &= 1, \dots, n/2, \\ N_i^n &= \int_{-1}^1 \frac{V_i^{n+\bar{i}}(x) \bar{V}_i^{n+\bar{i}}(x)}{\sqrt{1-x^2}} dx, & i &= 1, \dots, n \end{aligned}$$

are the standard of biorthogonal functions [10, 11].

Let us substitute (13) into the equation (12) and obtain

$$\frac{4c^2}{l^2} V_{xx}''(x) G(t) + \left( \frac{8c^2}{l^2} \beta + \frac{2c^2 A_3}{l} \right) V_x'(x) G(t) + \left( \frac{4c^2}{l^2} \beta^2 G(t) - G_{tt}''(t) + \frac{2c^2 A_3}{l} \beta G(t) \right) V(x) = 0.$$

Thence,  $\beta = -\frac{lc_3}{4}$ .

As follows we obtain

$$\frac{4c^2}{l^2} V_{xx}''(x) G(t) - \left( \frac{c^2 A_3}{4} G(t) + G_{tt}''(t) \right) V(x) = 0. \quad (14)$$

The product  $V(x)G(t)$  one can find in the form

$$V(x)G(t) = \sum_{i=1}^{n+2} V_i^{n+\bar{i}}(x) G_i(t). \quad (15)$$

Then,

$$\begin{aligned} V(x)G''(t) &= \sum_{i=1}^{n+2} V_i^{n+\bar{i}} G_i''(t), \\ V_{xx}''(x)G(t) &= \sum_{i=1}^n \frac{d^2 V_i^{n+\bar{i}}(x)}{dx^2} G_i(t) + \frac{d^2 V_{n+1}^{n+1}(x)}{dx^2} G_{n+1}(t) + \frac{d^2 V_{n+2}^n(x)}{dx^2} G_{n+2}(t). \end{aligned} \quad (16)$$

From the boundary conditions (11), considering that  $V_i(-1) = V_i(1) = 0$ , we have

$$\frac{e^{\frac{lc_3}{4}} W(1, t) + e^{-\frac{lc_3}{4}} W(-1, t)}{2(n+1)^2} = G_{n+2}(t), \quad \frac{e^{\frac{lc_3}{4}} W(1, t) - e^{-\frac{lc_3}{4}} W(-1, t)}{2(n+2)^2} = G_{n+1}(t). \quad (17)$$

Expressions (15), (16) we substitute into the equation (14) and obtain the following

$$\begin{aligned} \frac{4c^2}{l^2} \left( \sum_{i=1}^n \frac{d^2 V_i^{n+i}(x)}{dx^2} G_i(t) + \frac{d^2 V_{n+1}^{n+1}(x)}{dx^2} G_{n+1}(t) + \frac{d^2 V_{n+2}^n(x)}{dx^2} G_{n+2}(t) \right) \\ - \frac{c^2 A_3}{4} \sum_{i=1}^{n+2} V_i(x) G_i(t) - \sum_{i=1}^{n+2} V_i(x) G_i''(t) = 0. \quad (18) \end{aligned}$$

The equation (18) we multiply by  $\frac{\bar{V}_i(x)}{\sqrt{1-x^2}}$ ,  $i = 1, \dots, n$  and integrate in accordance to  $x$  within from  $-1$  to  $1$ . If using equations (17) one will receive the system from  $n$  of the equations of the kind

$$\begin{aligned} G_{2i-\bar{i}}''(t) + \left( \frac{4c^2}{l^2} \frac{1}{\lambda_{2i-1+\bar{i}}^n} + \frac{c^2 A_3^2}{4} \right) G_{2i-\bar{i}}(t) \\ = \frac{c^2 \pi}{l^2} \frac{\bar{c}_{i+1}^{2i-1+\bar{i}}}{4^i \lambda_{2i-1+\bar{i}}^n} \frac{1}{N_{2i-\bar{i}}^n} \left[ e^{\frac{lc_3}{4}} W(1, t) + (-1)^{\bar{i}} e^{-\frac{lc_3}{4}} W(-1, t) \right], \quad i = 1, \dots, k, \quad 2k = n, \quad (19) \end{aligned}$$

where  $\bar{i} = 0$  – for even functions, and  $\bar{i} = 1$  – for uneven functions.

The system (19) is made from  $n$  of non-homogeneous linear equations of the second order with the constant coefficients. The solutions of the characteristic equations will be the numbers  $k_{1,2}^{2i-\bar{i}} = \pm \sqrt{\frac{4c^2}{l^2} \frac{1}{\lambda_{2i-1+\bar{i}}^n} + \frac{c^2 A_3^2}{4}}$ . Since,

$$\begin{aligned} q_0(t) &= q_{0n} + (q_0 - q_{0n}) e^{-\gamma_0 t}, \\ q_l(t) &= q_{ln} + (q_l - q_{ln}) e^{-\gamma_l t}, \end{aligned}$$

then

$$\begin{aligned} e^{\frac{lc_3}{4}} W(1, t) + (-1)^{\bar{i}} e^{-\frac{lc_3}{4}} W(-1, t) \\ = \frac{\rho_s}{s} \left( q_{ln} e^{\frac{lc_3}{4}} + (q_l - q_{ln}) e^{-\gamma_l t + \frac{lc_3}{4}} + (-1)^{\bar{i}} \left( q_{0n} e^{-\frac{lc_3}{4}} + (q_0 - q_{0n}) e^{-\gamma_0 t - \frac{lc_3}{4}} \right) \right). \end{aligned}$$

Therefore, the following functions will be the solutions of the equations of the system (19)

$$G_{2i-\bar{i}}(t) = A_{2i-\bar{i}} e^{\frac{lc_3}{4}} + B_{2i-\bar{i}} e^{-\gamma_0 t - \frac{lc_3}{4}} + E_{2i-\bar{i}} e^{-\gamma_l t + \frac{lc_3}{4}} + D_{2i-\bar{i}} e^{-\frac{lc_3}{4}}, \quad (20)$$

where

$$\begin{aligned} A_{2i-\bar{i}} &= \frac{\frac{\rho_s}{s} q_{ln} \frac{c^2 \pi}{l^2} \frac{\bar{c}_{i+1}^{2i-1+\bar{i}}}{4^i \lambda_{2i-1+\bar{i}}^n} \frac{1}{N_{2i-\bar{i}}^n}}{\frac{4c^2}{l^2} \frac{1}{\lambda_{2i-1+\bar{i}}^n} + \frac{c^2 A_3^2}{4}}, \quad B_{2i-\bar{i}} = (-1)^{\bar{i}} \frac{\frac{\rho_s}{s} (q_0 - q_{0n}) \frac{c^2 \pi}{l^2} \frac{\bar{c}_{i+1}^{2i-1+\bar{i}}}{4^i \lambda_{2i-1+\bar{i}}^n} \frac{1}{N_{2i-\bar{i}}^n}}{\gamma_0^2 + \frac{4c^2}{l^2} \frac{1}{\lambda_{2i-1+\bar{i}}^n} + \frac{c^2 A_3^2}{4}}, \\ E_{2i-\bar{i}} &= \frac{\frac{\rho_s}{s} (q_l - q_{ln}) \frac{c^2 \pi}{l^2} \frac{\bar{c}_{i+1}^{2i-1+\bar{i}}}{4^i \lambda_{2i-1+\bar{i}}^n} \frac{1}{N_{2i-\bar{i}}^n}}{\gamma_l^2 + \frac{4c^2}{l^2} \frac{1}{\lambda_{2i-1+\bar{i}}^n} + \frac{c^2 A_3^2}{4}}, \quad D_{2i-\bar{i}} = (-1)^{\bar{i}} \frac{\frac{\rho_s}{s} q_{0n} \frac{c^2 \pi}{l^2} \frac{\bar{c}_{i+1}^{2i-1+\bar{i}}}{4^i \lambda_{2i-1+\bar{i}}^n} \frac{1}{N_{2i-\bar{i}}^n}}{\frac{4c^2}{l^2} \frac{1}{\lambda_{2i-1+\bar{i}}^n} + \frac{c^2 A_3^2}{4}}. \end{aligned}$$

Thus, we found solutions of the system (9) on the interval  $x \in [-1, 1]$ , namely

$$W(x, t) = e^{-\frac{lc_3}{4}x} \sum_{i=1}^{n+2} V_i^{n+\bar{i}}(x) G_i(t),$$

$$P(x, t) = -\frac{2c^2}{l} e^{-\frac{lc_3}{4}x} \sum_{i=1}^{n+2} \left( -\frac{lc_3}{4} V_i^{n+\bar{i}}(x) + U_{i-(-1)^{\bar{i}}}^{n-\bar{i}-(-1)^{\bar{i}}}(x) \right) \int_0^t G_i(t) dt + P(x, 0),$$

where  $U_{n+2}^n(x) = T_{n+2}''(x)$ ,  $U_{n+1}^{n-1}(x) = T_{n+1}''(x)$ .

If one can return to the variable  $y$  in obtained solutions, it is possible to obtain the solutions of the system (7)

$$\omega(y, t) = e^{-\frac{lc_3}{4} \frac{2y-l}{l}} \sum_{i=1}^{n+2} V_i^{n+\bar{i}}(y) G_i(t),$$

$$p(y, t) = -\frac{2c^2}{l} e^{-\frac{lc_3}{4} \frac{2y-l}{l}} \sum_{i=1}^{n+2} \left( -\frac{lc_3}{4} V_i^{n+\bar{i}}(y) + U_{i-(-1)^{\bar{i}}}^{n-\bar{i}-(-1)^{\bar{i}}}(y) \right) \int_0^t G_i(t) dt + p(y, 0).$$

## 5. Approximate solution

We will find the product  $V(x)G(t)$  in the form

$$V(x)G(t) = \sum_{i=1}^{n+2} V_i^{n+\bar{i}}(x) \sum_{j=0}^{\infty} q_{ij} L_j(t), \quad (21)$$

where  $L_j(t)$  are Laggerr polynomials.

If  $s \rightarrow \infty$ , then

$$V_{xx}''(x)G(t) = \sum_{i=1}^n \frac{d^2 V_i^{n+\bar{i}}(x)}{dx^2} \sum_{j=0}^s q_{ij} L_j(t) + \frac{d^2 V_{n+1}^{n+1}(x)}{dx^2} \sum_{j=0}^s q_{n+1,j} L_j(t) + \frac{d^2 V_{n+2}^n(x)}{dx^2} \sum_{j=0}^s q_{n+2,j} L_j(t),$$

$$V(x)G_{tt}''(t) = \sum_{i=1}^{n+2} V_i^{n+\bar{i}}(x) \sum_{j=0}^s q_{i,j+2} \sum_{k=0}^j (j+1-k) L_k(t). \quad (22)$$

From the boundary conditions (11), considering that  $V_i(-1) = V_i(1) = 0$  [10, 11], we have

$$\frac{e^{\frac{lc_3}{4}} W(1, t) + e^{-\frac{lc_3}{4}} W(-1, t)}{2(n+1)^2} = \sum_{j=0}^s q_{n+2,j} L_j(t), \quad \frac{e^{\frac{lc_3}{4}} W(1, t) - e^{-\frac{lc_3}{4}} W(-1, t)}{2(n+2)^2} = \sum_{j=0}^s q_{n+1,j} L_j(t). \quad (23)$$

If we multiply the latter two equations by  $e^{-t}L_j(t)$  and integrate with respect to  $t$  from 0 to  $\infty$  we can find the unknown coefficients  $q_{n+1,j}$ ,  $q_{n+2,j}$ ,  $j = 1, \dots, s$  for (21). We substitute the expressions (22), (23) into the equation (14) and obtain

$$\frac{4c^2}{l^2} \left( \sum_{i=1}^n \frac{d^2 V_i^{n+\bar{i}}(x)}{dx^2} \sum_{j=0}^s q_{ij} L_j(t) + \frac{d^2 V_{n+1}^{n+1}(x)}{dx^2} \sum_{j=0}^s q_{n+1,j} L_j(t) + \frac{d^2 V_{n+2}^n(x)}{dx^2} \sum_{j=0}^s q_{n+2,j} L_j(t) \right) - \frac{c^2 A_3^2}{4} \sum_{i=1}^{n+2} V_i(x) \sum_{j=0}^s q_{ij} L_j(t) - \sum_{i=1}^{n+2} V_i(x) \sum_{j=0}^s q_{i,j+2} \sum_{k=0}^j (j+1-k) L_k(t) = 0. \quad (24)$$

The equation (24) is multiplied by  $\frac{\bar{V}_i(x)}{\sqrt{1-x^2}}e^{-t}L_j(t)$ ,  $i = 1, \dots, n$ ,  $j = 1, \dots, s$  and integrated with respect to  $t$  from 0 to  $\infty$  and with respect to  $x$  from  $-1$  to  $1$ . We obtain  $i = n$  of the systems, each of them contains  $j = s$  of equations and  $j = s$  of the unknowns, from which we can find the rest of unknown coefficients  $q_{i,j}$ ,  $i = 1, \dots, n$ ,  $j = 1, \dots, s$  for (21). Thus, we have found the solution of the system (9) on the interval  $x \in [-1, 1]$ , namely

$$W(x, t) = e^{-\frac{lc_3}{4}x} \sum_{i=1}^{n+2} V_i^{n+i}(x) \sum_{j=0}^s q_{ij}L_j(t),$$

$$P(x, t) = -\frac{2c^2}{l}e^{-\frac{lc_3}{4}x} \sum_{i=1}^{n+2} \left( -\frac{lc_3}{4}V_i^{n+i}(x) + U_{i-(-1)^i}^{n-i-(-1)^i}(x) \right) \sum_{j=0}^s q_{ij} [L_j(t) - L_{j+1}(t)] + P(x, 0),$$

where  $U_{n+2}^n(x) = T''_{n+2}(x)$ ,  $U_{n+1}^{n-1}(x) = T''_{n+1}(x)$ . We return to the variable  $y$  in the obtained solutions and get the solutions of the system (7)

$$\omega(y, t) = e^{-\frac{lc_3}{4}\frac{2y-l}{l}} \sum_{i=1}^{n+2} V_i^{n+i}(y) \sum_{j=0}^s q_{ij}L_j(t),$$

$$p(y, t) = -\frac{2c^2}{l}e^{-\frac{lc_3}{4}\frac{2y-l}{l}} \sum_{i=1}^{n+2} \left( -\frac{lc_3}{4}V_i^{n+i}(y) + U_{i-(-1)^i}^{n-i-(-1)^i}(y) \right) \sum_{j=0}^s q_{ij} [L_j(t) - L_{j+1}(t)] + p(y, 0).$$

The results of the solution of the system of equations (7) for  $l = 100000$  m,  $\lambda = 0.01$ ,  $z = 0.908$ ,  $R = 500$  Joul/(kg·K),  $D = 1.4$  m,  $T = 300$  K,  $q_0 = 894$  m<sup>3</sup>/s,  $q_{0n} = 993$  m<sup>3</sup>/s,  $q_l = 894$  m<sup>3</sup>/s,  $q_{ln} = 993$  m<sup>3</sup>/s,  $p_0 = 70$  atm,  $p_1 = 58.4$  atm,  $\rho_s = \rho_0 = 0.682$  kg/m<sup>3</sup>,  $v_1 = 6$  m/s,  $v_2 = 12$  m/s,  $\gamma_0 = 0.00069$ ,  $\gamma_l = 0.00075$ ,  $c = 500$  m/s,  $\Delta x = 5000$  m,  $\Delta t = 420$  s.

**Table 1.** Value of mass consumption for the pipeline with length  $l = 100000$  m at different values of time  $t$  and the coordinate  $y_1$  for analytical-approximate solution of the problem from the Item 4 and the coordinate  $y_2$  for approximate solution of the task from the Item 5 at  $n = 10$ , the value  $t$  is indicated in seconds in Table.

t	y							
	10000		30000		60000		90000	
	y <sub>1</sub>	y <sub>2</sub>	y <sub>1</sub>	y <sub>2</sub>	y <sub>1</sub>	y <sub>2</sub>	y <sub>1</sub>	y <sub>2</sub>
420	407.22	407.22	407.42	407.42	407.68	407.68	407.87	407.87
1260	421.70	421.70	421.99	421.99	422.40	422.40	422.77	422.77
2100	429.77	429.77	430.02	430.02	430.39	430.39	430.74	430.74
2940	434.26	434.26	434.46	434.46	434.74	434.74	435.01	435.01
3780	436.77	436.77	436.91	436.91	437.10	437.10	437.30	437.30
4620	438.17	438.17	438.26	438.26	438.39	438.39	438.52	438.52
5460	438.95	438.95	439.01	439.01	439.09	439.09	439.18	439.18
6300	439.38	439.38	439.42	439.42	439.47	439.47	439.53	439.53
7140	439.63	439.63	439.65	439.65	439.68	439.68	439.72	439.71
7980	439.76	439.76	439.78	439.77	439.80	439.79	439.82	439.81
8820	439.84	439.81	439.85	439.80	439.86	439.79	439.87	439.78
9660	439.88	439.70	439.89	439.60	439.89	439.43	439.90	439.26

**Table 2.** Value of pressure for the pipeline with length  $l = 100000$  m at different values of time  $t$  and the coordinate  $y_1$  for analytical-approximate solution of the problem from the Item 4 and the coordinate  $y_2$  for approximate solution of the task from the Item 5 at  $n = 10$ , the value  $t$  is indicated in seconds in Table.

$t$	$y$							
	10000		30000		60000		90000	
	$y_1$	$y_2$	$y_1$	$y_2$	$y_1$	$y_2$	$y_1$	$y_2$
420	68.92	68.92	66.73	66.73	63.31	63.31	59.68	59.68
1260	68.89	68.89	66.71	66.71	63.28	63.28	59.66	59.66
2100	68.86	68.86	66.68	66.68	63.26	63.26	59.64	59.64
2940	68.84	68.84	66.65	66.65	63.23	63.23	59.62	59.62
3780	68.82	68.82	66.64	66.64	63.22	63.22	59.60	59.60
4620	68.81	68.81	66.63	66.63	63.21	63.21	59.59	59.59
5460	68.80	68.80	66.62	66.62	63.20	63.20	59.58	59.58
6300	68.80	68.80	66.61	66.61	63.19	63.19	59.58	59.58
7140	68.80	68.79	66.61	66.61	63.19	63.19	59.58	59.58
7980	68.79	68.78	66.61	66.61	63.19	63.19	59.57	59.58
8820	68.79	68.77	66.61	66.61	63.19	63.19	59.57	59.59
9660	68.79	68.75	66.61	66.61	63.19	63.19	59.57	59.60

## 6. Conclusions

The obtained results confirm effective usage of the built biorthogonal polynomials for solving problems of the mathematical physics. Not only the number of products  $n$  of the corresponding series sum, accuracy of calculation have an essential influence on finding of approximate solution of our problem, but the method of the function by time. If one can find the function by time by means of the analytical method, then we obtain analytical-approximate solution from the Item 4, which helps us to find the solution of system (7) on the random period of time, contrary to the approximate solution from the Item 5, which will be correct only comparable on small intervals of time  $t \leq 2.5$  hour, where the function concerning time is found through the approximate method.

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## Методи знаходження розподілу тиску в трубопроводі

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У праці на базі побудованих авторами біортогональних поліномів запропоновано метод розв'язування задач математичної фізики, зокрема для розрахунку нестационарного руху газу в трубопроводах. Досліджено спосіб розв'язування задачі методом розділення змінних у базисі біортогональних поліномів. Знайдено аналітично-наближений та наближений розв'язки задачі у вигляді суми ряду біортогональних та квазіспектральних поліномів. Проведено порівняльний аналіз між отриманими наближеним та аналітично-наближеним розв'язками. Вивчено вплив параметрів методів, зокрема порядку часткової суми, розрядної сітки та похибки обчислення на точність отриманого розв'язку. Результати обчислень подано у вигляді таблиць.

**Ключові слова:** *спектральні методи, математична модель, нестационарний рух газу, лінеаризація, біортогональні та квазіортогональні поліноми.*

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