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Amplitude equations for activator-inhibitor system with superdiffusion

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The generalized activator-inhibitor model with cubic nonlinearity, in which the classical Laplacian is replaced by fractional operator has been studied. The fractional operator reflects the nonlocal behavior of superdiffusion. A spatially homogeneous, time independent solution has been found and its linear stability was studied. We have also performed a weakly nonlinear analysis and obtained a system of amplitude equations that are the basis for analysing pattern formation as well as parameter regimes for which various steady-state patterns would exist.

Keywords: *reaction-diffusion system, cubic nonlinearity, fractional operator, superdiffusion.*

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1. Introduction

Observations of different spatially nonhomogeneous patterns with complicated symmetries in many physical, chemical, and biological media have made the reaction-diffusion systems to be a subject of numerous investigations [1–4]. Recently, many scientists noticed that the diffusion in real-life systems has got an anomalous character [5–8]. Although the anomaly order appeared to be rather insignificant in the vast majority of examples, there exists a bunch of complex systems, (e.g., composite or amorphous materials, complex micro-emulsions, living tissues, etc.), which call for the models with substantial diffusion anomaly.

The investigation of superdiffusion becomes important because it has been detected experimentally in several systems. In particular, the superdiffusion has been observed in transport in nonhomogeneous rocks [9, 10], turbulent flows [11, 12], optics [13], single-molecule spectroscopy [14], etc.

The effect of superdiffusion on pattern formation and pattern selection in the Brusselator model is studied in [15]. The authors have performed a weakly nonlinear analysis and obtained a system of amplitude equations. The analysis of these equations allowed them to predict the parameter regimes where hexagons, stripes and their coexistence are expected.

Pattern selection in the formation of hexagons and stripes in the activator-inhibitor system with superdiffusion is also studied in [16]. Note that the considered activator-inhibitor model with, however, the normal diffusion, was studied by Dufiet and Boissonade [17] in order to describe the chlorite-iodine-malonic acid reaction. In [16] the linear stability analysis allowed the authors to show, in particular, that the superdiffusive exponent has a significant effect on the wave number of Turing patterns.

Due to the foregoing facts, we can conclude that the investigation of nonlinear dynamics and Turing pattern formation in activator-inhibitor systems with superdiffusion remains to be a very important problem.

It was shown in [18] that by the decrement of fractional derivative order i.e., when the level of anomalous diffusion is essential, the qualitatively different types of spatio-temporal nonlinear dynamics can occur in these systems. There the Brusselator model and the model with cubic nonlinearity were considered.

The aim of this paper is to study the generalized activator-inhibitor model with cubic nonlinearity, in which the classical Laplacian is replaced by a fractional operator (the case of superdiffusion). We focus on the obtaining, by means of a weakly nonlinear analysis, a system of amplitude equations that can serve as a basis for the analysing pattern formation.

2. Mathematical model

We consider the reaction-diffusion model with cubic nonlinearity, in which the classical spatial differential operator is replaced by Δ^α (the operator representing the superdiffusion)

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= D_1\Delta^\alpha u(x,t) + u - \frac{1}{3}u^3 - v, \\ \frac{\partial v(x,t)}{\partial t} &= D_2\Delta^\alpha v(x,t) + u - v + A. \end{aligned} \tag{1}$$

The system (1) must be completed by the following Neumann boundary conditions

$$\begin{aligned} \left.\frac{\partial u}{\partial x}\right|_{x=0} &= \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0, \\ \left.\frac{\partial v}{\partial x}\right|_{x=0} &= \left.\frac{\partial v}{\partial x}\right|_{x=L} = 0, \end{aligned} \tag{2}$$

or periodic boundary conditions

$$\begin{aligned} u|_{x=0} = u|_{x=L}\,, &\quad v|_{x=0} = v|_{x=L}\,, \\ \left.\frac{\partial u}{\partial x}\right|_{x=0} = \left.\frac{\partial u}{\partial x}\right|_{x=L}, &\quad \left.\frac{\partial v}{\partial x}\right|_{x=0} = \left.\frac{\partial v}{\partial x}\right|_{x=L} \end{aligned} \tag{3}$$

with certain initial conditions. Here $u = u(x,t)$ is an activator variable and $v = v(x,t)$ is inhibitor one; D_1 and D_2 are the diffusion coefficients; A and B are the external bifurcation parameters; $x \in [0, L]$ is a space coordinate; t is a time; α is the exponent of fractional operator. Besides, $1 < \alpha < 2$ (the case of superdiffusion).

In one dimension, the fractional operator has the form [19–22]

$$\frac{\partial^\alpha f(x,t)}{\partial x^\alpha} = -\frac{1}{2\cos(\pi\alpha/2)}\left[D_+^\alpha f(x,t) + D_-^\alpha f(x,t)\right],$$

where for $1 < \alpha < 2$

$$\begin{aligned} D_+^\alpha f(x,t) &= \frac{1}{(2-\alpha)}\frac{d^2}{dx^2}\int_{-\infty}^{x}\frac{f(\xi,t)}{(x-\xi)^{\alpha-1}}d\xi, \\ D_-^\alpha f(x,t) &= \frac{1}{(2-\alpha)}\frac{d^2}{dx^2}\int_{x}^{\infty}\frac{f(\xi,t)}{(\xi-x)^{\alpha-1}}d\xi, \end{aligned}$$

or in a form defined by its action in Fourier space $F[\frac{\partial^\alpha f}{\partial x^\alpha}](k) = -k^\alpha F[f](k)$. In higher dimensions, the Laplacian is replaced by the operator [19]

$$\Delta^\alpha \equiv -(-\Delta)^{\alpha/2}(1 < \alpha < 2),$$

defined by its action in Fourier space

$$F[\Delta^\alpha f](\mathbf{k}) = -\mathbf{k}^\alpha F[f](\mathbf{k}),$$

where $(-\Delta)^{\alpha/2}$ is Riesz derivative [19] and $(2-\alpha)$ is the Gamma function.

The spatially homogeneous and stationary solution of the system (1) with the boundary conditions (2) or (3) is obtained as solution of the system of algebraic equations

$$u - \frac{1}{3}u^3 - v = 0,$$

$$u - v + A = 0.$$

So the critical point of the system (1) corresponding to a homogeneous stationary solution, is

$$u_s = \sqrt[3]{-3A}, \quad v_s = \sqrt[3]{-3A} + A.$$

If we consider the deviation of the solution from the critical point

$$U = u - \sqrt[3]{-3A}, \quad V = v - \sqrt[3]{-3A} - A,$$

then, as a result, we can obtain

$$\begin{aligned} \frac{\partial U}{\partial t} &= D_1\Delta^\alpha U + (1 - \sqrt[3]{9A^2})U - V + \sqrt[3]{3A}U^2 - \frac{1}{3}U^3, \\ \frac{\partial V}{\partial t} &= D_2\Delta^\alpha V + U - V. \end{aligned} \tag{4}$$

The critical point is now given by $U = V = 0$. Stability of homogeneous stationary solution of the system can be analyzed by linearization of the system nearby this solution. So we decompose the nonlinear functions in the right-hand sides of system (4) into Taylor series in the vicinity of the critical point $U = V = 0$.

Then the system can be transformed to a linear system which has the form

$$\frac{\partial \mathbf{u}(x,t)}{\partial t} = \widehat{F}(u)\,\mathbf{u}(x,t), \tag{5}$$

where

$$\mathbf{u}(x,t) = \begin{pmatrix} U(x,t) \\ V(x,t) \end{pmatrix}, \quad \widehat{F}(u) = \begin{pmatrix} D_1\Delta^\alpha + 1 - \sqrt[3]{9A^2} & -1 \\ 1 & D_2\Delta^\alpha - 1 \end{pmatrix},$$

$\widehat{F}(u)$ is the Frechet derivative.

3. Linear stability analysis

In order to study the linear stability of the solution $U = V = 0$, we substitute the solution, given in the form

$$\mathbf{u}(x,t) = \begin{pmatrix} U(x,t) \\ V(x,t) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} e^{\lambda t + ikx}, \tag{6}$$

into the linear system (5). As a result, we can obtain the dispersion relation

$$\lambda^2 + \left[\sqrt[3]{9A^2} + (D_1 + D_2)\,k^\alpha\right]\lambda + \sqrt[3]{9A^2} + \left[D_1 - \left(1 - \sqrt[3]{9A^2}\right)D_2\right]k^\alpha + D_1D_2k^{2\alpha} = 0.$$

Here k is the wave number.

We are particularly interested in the Turing stability boundary, which corresponds to $\lambda = 0$. Then the neutral stability curve can be written in the form

$$A = \frac{1}{3}\sqrt{\left(\frac{D_2k^\alpha - D_1k^\alpha - D_1D_2k^{2\alpha}}{1 + D_2k^\alpha}\right)^3}. \tag{7}$$

The curve (7) has a single minimum: (A_{cr}, k_{cr}), where

$$A_{cr} = \frac{1}{3}\left[1 + \frac{D_1}{D_2} - 2\sqrt{\frac{D_1}{D_2}}\right]^{3/2}, \quad k_{cr} = \frac{1}{3}\left[\frac{1}{\sqrt{D_1 D_2}} - \frac{1}{D_2}\right]^{1/\alpha}.$$

For $\lambda = 0$, $k = k_{cr}$, and $A = A_{cr}$ we can introduce the eigenvector $\begin{pmatrix} a \\ b \end{pmatrix}$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{D_1/D_2} \end{pmatrix}.$$

In such manner, we obtained the Turing instability threshold A_{cr} and also the critical value of the wave number k_{cr}, which depends on exponent α.

4. Weakly nonlinear analysis

We perform a weakly nonlinear analysis of the system (4) near the instability threshold to study the pattern formation. We are interested in the formation of hexagons and stripes.

We introduce the slow time $T = \varepsilon^2 t$, and variables U and V as well as the bifurcation parameter A as

$$\begin{aligned} &U \sim \varepsilon U_1 + \varepsilon^2 U_2 + \varepsilon^3 U_3 + \ldots, \\ &V \sim \varepsilon V_1 + \varepsilon^2 V_2 + \varepsilon^3 V_3 + \ldots, \\ &A = A_{cr} + \varepsilon^2 \mu. \end{aligned} \tag{8}$$

Here U_i and V_i $(i = 1, 2, 3)$ are functions of T and x.

Substituting the expansions (8) into the system (4) and collecting like powers of ε, we obtain at orders ε^i $(i = 1, 2, 3)$ the sequence of problems

$$O(\varepsilon): \quad \begin{aligned} &D_1 \Delta^\alpha U_1 + \left(1 - \sqrt[3]{9A_{cr}^2}\right) U_1 - V_1 = 0, \\ &D_2 \Delta^\alpha V_1 + U_1 - V_1 = 0; \end{aligned} \tag{9}$$

$$O(\varepsilon^2): \quad \begin{aligned} &D_1 \Delta^\alpha U_2 + \left(1 - \sqrt[3]{9A_{cr}^2}\right) U_2 - V_2 = -R_2, \\ &D_2 \Delta^\alpha V_2 + U_2 - V_2 = 0; \end{aligned} \tag{10}$$

$$O(\varepsilon^3): \quad \begin{aligned} &D_1 \Delta^\alpha U_3 + \left(1 - \sqrt[3]{9A_{cr}^2}\right) U_3 - V_3 = \frac{\partial U_1}{\partial T} - R_3, \\ &D_2 \Delta^\alpha V_3 + U_3 - V_3 = \frac{\partial V_1}{\partial T}; \end{aligned} \tag{11}$$

where $R_2 = \sqrt[3]{3A_{cr}}\, U_1^2$, $R_3 = -\frac{2}{3}\sqrt[3]{\frac{9}{A_{cr}}}\,\mu U_1 + 2\sqrt[3]{3A_{cr}}\, U_1 U_2 - \frac{1}{3}U_1^3$.

Now our intend is the solutions to linearized system in the form [15] for the description of the appearance of both hexagons and stripes

$$\begin{pmatrix} U_1 \\ V_1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} E, \tag{12}$$

where

$$E = L_1 e_1 + L_2 e_2 + L_3 e_3 + c.c., \tag{13}$$

$$e_1 = e^{ik_{cr}x}, \quad e_2 = e_3 = e^{-\frac{1}{2}ik_{cr}x}.$$

Here, the amplitudes L_1, L_2, L_3 are functions of the slow time T; *c.c.* denotes complex conjugate terms.

The right-hand side R_2 in the $O\left(\varepsilon^2\right)$ problem can be written in the form

$$R_2 = PE^2, \quad P \equiv \sqrt[3]{3A_{cr}}\, a^2,$$

and can be represented as [15]

$$R_2 = \left(E_1 + E_2 + 2E_3 + 2E_4\right) P,$$

where

$$\begin{aligned} E_1 &= L_1^2 e_1^2 + L_2^2 e_2^2 + L_3^2 e_3^2 + c.c., \\ E_2 &= 2\big(|L_1|^2 + |L_2|^2 + |L_3|^2 \big), \\ E_3 &= L_1 L_2^* e_1 e_2^* + L_1 L_3^* e_1 e_3^* + L_2 L_3^* e_2 e_3^* + c.c., \\ E_4 &= L_1 L_2 e_3^* + L_1 L_3 e_2^* + L_2 L_3 e_1^* + c.c. \end{aligned}$$

Here, the asterisk denotes the complex conjugate. It should be noted that the terms proportional to E_4 are secular terms that appear in the $O\left(\varepsilon^2\right)$ problem and are regarded to be small [15, 23]. Therefore they contribute to the solvability condition at $O\left(\varepsilon^3\right)$.

As a result, the solution of the $O\left(\varepsilon^2\right)$ problem has the form

$$\begin{pmatrix} U_2 \\ V_2 \end{pmatrix} = \left[E_1 \begin{pmatrix} U_{21} \\ V_{21} \end{pmatrix} + E_2 \begin{pmatrix} U_{22} \\ V_{22} \end{pmatrix} + 2E_3 \begin{pmatrix} U_{23} \\ V_{23} \end{pmatrix} \right] P,$$

where the coefficients U_{2i}, V_{2i} are

$$\begin{aligned} U_{21} &= \frac{1 + 2^\alpha k_{cr}^\alpha D_2}{(1 + 2^\alpha k_{cr}^\alpha D_2)\,(\sqrt[3]{9A_{cr}^2} + 2^\alpha k_{cr}^\alpha D_1) - 2^\alpha k_{cr}^\alpha D_2}, \\ V_{21} &= \frac{1}{(1 + 2^\alpha k_{cr}^\alpha D_2)\,(\sqrt[3]{9A_{cr}^2} + 2^\alpha k_{cr}^\alpha D_1) - 2^\alpha k_{cr}^\alpha D_2}, \\ U_{22} &= V_{22} = \frac{1}{\sqrt[3]{9A_{cr}^2}}, \\ U_{23} &= \frac{1 + 3^{\alpha/2} k_{cr}^\alpha D_2}{\left(1 + 3^{\alpha/2} k_{cr}^\alpha D_2\right)(\sqrt[3]{9A_{cr}^2} + 3^{\alpha/2} k_{cr}^\alpha D_1) - 3^{\alpha/2} k_{cr}^\alpha D_2}, \\ V_{23} &= \frac{1}{\left(1 + 3^{\alpha/2} k_{cr}^\alpha D_2\right)(\sqrt[3]{9A_{cr}^2} + 3^{\alpha/2} k_{cr}^\alpha D_1) - 3^{\alpha/2} k_{cr}^\alpha D_2}. \end{aligned}$$

Then return to the $O\left(\varepsilon^3\right)$ problem. Putting the solutions U_1, V_1, U_2, V_2 into the right-hand side of this problem, yields

$$R_3 = 2PK_1 EE_1 + 2PK_2 EE_2 + 4PK_3 EE_3 - \frac{1}{3} a^3 E^3 - \frac{2}{3} \sqrt[3]{\frac{9}{A_{cr}}} \mu a E.$$

Here, $K_1 = \sqrt[3]{3A_{cr}} a U_{21}$, $K_2 = \sqrt[3]{3A_{cr}} a U_{22}$, $K_3 = \sqrt[3]{3A_{cr}} a U_{23}$.

We can represent the secular terms in the above products EE_1, EE_2, EE_3 and E^3 in such a form [15]

$$\begin{aligned} &\text{in } EE_1: && L_1 |L_1|^2 e_1 + L_2 |L_2|^2 e_2 + L_3 |L_3|^2 e_3 + c.c. \equiv E_0, \\ &\text{in } EE_2: && 2EF, \quad F = |L_1|^2 + |L_2|^2 + |L_3|^2, \\ &\text{in } EE_3: && EF - E_0, \\ &\text{in } \ \ E^3: && 6EF - 3E_0. \end{aligned}$$

Hence, the right-hand side R_3 can be written as

$$R_3 = -\frac{2}{3}\sqrt[3]{\frac{9}{A_{cr}}}\mu aE + E_0\left(2PK_1 + 4PK_2 - a^3\right) + \left(EF - E_0\right)\left(4PK_2 + 4PK_3 - 2a^3\right).$$

The equations of the system (11) are inhomogeneous. The right-hand sides of these equations contain solutions to the systems of equations of lower orders, namely U_1, U_2 та V_1. Now we use the Fredholm alternative, i.e. the right-hand sides of equations must be orthogonal to vector $\mathbf{U}^+$ that satisfy such an equation

$$\Lambda \cdot \mathbf{U}^+ = 0,$$

where

$$\Lambda = \begin{pmatrix} -D_1 k_{cr}^{\alpha} + 1 - \sqrt[3]{9A_{cr}^2} & 1 \\ -1 & -D_2 k_{cr}^{\alpha} - 1 \end{pmatrix}$$

is the conjugate operator.

The Fredholm alternative can be written as

$$\mathbf{U}^+ \cdot \mathbf{q} = 0, \tag{14}$$

where $\mathbf{q}$ is the vector of the right-hand sides of equations, in particular, in the considered $O\left(\varepsilon^3\right)$ problem it has the form

$$\mathbf{q} = \begin{pmatrix} a\dfrac{\partial E}{\partial T} - R_3 \\ b\dfrac{\partial E}{\partial T} \end{pmatrix}, \tag{15}$$

and the vector $\mathbf{U}^+$ is written by

$$\mathbf{U}^+ = \begin{pmatrix} a^+E \\ b^+E \end{pmatrix}, \quad \begin{pmatrix} a^+ \\ b^+ \end{pmatrix} = \begin{pmatrix} -\sqrt{\frac{D_2}{D_1}} \\ 1 \end{pmatrix}. \tag{16}$$

As a result, using the Fredholm alternative (14), relations (15), (16), and also (13), we obtain the system of amplitude equations

$$\begin{aligned} C_0\frac{\partial L_1}{\partial T} &= \mu C_1 L_1 + C_2 L_2^* L_3^* + C_3 L_1 |L_1|^2 + C_4 L_1\left(|L_2|^2 + |L_3|^2\right), \\ C_0\frac{\partial L_2}{\partial T} &= \mu C_1 L_2 + C_2 L_1^* L_3^* + C_3 L_2 |L_2|^2 + C_4 L_2\left(|L_1|^2 + |L_3|^2\right), \\ C_0\frac{\partial L_3}{\partial T} &= \mu C_1 L_3 + C_2 L_1^* L_2^* + C_3 L_3 |L_3|^2 + C_4 L_3\left(|L_1|^2 + |L_2|^2\right), \end{aligned} \tag{17}$$

where the coefficients C_k, $k = 0, 1, 2, 3, 4$, are given by

$$\begin{aligned} C_0 &= \frac{a^+a + b^+b}{a^+} = \frac{D_2 - D_1}{D_2}, \\ C_1 &= -\frac{2}{3}\sqrt[3]{\frac{9}{A_{cr}}}a = -\frac{2}{\sqrt{1 + \frac{D_1}{D_2} - 2\sqrt{\frac{D_1}{D_2}}}}, \\ C_2 &= 2P = 2\sqrt[3]{3A_{cr}}\,a^2 = 2\sqrt{1 + \frac{D_1}{D_2} - 2\sqrt{\frac{D_1}{D_2}}}, \end{aligned}$$

$$
\begin{aligned}
C_3 &= 2PK_1 + 4PK_2 - a^3 \\
&= \frac{\left(5 - 2^{3+\alpha} + 3\cdot 2^{2\alpha}\right)\left(1 + \frac{D_1}{D_2} - 2\sqrt{\frac{D_1}{D_2}}\right) + 2^{\alpha+1}\left(\sqrt{\frac{D_1}{D_2}} + \sqrt{\frac{D_2}{D_1}} - 2\right)}{\left(1 - 2^{1+\alpha} + 2^{2\alpha}\right)\left(1 + \frac{D_1}{D_2} - 2\sqrt{\frac{D_1}{D_2}}\right)}, \\
C_4 &= 4PK_2 + 4PK_3 - 2a^3 \\
&= \frac{2\left[\left(3 - 4\cdot 3^{\alpha/2} + 3^{\alpha}\right)\left(1 + \frac{D_1}{D_2} - 2\sqrt{\frac{D_1}{D_2}}\right) + 2\cdot 3^{\alpha/2}\left(\sqrt{\frac{D_1}{D_2}} + \sqrt{\frac{D_2}{D_1}} - 2\right)\right]}{\left(1 - 2\cdot 3^{\alpha/2} + 3^{\alpha}\right)\left(1 + \frac{D_1}{D_2} - 2\sqrt{\frac{D_1}{D_2}}\right)}.
\end{aligned}
\tag{18}
$$

In conclusion, by means of a weakly nonlinear analysis, we obtained the system of amplitude equations (17), with coefficients (18). These amplitude equations are present a basis for the analysis of pattern formation. The analysis of these equations can be a matter of further publications.

5. Conclusions

The generalized activator-inhibitor model with cubic nonlinearity, in which the classical Laplacian is replaced by a fractional operator, has been studied. The fractional operator reflects the nonlocal behavior of superdiffusion. The spatially homogeneous, time independent solution has been found and we have also studied its linear stability. We have obtained the Turing instability threshold A_{cr} and also the critical value of the wave number k_{cr}, which depends on superdiffusive exponent α.

We performed a weakly nonlinear analysis and obtained a system of amplitude equations. It should be noticed that the weakly nonlinear analysis gives an indication of what type of patterns to expect as well as parameter regimes for which various steady-state patterns would exist.

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Амплітудні рівняння для системи типу активатор-інгібітор із супердифузією

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Досліджено узагальнену модель типу активатор-інгібітор із кубічною нелінійністю, в якій класичний оператор Лапласа замінено дробовим аналогом. Дробовий оператор відображує нелокальну поведінку супердифузії. Знайдено просторово-однорідний стаціонарний розв'язок та вивчено його лінійну стійкість. Проведено також слабконелінійний аналіз та отримано систему амплітудних рівнянь. Отримані рівняння дають можливість аналізувати типи структур, які виникають у розглядуваній реакційно-дифузійній системі.

Ключові слова: *система реакції-дифузії, кубічна нелінійність, дробовий оператор, супердифузія.*

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