

# ATOMIC AND R-FUNCTIONS IN RADIOPHYSICAL APPLICATIONS

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## REVIEW

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### Abstract

This review consist of two parts. First part of review concerned with atomic functions theory and it's applications in digital signal processing problems, antennas synthesis and sampling theorem. In second part R-functions theory and WA-system functions are considered in applications to boundary-value problems of elliptic type.

**Keywords:** Atomic and R-functions, synthesis and analysis, WA-system functions.

## 1. INTRODUCTION



V.L. Rvachev  
1926-2005

The atomic functions (AFs) began to be studied in 1971, after construction of the function  $up(x)$  (the term «atomic function» was introduced only in 1975 in V.L. and V.A. Rvachev's paper *Atomic Functions in Mathematical Physics*) published in the book *Mathematization of Knowledge and Scientific-Engineering Progress, Kiev: Naukova Dumka, 1975, pp.188-199*.

Subsequently, the theory of AFs

was considered in detail in the following monographs:

- V.L. Rvachev and V.A. Rvachev *Non-classical Methods of Approximation Theory in Boundary-Value Problems, Kiev: Naukova Dumka, 1979.*
- V.L. Rvachev *The R-function Theory and Its Some Applications, Kiev: Naukova Dumka, 1982.*
- V.F. Kravchenko, *Lectures on the Theory of Atomic Functions and Their Some Applications, Moscow, Radiotekhnika, 2003.*
- V.F. Kravchenko and V.L. Rvachev, *Algebra Logic. Atomic Functions and Wavelets in Physical Applications, Fizmatlit, Moscow, 2006.*
- *Digital Signal and Image Processing in Radiophysical Applications, Edited by V.F. Kravchenko, Fizmatlit, Moscow, 2007.*

The history of the function

$$up(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\{-itx\} \prod_{k=1}^{\infty} \frac{\sin(t \cdot 2^{-k})}{t \cdot 2^{-k}} dt$$

is follows. In 1967 V.L. Rvachev stated the next problem. Let  $\varphi(x)$  be a compactly supported differentiable function increasing on one part of its support and decreasing on another part, i.e., having one maximum ("hump"). So, the plot of its derivate consists

of one "hump" and one "hole". The problem is to find a function  $\varphi(x)$  whose derivate's "hump" and "hole" are similar to the "hump" of the function. Mathematically it means the following: does a compactly supported solution to the equation  $y'(x) = a[y(2x+1) - y(2x-1)]$  exist (for definiteness we suppose  $\varphi(x)$  is compactly supported on the interval  $[-1;1]$ )? In V.L. and V.A. Rvachev's work "On One Compactly Supported Function", *Dokl. Akad. Nauk Ukr. SSR, Ser. A, 1971, pp. 705-707*, existence and uniqueness of such a compactly supported solution were proved. This function was denoted by  $up(x)$ . Thus, while the classical algebraic or trigonometric polynomials satisfy homogeneous linear differential equations with constant coefficients, the function  $up(x)$  and other analogous functions

satisfy equations  $Ly(x) = \sum_{k=1}^M c_k y(ax - b_k)$ , where  $L$  is a linear differential operator with constant coefficients. Similarly to linear differential equations with constant coefficients, these equations also can be effectively studied with the use of the Fourier transform. Such equations were studied insufficiently up to now that possibly can be explained by absence of their immediate physical interpretation. It also was shown that AFs take intermediate place between splines and classical polynomials. They are smoother than splines but are non-analytical unlike the polynomials. Similarly to B-splines AFs are compactly supported and similarly to polynomials they are universal from the point of view of their approximation properties (approximate universality). Splines are local but not approximately universal (spline's degree must depend on the smoothness of an approximated function to provide optimal rate of approximation); at the same time they are local unlike the polynomials. Conversely, the latter are universal from the point of view of approximation

theory (namely, analytical functions are approximated by polynomials better than by AFs) but are not compactly supported. From one side, AFs are situated between polynomials and splines and, from another side, splines are closer to polynomials. Therefore, AFs can be considered as a natural extension to the class of elementary functions. Their study is very important now when the compactly supported functions are widely being used. AFs are useful in numerical analysis, in the cases when an approximated function is smooth enough and the use of polynomials is inconvenient due to the fact that they are not compactly supported. Here, to approximate a function of  $n$  variables it is advisable to use the tensor product

$$up(n, x) = \prod_{i=1}^n up(x_i).$$

Possibly, AFs will be popular not only in approximation theory and numerical analysis but also in other branches of mathematics. Recently, investigations in the domain of AFs are carried out in many scientific directions (Fig. 1).

## 2. MAIN PHYSICAL APPLICATIONS OF AF

- Kravchenko-Kotel'nikov generalized series on basis of AF  $h_a(\omega)$  and  $fup_N(\omega)$ .
- Levitan polynomial and Strang-Fix on the basis of AF.
- A new class of WA-systems of Kravchenko-Rvachev functions.
- Some estimates for the spectral density of a time series on basis of the AF family.
- Atomic functions and N-D Whittaker-Kotel'nikov-Shannon Theorem.
- Kravchenko-Kotel'nikov analytical wavelets in digital signal processing.
- Kravchenko-Wigner transformation in nonlinear digital signal processing.
- Kravchenko-Kotel'nikov weight functions in spectroscopy of digital signal processing.
- Analytical wavelets Kravchenko-Kotel'nikov and Kravchenko-Levitan digital ultra wideband signal processing.
- Synthesis of two-dimensional digital filters with non standard geometry of basic area.
- Problems of multidimensional digital signal processing.
- Signal processing antennas.
- A new technique for electronic antennas.
- Constructing of Kravchenko-Kotel'nikov-Gauss and Kravchenko-Levitan-Gauss weight functions.
- New class of wavelets on basis of AF  $h_a(x)$ .

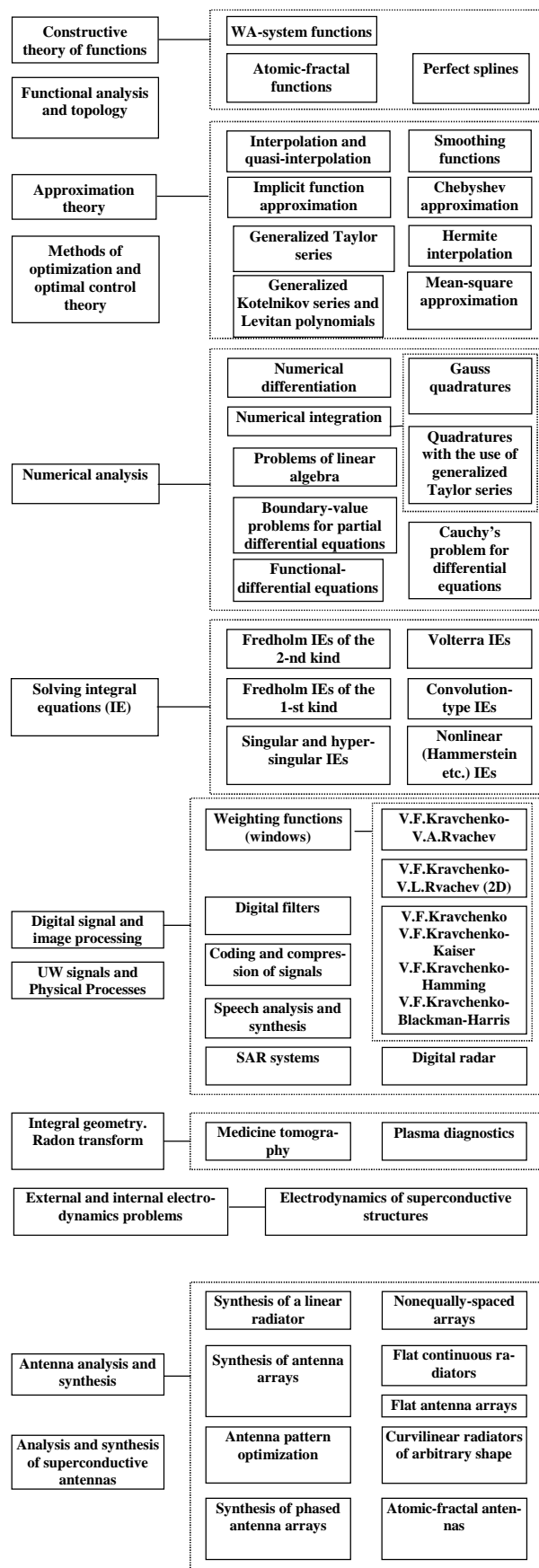


Fig. 1. Main applications of atomic functions.

- Digital processing and spectral estimation of ultra-wideband signals by AF and wavelets.
- Atomic functions of the new of atomic-fractal functions to problems of antenna synthesis.
- Kravchenko and Kravchenko-Rvachev weight functions in construction problems of radar images in aperture synthesizing.
- Atomic functions in the probability theory and stochastic processes.
- The application of R-functions theory and wavelets to the boundary value problems solving of the elliptic type.

### 3. SPECTRAL PROPERTIES OF NEW WEIGHTING FUNCTIONS IN DIGITAL SIGNAL PROCESSING

As is known, one of the main questions, common for all classic problems of signal spectral estimation, is the use of weighting functions (windows). Digital signal processing by means of windows is used in practice for control of physical effects caused by presence of side lobes in spectral estimates. In on the basis of ideas and results presented earlier, a new method for constructing weighting functions is developed and justified. It is based on combination (direct product) of AF  $fup_N(x)$  with classical Gauss, Bernstein, and Dolph-Chebyshev functions (Tables 1 and 2). Characteristics of the new weighting functions as well as of classical Hamming, Blackman-Harris, Natoll, and Kaiser windows are presented. Numerical experiments and physical analysis of results showed that parameters of the new synthesized Kravchenko, Kravchenko-Gauss Kravchenko-Dolph-Chebyshev windows are comparable with those of classical windows, and some of them are even better. These results are basic ones for realization of digital spectral processing of multivariate signals in Doppler radar, synthesized aperture radar, in problems of signal resolution and compression, computer tomography and thermography, and medicine diagnostics.

### 4. ATOMIC FUNCTIONS AND NUMERICAL METHODS OF THE ANTENNAS THEORY SYNTHESIS

In the work **E.G. Zelkin** and **V.F. Kravchenko** (*Atomic Functions in Antenna Synthesis Problems and New Windows, Review, Journal of Communications Technology and Electronics, Vol. 46, No. 8, 2001, pp. 829-857. Translated from Radiotekhnika and Electronica, Vol. 46, No. 8, 2001, pp.903-931*) is devoted to analysis of Atomic Functions (AF) applications to principal problems of antennas theory. It is well known that antenna synthesis refers to

**Table 1.** Main physical parameters of the new Kravchenko windows. List of symbols: K – Kravchenko, KG – Kravchenko-Gauss, KC – Kravchenko–Dolph-Chebyshev

| Weight functions (Windows) | Equivalent noise bandwidth, bin | Overlap correlation (for the 50% overlap), % | Spurious amplitude modulation, dB | Maximum transformation loss, dB | Maximum sidelobe level, dB | 6-dB bandwidth, bin | Coherent gain |
|----------------------------|---------------------------------|--|-----------------------------------|---------------------------------|----------------------------|---------------------|---------------|
| $K_2^4$                    | 1.9861                          | 4.2498                                       | 0.8518                            | 3.8318                          | -51.6112                   | 2.6276              | 0.3610        |
| $K_2^2G_2$                 | 1.8105                          | 7.4054                                       | 1.0259                            | 3.6038                          | -53.7964                   | 2.4255              | 0.3944        |
| $K_2G_3$                   | 1.9643                          | 4.7297                                       | 0.8781                            | 3.8101                          | -68.8390                   | 2.6276              | 0.3614        |
| $K_4^2G_2^2$               | 1.9631                          | 4.7869                                       | 0.8809                            | 3.8103                          | -70.6203                   | 2.6276              | 0.3607        |
| $K_4^4G_2$                 | 1.9696                          | 4.6700                                       | 0.8742                            | 3.8180                          | -71.2806                   | 2.6276              | 0.3598        |
| $K_4^2G_3$                 | 2.0415                          | 3.7429                                       | 0.8156                            | 3.9152                          | -74.8054                   | 2.6276              | 0.3467        |
| $K_4C_{3.5}$               | 1.8007                          | 7.3910                                       | 1.0249                            | 3.5793                          | -74.9523                   | 2.4255              | 0.3988        |

**Table 2.** Main physical parameters of classical windows.

| Weight functions (Windows)    | Equivalent noise bandwidth, bin | Overlap correlation (for the 50% overlap), % | Spurious amplitude modulation, dB | Maximum transformation loss, dB | Maximum sidelobe level, dB | 6-dB bandwidth, bin | Coherent gain |
|-------------------------------|---------------------------------|--|-----------------------------------|---------------------------------|----------------------------|---------------------|---------------|
| Rectangular                   | 1.0000                          | 50.0000                                      | 3.9224                            | 3.9224                          | -13.2799                   | 1.2128              | 1.0000        |
| Triangular                    | 1.3333                          | 25.0001                                      | 1.8242                            | 3.0736                          | -26.5077                   | 1.8191              | 0.5000        |
| Gauss $\alpha=3.5$            | 1.9765                          | 4.6147                                       | 0.8702                            | 3.8292                          | -71.0006                   | 2.6276              | 0.3579        |
| Hamming                       | 1.3638                          | 23.3241                                      | 1.7492                            | 3.0967                          | -45.9347                   | 1.8191              | 0.5395        |
| Blackman-Harris (four-termed) | 2.0044                          | 3.7602                                       | 0.8256                            | 3.8453                          | -92.0271                   | 2.6276              | 0.3587        |
| Natoll (four-termed)          | 1.9761                          | 4.1760                                       | 0.8506                            | 3.8087                          | -97.8587                   | 2.6276              | 0.3636        |
| Dolph-Chebyshev $\alpha=3.5$  | 1.6328                          | 11.8490                                      | 1.2344                            | 3.3636                          | -70.0161                   | 2.2234              | 0.4434        |
| Bernstein-Rogozinskii         | 1.2337                          | 31.8309                                      | 2.0982                            | 3.0103                          | -23.0101                   | 1.6170              | 0.6366        |
| Kaiser $\alpha=3$             | 1.7952                          | 7.3534                                       | 1.0226                            | 3.5639                          | -69.6568                   | 2.4255              | 0.4025        |

inverse problems of mathematical physics. The pioneering results on the mathematical theory of radiating structure synthesis appeared in the late 1930s in works by E.H. Brown, I. Wolf, G.S. Ramm, and A.A. Pistol'kors, where the synthesis problem was solved using the Fourier-series expansion. Later, this problem was traditionally solved using partial patterns, eigen-functions, and the Fourier integral methods. A great contribution to solving antenna synthesis problems was made by G.S. Ramm, A.A. Pistol'kors, A.Z. Fradin, I.I. Vol'man, E.G. Zelkin, Ya.N. Fel'd, L.D. Bakhrakh, and many others. Originally, the Fourier integral method was applied by I.I. Vol'man to the synthesis of linear antennas. This method was developed and rigorously substantiated by E.G. Zelkin. The eigenfunction method was developed by A.Z. Fradin and A.A. Pistol'kors. L.D. Bakhrakh has shown that the main results given by the eigenfunction method can be obtained by the direct solution of the integral equation arising in the synthesis problem. The method of partial patterns is one of the basis methods for solving antenna synthesis problems. In Russia, this method was developed mainly by E.G. Zelkin and, in England, by P. Woodward. The mathematical theory of radiating structure synthesis was strongly influenced by the theory of regularization of ill-posed problems, which was created in the late 1960s by A.N. Tikhonov, V.K. Ivanov, and M.M. Lavrent'ev. The ideals of A.N. Tikhonov were developed further by V.I. Dmitriev, A.V. Chechkin, A.S. Il'inskii, V.I. Popovkin, A.G. Sveshnikov, L.D. Bakhrakh, and S.D. Kremenetskii.

The general antenna synthesis theory was presented in the monograph written by B.M. Minkovich and V.P. Yakovlev. Of certain interests are the investigations performed by A.F. Chaplin on the analysis and synthesis of antenna arrays as well as the work by B.Z. Katselenbaum, A.N. Sivov, and their colleagues. In these works, the synthesis problems are considered as variational problems (the optimum values of the physical antenna characteristics are sought).

The monograph by M.I. Andriyчук, N.N. Voitovich, P.A. Savenko, and V.P. Tkachuk deals with antenna synthesis problems in which antennas of different types are synthesized from a given amplitude pattern. The synthesis problem is formulated as a variational problem with the mean-square deviation criterion or its modifications. Despite a lot of mathematical methods and physical approaches developed by different scientific schools for solving this topical and important problem, many synthesis problems (comprising, e.g., synthesis of nonuniform antenna arrays and mixed synthesis problems) require developing analytical and numerical methods of modern electrodynamics based on system approaches and original ideas. Recently, a new mathematical technique of atomic functions (AFs) has found application in boundary value problems, remote-sensing data processing, image reconstruction, physical

electronics, and antenna analysis and synthesis problems. Now it is a full-blown theory, which is used here to obtain the results presented below. Originally, atomic functions had been applied to the antenna analysis and synthesis problems by V.F. Kravchenko (*Approximation of diagram direction and synthesis of linear radiator on basis of atomic functions, Zarubezhnaya Radioelektronika. Uspekhy Sovremennoy Radioelektroniki, 1996, No. 8, pp. 23-28*). These investigations were continued in (E.G. Zelkin, V.F. Kravchenko, and V.I. Gusevskii, *Constructive Approximation Methods in Antenna Theory, Sains-Press, Moscow, 2005*). Thus, the active applications of AF for the antenna synthesis problems have started.

## 5. R-FUNCTIONS THEORY AND WAVELETS FUNCTIONS IN THE BOUNDARY-VALUE PROBLEMS SOLVING OF THE ELLIPTIC TYPE

Let's consider a new approach of solving boundary value problems for differential equations of the elliptic type partial derivatives is represented. It is based on Galerkin classic variation method which is converted with the help of R-functions structural method (V.F. Kravchenko, V.L. Rvachev. *Boolean Algebra, Atomic Functions and Wavelets in Physical Applications, Moscow, Fizmatlit, 2006*) and wavelet-basis properties. The main point of such approach is the construction of the computational algorithm concerning the wavelet approximation of the analytic and geometric components of the boundary value problem. To convert the geometric information into analytic one as well as to satisfy the boundary conditions of the problem using the structures of solution helps the R-functions body of mathematics. The basic element of the obtained functional is the wavelet-basis (H.L. Resnikoff, R.O. Wells, *Wavelet Analysis: the Scalable Structure of Information, New York, Springer, 1998*), the expansion coefficients of the domain function, the function of the right part of the equation and the function of boundary conditions of the wavelet-basis. As a result while matrix system compiling we obtain some calculating advantages: matrixes of the system are discharged, the calculation of matrix elements does not demand the integration and is carried out with the help of finite number of elementary mathematical operations over the coupling coefficients of the corresponding wavelet system. New fast computational algorithms based on fundamental wavelet properties for coupling coefficients are also introduced and founded in (V.F. Kravchenko, A.V. Yurin, *The application of R-functions theory and wavelets to the boundary value problems solving of the elliptic type, An International Journal Electromagnetic Waves and Electronic Systems, 2009, Vol. 14, No. 3, pp. 4-39*. V.F. Kravchenko,

O.S. Labun'ko, A.M.Lerer, and G.P.Sinyavsky, *Computing Methods in the Modern Radio Physics*, Edited by V.F.Kravchenko, Moscow, Fizmatlit, 2009). The inverse problem of the analytical geometry and the R-function method.

Owing to (V.L.Rvachev and V.A.Rvachev *Non-classical Methods of Approximation Theory in Boundary-Value problems*, Kiev: Naukova Dumka, 1979. V.L.Rvachev *The R-function Theory and Its Some Applications*, Kiev: Naukova Dumka, 1982. V.F.Kravchenko and V.L.Rvachev, *Algebra Logic. Atomic functions and Wavelets in Physical Applications*, Fizmatlit, Moscow, 2006) it is possible to examine the construction of complex shaped locuses at the analytic level. The methods of construction of the function  $(x,y)$  are based on the theory of R-functions. In accordance with this theory, firstly it is necessary to derive the logical formula (predicative equation) for construction the boundary equation for some area. Let area  $S$  be given by  $R^2$  with the sectionally smooth boundary  $S$ . It is necessary to build the function  $(x,y)$  which is positive inside  $S$  negative outside  $S$  and vanishing at  $S$ . The obtained equation  $(x,y)=0$  will define the locus which is represented by the domain boundary. We symbolize the characteristic function which is conformed to the area  $S$  as  $\chi_i = [\omega_i(x, y) \geq 0]$ . Gaining some system  $\chi_i = [\omega_i(x, y) \geq 0]$  of the characteristic functions and Boolean function  $Y=F(x_1...x_m)$  we can build the predicate

$$\chi = F(\chi_1, \dots, \chi_m) = F([\omega_1(x, y) \geq 0], \dots, [\omega_m(x, y) \geq 0])$$

determining the area  $S$  which is built of the  $S_1...S_m$  auxiliary areas in compliance with the logical rules determined by the Boolean function  $F$ .

Proposing that area  $S$  is got from reference areas  $S_1, \dots, S_m$  with the help of the following logical set operations " $\cap$ " is intersection, " $\cup$ " is unification " $\neg$ " is complement note down

$$S = F([S_1, \dots, S_m], [\cap, \cup, \neg]) .$$

But for all that consider that reference areas  $S_1...S_m$  have got simpler shape, than  $S$  and for each of them the boundary equation is given  $\omega_i(x, y) = 0$  ( $i=1...m$ ).

The R-function method gives the opportunity to gain the boundary equation of the area  $S$  in the analytic form from its set is theoretic description  $\omega(x, y) = 0$ .

**The definition 1.** The R-function method ( the function of V.L. Rvachev) corresponding to the discretization of the number scale into the intervals  $(-\infty, 0)$  and  $[0, \infty)$  is called the function the sign of which is completely determined by the sign of its arguments.

**The definition 2.** The function  $z=f(x,y)$  is called R-function if there exists such Boolean function  $S$ , that  $S[z(x,y)] = [S(x), S(y)]$ , where the two-valued predicate is

$$S(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

Each R-function corresponds the only going with it Boolean function. The opposite is not true. The same Boolean function corresponds the infinite set of R-functions. The set of R-functions is closed that means that the superposition of R-functions serves as R-functions as well.

**The definition 3.** The system of functions H which consists of R-functions is called enough complete if the set of all superpositions of the elements H (the set of H-realized functions) obtains nonempty intersection with each branch of the R-function set.

**Table 3.** The main systems of R-functions.

| The attendant Boolean function | The system $R_\alpha$   | The system $R_0^m$   |
|--------------------------------|---|--|
| $F_1(a,b) = a \wedge b$        | $y_1 = x_1 \wedge_\alpha x_2 = \frac{1}{1+\alpha} \cdot (x_1 + x_2 - \sqrt{x_1^2 + x_2^2 - 2\alpha x_1 x_2})$ | $y_1 = x_1 \wedge_0^m x_2 = \left( x_1 + x_2 - \sqrt{x_1^2 + x_2^2} \right) \cdot (x_1^2 + x_2^2)^{m/2}$ |
| $F_2(a,b) = a \vee b$          | $y_2 = x_1 \vee_\alpha x_2 = \frac{1}{1+\alpha} \cdot (x_1 + x_2 + \sqrt{x_1^2 + x_2^2 - 2\alpha x_1 x_2})$   | $y_2 = x_1 \vee_0^m x_2 = \left( x_1 + x_2 + \sqrt{x_1^2 + x_2^2} \right) \cdot (x_1^2 + x_2^2)^{m/2}$   |
| $F_3(b) = \bar{b}$             | $y_3 = \bar{x} = -x$  | $y_3 = \bar{x} = -x$   |

The sufficient condition of the completeness of H system is the completeness of H-system if Boolean functions are corresponding and going with it. Some of the sets of R-functions are given in the Table 3.

The system R which was mentioned in the Table 3

**Table 4.** The  $R_0$  and  $R_1$  systems.

| The system $R_0$   | The system $R_1$  |
|--|---|
| $y_1 \equiv x \wedge_0 y \equiv \left( x + y - \sqrt{x^2 + y^2} \right)$ | $y_1 \equiv x \wedge_1 y \equiv \frac{1}{2} \left( x + y - \sqrt{x^2 + y^2 - 2xy} \right) \equiv \frac{1}{2} (x + y -  x - y ) \equiv \min(x, y)$ |
| $y_2 \equiv x \vee_0 y \equiv \left( x + y + \sqrt{x^2 + y^2} \right)$   | $y_2 \equiv x \vee_1 y \equiv \frac{1}{2} (x + y +  x - y ) \equiv \max(x, y)$  |
| $y_3 \equiv \bar{x} = -x$  | $y_3 \equiv -x$   |

conveys the parameters which can obtain the values from the interval  $(-1, 1]$ . This parameter also can be presented as the function of the coordinates. Especially the special cases are of the following interest:  $R_0$  when  $=0$  and  $R_1$  when  $=1$ . The functions from these systems are given in the Table 4. The system  $R_0^m$  is made of  $C^m$  class function.

Note that there are some disadvantages of the systems  $R_0$  and  $R_1$ . The first one is non-differentiable at the coordinate origin and the second one is non-

differentiable at the bisector of the first and third quadrantal angles.

Consider an example (*Digital Signal and Image Processing in Radio Physical Applications*, Edited by V.F. Kravchenko, Fizmatlit. Moscow, 2007). Let us construct the locus of the pawn. While constructing the locus of the pawn we divide it into the range of the subregions  $\omega_1 \dots \omega_5$  (Fig. 2).

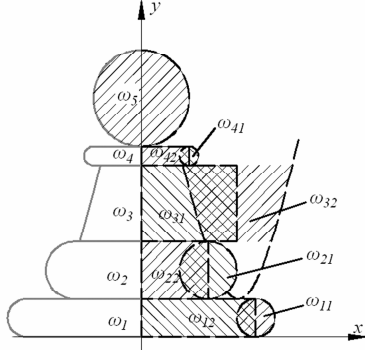


Fig. 2. The image of the pawn.

The function, determining the locus has the following view:

$$\omega_{\text{pawn}}(x, y) = \omega_1^{\text{pawn}}(x, y) \vee_{\alpha} \omega_2^{\text{pawn}}(x, y) \vee_{\alpha} \omega_3^{\text{pawn}}(x, y) \vee_{\alpha} \omega_4^{\text{pawn}}(x, y) \vee_{\alpha} \omega_5^{\text{pawn}}(x, y)$$

For the purpose of the clearness of the axial symmetry it is necessary to delimit the subregions:  $\omega_1^{\text{pawn}}$ ,  $\omega_2^{\text{pawn}}$ ,  $\omega_3^{\text{pawn}}$ ,  $\omega_4^{\text{pawn}}$ ,  $\omega_5^{\text{pawn}}$ . It is carried out by the division of the neighboring region by the parameter  $(\varepsilon = 0, >0)$ .

1.  $\omega_1^{\text{pawn}}$  is constructed from the circle  $\omega_{11}^{\text{pawn}}$  and the rectangle  $\omega_{12}^{\text{pawn}}$

$\omega_{11}^{\text{pawn}}(x, y) = \omega_{11}^{\text{circle}}(x, y) \vee_{\alpha} \omega_{12}^{\text{rect}}(x, y)$   
 $\omega_{11}^{\text{circle}}(x, y) = \frac{1}{2} [1 - (x-6)^2 - (y-1)^2] = 0$  is the normalized first-order circle equation, and

$\omega_{12}^{\text{rect}}(x, y) = \left( \frac{9 - (x-3)^2}{6} \right) \wedge_{\alpha} \left( \frac{1 - (y-1)^2}{2} \right) = 0$  is the normalized first-order rectangle equation.

2.  $\omega_2^{\text{pawn}}(x, y)$  is got in a similar way:

$$\omega_{21}^{\text{pawn}}(x, y) = \frac{1}{3} [2.25 - (x-3.5)^2 - (y-3.5-\varepsilon)^2] = 0,$$

$$\omega_{22}^{\text{pawn}}(x, y) = \left( \frac{3.0625 - (x-1.75)^2}{3.5} \right) \wedge_{\alpha} \left( \frac{2.25 - (y-3.5-\varepsilon)^2}{3} \right) = 0,$$

and

$$\omega_2^{\text{pawn}}(x, y) = \omega_{21}^{\text{pawn}}(x, y) \vee_{\alpha} \omega_{22}^{\text{pawn}}(x, y)$$

3. The construction  $\omega_3^{\text{pawn}}(x, y)$  is the intersection of the sets of the rectangle  $\omega_{31}^{\text{pawn}}$  and the parabola  $\omega_{32}^{\text{pawn}}$ :

$$\omega_{31}^{\text{pawn}}(x, y) = \left( \frac{6.25 - (x-2.5)^2}{5} \right) \wedge_{\alpha} \left( \frac{4 - (y-7-\varepsilon)^2}{14} \right) = 0.$$

For the derivation of the normalized first-order parabola equation  $\omega_{32}^{\text{pawn}}(x, y) = (x-5)^2 - y + 2 = 0$  we use the well-known formula  $\omega(x, y) = \omega_1(\omega_1^2 + |\text{grad}\omega_1|^2)^{-\frac{1}{2}}$ .

$$\text{Then } \omega_{32}^{\text{pawn}}(x, y) = \frac{(x-5)^2 - y + 2}{\sqrt{((x-5)^2 - y + 2)^2 + (2x-11)^2}},$$

$$\omega_3^{\text{pawn}}(x, y) = \omega_{31}^{\text{pawn}}(x, y) \wedge_{\alpha} \omega_{32}^{\text{pawn}}(x, y),$$

$$\omega_4^{\text{pawn}}(x, y) = \omega_{41}^{\text{pawn}}(x, y) \vee_{\alpha} \omega_{42}^{\text{pawn}}(x, y),$$

where

$$\omega_{41}^{\text{pawn}}(x, y) = 0.25 - (x-2.5)^2 - (y-9.5-\varepsilon)^2,$$

$$\omega_{42}^{\text{pawn}}(x, y) = \left( \frac{1.5625 - (x-1.25)^2}{2.5} \right) \wedge_{\alpha} (0.25 - (y-9.5-\varepsilon)^2).$$

$$\omega_5^{\text{pawn}}(x, y) = \frac{1}{5} [6.25 - x^2 - (y-12.5-\varepsilon)^2].$$

Considering the combination of the carried out operations we can deduce the scheme of construction

$$\omega_{\text{pawn}}(x, y) = (\omega_{11}^{\text{pawn}}(x, y) \vee_{\alpha} \omega_{12}^{\text{pawn}}(x, y)) \vee_{\alpha} (\omega_{21}^{\text{pawn}}(x, y) \vee_{\alpha} \omega_{22}^{\text{pawn}}(x, y)) \vee_{\alpha} (\omega_{31}^{\text{pawn}}(x, y) \wedge_{\alpha} \omega_{32}^{\text{pawn}}(x, y)) \vee_{\alpha} (\omega_{41}^{\text{pawn}}(x, y) \vee_{\alpha} \omega_{42}^{\text{pawn}}(x, y)) \vee_{\alpha} \omega_5^{\text{pawn}}(x, y).$$

We obtain quite a bulky expression. Nevertheless the combining function describing the pawn is represented by the class of elementary functions: by the circle, the line and parabola functions. For building another side of the pawn it is necessary to use the transformation of coordinates. As the function  $\omega_{\text{pawn}}(x, y)$  must be symmetrical with regard to the ordinate axis then it must be even endwise the abscissa as well. Thus, having carried out the transformation  $\omega_{\text{pawn}}^{\text{sym}}(x, y) = \omega_{\text{pawn}}(-x, y)$  we shall obtain the locus which is symmetrical with regard to the ordinate axis.

For the description of the whole chess piece it is necessary to combine the initial area and it's symmetrical one by R-operations:

$$\omega_{\text{pawn}}^{\text{completed}}(x, y) = \omega_{\text{pawn}}(x, y) \vee_{\alpha} \omega_{\text{pawn}}^{\text{sym}}(x, y) = \omega_{\text{pawn}}(x, y) \vee_{\alpha} \omega_{\text{pawn}}(-x, y).$$

Thus, derived function is vanishing on the symmetric axis. For the removal of the "special point" it is necessary to move the locus  $\omega(x, y)$  over the ordinate axis on some minor value  $(\varepsilon > 0, \varepsilon = 0)$  and then to carry out the transformation of the symmetry:

$$\omega_{\text{pawn}}^{\text{completed}}(x, y, \varepsilon) = \omega_{\text{pawn}}(x - \varepsilon, y) \vee_{\alpha} \omega_{\text{pawn}}(-x + \varepsilon, y).$$

Therefore the value of the gained function about the point  $x=0$  will be non-zero. At the same time  $\lim_{\substack{\varepsilon \rightarrow 0 \\ x \rightarrow 0}} \omega_{\text{pawn}}^{\text{completed}}(x, y, \varepsilon) = 0$  this fact leads to the deformation

of the locus. For the correction of the value of the function  $\omega_{\text{pawn}}^{\text{completed}}(x, y, \varepsilon)$  it is necessary to multiply it by function  $\chi(x)$  which would be equal to zero under all  $x$  except about the point  $x=0$  and  $\lim_{x \rightarrow 0} \chi(x) = \infty$ . As the result we gain

$$\lim_{\varepsilon \rightarrow 0} \chi(x) \cdot \omega_{\text{pawn}}^{\text{completed}}(x, y, \varepsilon) = \omega_{\text{pawn}}^{\text{completed}}(x, y).$$

Now the function  $\chi(x) = \frac{x^2 + 1}{x^2}$  meets the given demands is even and therefore,  $\chi(x) \cdot \omega_{\text{pawn}}^{\text{completed}}(x, y, \varepsilon)$  - is R-function. Then

$$\omega_{\text{pawn}}^{\text{completed}}(x, y) \Big|_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon > 0}} = \frac{x^2 + 1}{x^2} \left[ \omega_{\text{pawn}}(x - \varepsilon, y) \vee_{\alpha} \omega_{\text{pawn}}(-x + \varepsilon, y) \right].$$

The built chess piece is represented on the Fig. 3.

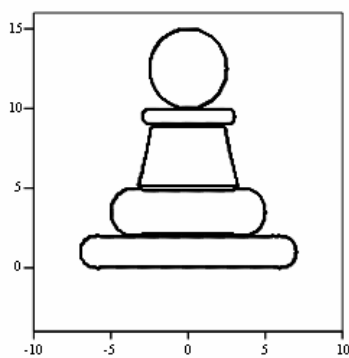


Fig. 3. The locus  $\omega_{\text{pawn}}^{\text{completed}}(x, y)$ .

### The Solution of Boundary-Value Problems with help V.L. Rvachev functions and WA-system functions.

With R-functions method in the mathematical physics theory for complex domain appears the possibility of creating a constructive mathematical tool, which incorporates the capabilities of classical continuous analysis and logic algebra. This allows one to overcome the main obstacle which hinders the use of variational methods when solving boundary value problems in domains of complex shape with complex boundary conditions, this obstacle being connected with the construction of so-called coordinate sequences. In contrast to widely used methods of the network type (finite difference, finite and boundary elements), in the R-functions method all the geometric information present in the boundary value problem statement is reduced to analytical form, which allows one to search for a solutions in the form of formulae called solution structures containing some indefinite functional components. A method (Fig.4.) of constructing solution structures satisfying the required conditions of completeness has been developed. The structural formulae include the left-hand sides of the normalized equations of the boundaries of the domains or their regions being considered, thus allowing one to change the solution structure expeditiously when changing the geometric shape. Given in the work is a definition of the basis class of R-functions, solution with their help of the inverse problem of analytical geometry. Thus, the main idea of the RFM consists in two steps. In the first one, which is of an analytical nature, the general solution of the problem is generated in the form of a so-called “general structure of the solution” (GSS) which strictly satisfies all the prescribed boundary conditions and contains some undetermined functions. In the second step these functions are found by means of any known numerical method in order to satisfy the governing differential equation or to minimize a corresponding functional. Let us find the solution  $u$  of the differential equation

$$Au=f \quad (f \in H) \quad (1)$$

inside the bounded domain  $\Omega \subset \mathbb{R}^2$  with given boundary conditions

$$L_i u|_{\partial\Omega} = g_i, \quad i = \overline{1, N} \quad (2)$$

on the parts  $\Gamma_i$  of the boundary  $\partial\Omega$ . In Eq. 1  $A$  is the elliptic differential operator, the range of definition  $D(A)$  and actual range  $R(A)$  of which are dense subsets of the given Hilbert space  $H$ ;  $L_i$  are the operators of the boundary conditions;  $f, g_i$  are known functions. The mentioned in the problem definition Eq. 1, Eq. 2 functions  $u, f, g$ , and operators  $A, L$  are analytic components of the boundary problem, and the domain  $\Omega$  and its boundary  $\partial\Omega$  are geometric objects.

The R-function method is by its essence the constructive solution of the inversion of the geometric information into analytic one without approximation. Its basis is the formation of the solution structure

$$u = B\left(\Phi, \omega, \{\omega_i\}_{i=1}^N, \{g_i\}_{i=1}^N\right) \quad (3)$$

which at any selection of the indefinite component satisfies the boundary conditions Eq. 2. Here,  $B$  is the operator, depending on the geometry of domain  $\Omega$  and the boundary parts  $\Gamma_i$  as well as on the functions  $g_i$  and operators of boundary conditions  $L_i$ , but it does not depend on the kind of operator  $A$  and function  $f$ . Thus, the structure of solution Eq. 3 determines the sheaf of functions, meeting the demands Eq. 2. Having the solution structure Eq. 3 which takes into consideration the given edge conditions Eq. 2 it is necessary to choose only the component  $\omega$  for the best satisfaction of the fundamental equation, parts of the boundary conditions, the simplicity of the numerical realization or for the other demands, which are made towards the algorithm on the whole. For most of the approximate methods the expression of  $u$  as the finite series

$$\Phi(x, y) \approx \Phi_N(x, y) = \sum_{n=1}^N c_n \phi_n(x, y) \quad (4)$$

along some implicit basic function system is typical. For example, one can take the orthogonal polynomials (trigonometrical, power, Tchebyshev of I and II types, Legendre, Gegenbauer, atomic functions  $up(t), fup_2(t)$ , orthogonal wavelets Daubechies, Coifman, Kravchenko), which belong to functions with the infinite carrier, or finite functions (splines, atomic functions). The matter of the extent of the advantage of the chosen basis in R-function method is one of the dominant which influences the quality of the obtained solution. With the help of substitution Eq. 4 in Eq. 3 we obtain the following structural formula:

$$u(x, y) = B_1\left(x, y, \{c_n\}_{n=1}^N\right), \quad (5)$$

and for the linear edge conditions

$$u(x, y) = \sum_{n=1}^N c_n \chi_n(x, y) + \chi_0(x, y), \quad (6)$$

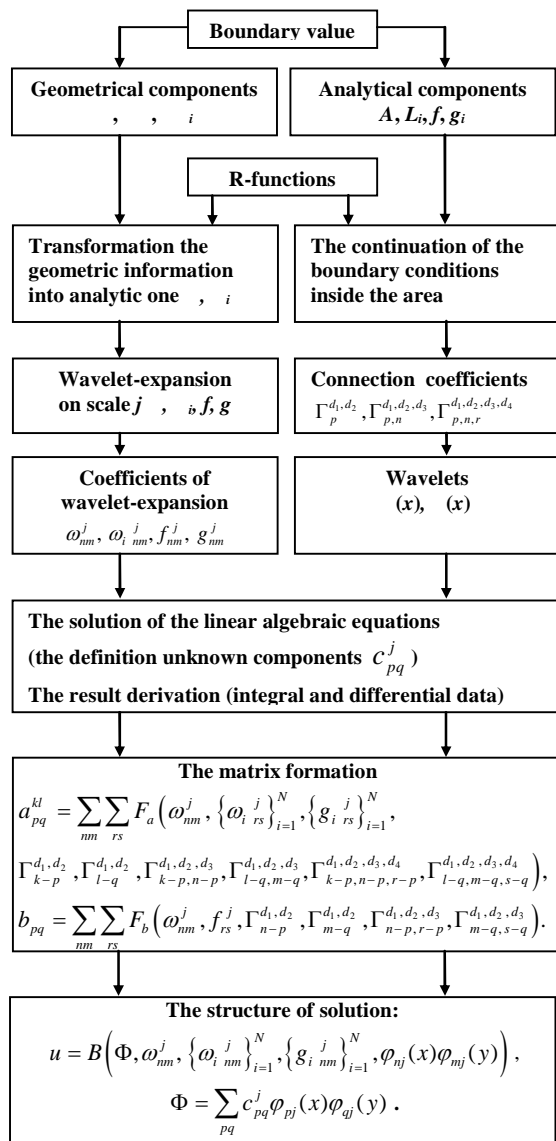
where  $\{\chi_n(x, y)\}_{n=1}^N$  is the known functional sequence;  $\chi_0(x, y)$  is the function, considering the heterogeneous boundary conditions. Functions Eq. 5, Eq. 6 are also the decision structures of the

corresponding boundary value problem, but the unknown component there is vector  $\mathbf{C} = (c_1, c_2, \dots, c_n)$ .

The structures of the elliptic differential equation solution in the second-order derivatives (Laplace, Poisson, Helmholtz) for basic types of boundary conditions are represented in Table 5.

**Table 5.** Main types of boundary conditions and corresponding solution structures.

| Boundary conditions  | Solution structures                           |
|--|---|
| $u _{\partial\Omega} = \varphi$<br>(Dirichlet condition)   | $u = \omega\Phi + \varphi$                    |
| $\partial u / \partial \mathbf{n} _{\partial\Omega} = \varphi$<br>(Neumann condition)  | $u = (1 - \omega D)\Phi - \omega\varphi$      |
| $(\partial u / \partial \mathbf{n} + hu) _{\partial\Omega} = \varphi$<br>(3 <sup>rd</sup> -kind condition)   | $u = [1 + \omega(h - D)]\Phi - \omega\varphi$ |
| Notes: $D = \left( \frac{\partial \omega}{\partial x} \frac{\partial}{\partial x} + \frac{\partial \omega}{\partial y} \frac{\partial}{\partial y} \right) = \frac{\partial}{\partial \mathbf{n}}$ ,<br>for condition 2 and 3 equation $(x, y)$ is normalized. |   |



Thus, in the **R-function** method it turned out to be rather simple:

- $\omega$  constructing
- to change a domain from one form to another (with the change of corresponding function)
- to change over from one kind of boundary condition to other ones (a change of the solution structure corresponds to this)
- to use the obtained results when solving other problems (eg, when a sequence of the problems is considered, viz, at first the temperature field is found, and then the stress field caused by it is found, or a nonlinear problem is solved as a sequence of linear problems)
- to use integral transforms at intermediate stages (eg, when solving non-stationary problems or reducing the solution of a 3D problem for prismatic bodies to 2D problems)
- to use various combinations of variational and difference methods, et. al.

## 6. CONCLUSIONS

In such a brief summary it is impossible to consider the capabilities of the theory AF. Attention here was focused primarily on the application of atomic to signal processing, image reconstruction, which is also the subject of the recent publications. However, the potential capabilities of this mathematical tool are quite diverse and they can find wide application in various boundary value problems in radiophysics.

The theory of R-functions allows one to develop a mathematical apparatus which combines the possibilities of classical continuous analysis and Boolean algebra. This made it possible to overcome the main obstacle to the use of variational methods in solving boundary value problems in domains with complex geometry. Variational methods which appeared well before net methods at the times of computerization gave way to numerical methods in which the geometry is taken into account by its discrete optimization. In contrast to the latter, in RFM all the geometric information contained in the mathematical models of the fields is transformed to the analytical form, allowing one to seek the solution in the form of formulae called solution structures, and to construct approximation sequences with the use of variation, projection, and, in principle, any other methods. In this respect, RFM is a type of amplifier for many methods, this being testified by its use in the finite element, boundary element, and finite difference methods. In the RFM, geometric information may be included in the solution structure in alphabetic form. This is especially important from the point of view of computer implementation of these methods, since the worked out and debugged programs for one collection of values of these parameters are the same simultaneously for a multitude of other values. This made it possible

**Fig. 4.** Diagram of direct methods realization.



(especially after the development of cortege algebra) to develop the highly intelligent POLYE system and then the RL language oriented to the solution of boundary problems in applied electrodynamics, thermal physics problems, computerized tomography, medicine termography, antenna analysis and synthesis, computational geometry, fractal design etc.

## ACKNOWLEDGMENTS

The investigations were supported by grant NSh-5708.2008.9.

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