A Fitted Numerical Method for Singularly Perturbed Integro Differential Equations with Delay

Fevzi Erdogan[1] , Mehmet Giyas Sakar[2]

1. Yuzuncu Yil University, Faculty of Sciences, Department of Mathematics, Van, Turkey, E-mail:ferdogan@yyu.edu.tr

2. Yuzuncu Yil University, Faculty of Sciences, Department of Mathematics, Van, Turkey, E-mail: giyassakar@hotmail.com

***Abstract* – This study deals with the singularly perturbed initial value problems for a quasilinear first-order integro-differential equations with delay. A numerical method is generated on a grid that is constructed adaptively from a knowledge of the exact solution, which involves appropriate piecewise-uniform mesh on each time subinterval. An error analysis shows that the discrete solutions are uniformly convergent with respect to the perturbation parameter. The parameter uniform convergence is confirmed by numerical computations.**

Keywords – Singularly perturbed problems, integro-differential equation, difference schemes,uniformly convergent.

I. Introduction

Consider an initial value problem for the linear second order singularly perturbed integro-differential equation with delay

$$\begin{aligned} &e u'(t)+f(t,u(t),u(t-r))+ \\ &+\int_0^t K(t,s)u(s-r)ds=0 \end{aligned} \quad t\in I, \qquad (1)$$

$$u(t)=j(t),\ t\in I_0, \qquad (2)$$

where $I=(0,T], I_0=(-r,0]$. $0<e\leq 1$ is the perturbation parameter, $a(t)\geq a>0$, $f(t)$ and $j(t)$ are given sufficiently smooth functions satisfying certain regularity conditions to be specified and r is a constant delay.

Volterra delay-integro-differential equations (VDIDEs) arise widely in scientific fields such as biology, ecology, medicine and physics This class of equations plays an important role in modelling diverse problems of engineering and natural science, and hence has led researchers to develop a theory and numerical analysis for VDIDEs.

Delay differential equations play an important role in themathematical modelling of various practical phenomena in thebiosciences and control theory. Any system involving a feedback control will almost always involve time delays. These arise because a finite time is required to sense information and then react to it. A singularly perturbed delay differential equation is an ordinary differential equation in which the highest derivative is multiplied by a small parameter and involving at least one delay term [1-4]. Such problems arise frequently in the mathematical modelling of various practical phenomena, for example, in the modelling of several physical and biological phenomena like the optically bistable devices [5], description of the human pupil-light reflex [6], a variety of models for physiological processes or diseases and variational problems in control theory where they provide the best and in many cases the only realistic simulation of the observed phenomena[7].An overview of numerical treatment for first and second order singularly perturbed delay differential equations, may be obtained in [8-15](see,also references therein).

The numerical analysis of singular perturbation cases has always been far from trivial because of the boundary layer behavior of the solution. Such problems undergo rapid changes within very thin layers near the boundary or inside the problem domain. It is well known that standard numerical methods for solving singular perturbation problems do not give satisfactory result when the perturbation parameter is sufficiently small. Therefore, it is important to develop suitable numerical methods for these problems, whose accuracy does not depend on the perturbation parameter, i.e. methods that are uniformly convergent with respect to the perturbation parameter [16-19].

In a singularly perturbed delay differential equation, one encounters with two difficulties, one is because of occurrence of the delay term and another one is due to presence of perturbation

parameter. To overcome the first difficulty, we employed the numerical method of steps [2] for the delay argument which converted the problem to a initial value problem for a singularly perturbed differential equation. Then we constructed a numerical scheme based on finite difference method on non uniform Shishkin mesh for the numerical solution.

In the present study we discretize the problem(1)-(2) using a numerical method, which is composed of a fitted difference scheme on piecewise uniform Shishkin mesh on each time-subinterval.

II. Discretization and Mesh

In this section,we construct a numerical scheme for solving the initial value problem(1)-(2).

We propose the following difference scheme for approximation (1)-(2)

$$\begin{aligned} &L_N y_i = e\frac{y_i-y_{i-1}}{h}+f(t_i,y_i,y_{i-N}) \\ &+\sum_{j=0}^{i-1} h_{j+1}K(t_i,t_j)y_{j-1}=0, \end{aligned} \quad ,\ i=1,2,...,N_0, \qquad (3)$$

$$y_i=j_i,\ -N\leq i\leq 0, \qquad (4)$$

The difference scheme (3)-(4), in order to be $e-$uniform convergent, we will use the Shishkin mesh. For the even number N , the piecewise uniform mesh $w_{N,p}$ divides each of the interval $[r_{p-1},s_p]$ and $[s_p,r_p]$ into N/2 equidistant subintervals, where the transition point s_p, which separates the fine and coarse portions of the mesh is obtained by $s_p=r_{p-1}+\min\{r/2,a^{-1}e\ln N\}$.

III. Convergence Analysis

We now estimate the approximate error $z_i = y_i - u_i$, which satisfies the discrete problem

$$L_N y_i = R_i,\ i = 0,1,2,...,N_0, \quad (5)$$

$$z_i = 0, -N \le i \le 0, \ , \quad (6)$$

where R_i is the truncation errors.

Theorem. The continuously differentiable function f(t,u,v) satisfies the regularity conditions and the derivative $f(t,u,v)$ is bounded for $\frac{\partial}{\partial t} f(t,u,v)$ is bounded for given interval. Then the following estimate holds

$$|y_i - u_i| \le CN^{-1} \ln N,\ 0 \le i \le N_0,$$

where u_i is exact solution of the problem (1)-(2) and u_i is the solution of the problem (3)-(4).

IV. Numerical Results

In this section, a simple numerical example is devised to verify the validity for the proposed method.

Consider the test problem

$$eu'(t) + u(t) + u(t-1) + \int_0^t u(s-1)ds = 0, t \in (0,2],$$
$$u(t) = 1,\ -1 \le t \le 0.$$

The exact solution given by

$$u(t) = \begin{cases} e^{-t}, t \in [0,1] \\ \frac{1}{e^2} e^{-t/e}[e e^{2/e} - e e^{(t+1)/e} - e^{t/e}(t-e)e^2 \\ +e^{1/e}(-1+t+e-et+2e^2-e^3)], t \in [1,2] \end{cases}$$

We define the exact errors $e^{N,p}$ as follows

$$e^{N,p} = \|y-u\|_{w_{N,p}},\ p = 1,2$$

where y is the numerical approximation of u for values of N and e.

Maximum errors and rates of convergence .

V. Numerical Results

In this section, a simple numerical example is devised to verify the validity for the proposed method.

Consider the test problem

$$eu'(t) + u(t) + u(t-1) + \int_0^t u(s-1)ds = 0, t \in (0,2],$$
$$u(t) = 1,\ -1 \le t \le 0.$$

The exact solution given

by $u(t) = \begin{cases} e^{-t}, t \in [0,1] \\ \frac{1}{e^2} e^{-t/e}[e e^{2/e} - e e^{(t+1)/e} - e^{t/e}(t-e)e^2 \\ +e^{1/e}(-1+t+e-et+2e^2-e^3)], t \in [1,2] \end{cases}$

We define the exact errors $e^{N,p}$ as follows

$$e^{N,p} = \|y-u\|_{w_{N,p}},\ p = 1,2$$

where y is the numerical approximation of u for values of N and e.

Maximum errors and rates of convergence .

e	N=64	N=128	N=256
2^{-2}	0,01328142	0,0640103	0,0322010
2^{-8}	0,01361427	0,0692344	0,0330323
2^{-16}	0,01482356	0,0713864	0,0342657

Conclusion

In this study we have presented a numerical approach to solve a quasilinear singularly perturbed first-order integro-differential equation with delay. We proposed a fitted difference scheme on piecewise-uniform Shishkin mesh on each time subinterval. We have shown that the method displays uniform convergence with respect to the perturbation parameter for numerical approximation of the solution.

References

[1] R. Bellman, K.L. Cooke, Differential-Difference Equations, Academy Press, New York, 1963.

[2] R.D. Driver, Ordinary and Delay Differential Equations, Belin-Heidelberg, New York, Springer, 1977.

[3] A. Bellen, M. Zennaro, Numerical methods for delay differential equations, Oxford University Press, Oxford, 2003.

[4] Y. Kuang, Delay Differential Equations with Applications in Population Dynamics, Academic Press, New York, 1993.

[5] S.N.Chow, J.Mallet-Paret, Singularly perturbed delay-differential equations, in: J.Chandra, A.C.Scott (Eds.), Coupled Nonlinear Oscillators, North-Holland, Amsterdam, 1983, pp.7-12.

[6] A. Longtin, J. Milton, Complex oscillations in the human pupil light reflex with mixed and delayed feedback, Math. Biosci. 90 (1988)183-199.

[7] M.C. Mackey, L. Glass, Oscillation and chaos in physiological control systems, Science, 197(1977)287-289.

[8] G.M. Amiraliyev, F. Erdogan, Uniform numerical method for singularly perturbed delay differential equations, J.Comput. Math. Appl. 53(2007)1251-1259.

[9] G.M. Amiraliyev, F. Erdogan, Difference schemes for a class of singularly perturbed initial value problems for delay differential equations, Numer. Algorithms, 52, 4(2009) 663-675.

[10] I.G. Amiraliyeva, Fevzi Erdogan, G.M.Amiraliyev, A uniform numerical method for dealing with a singularly perturbed delay initial value problem, Applied Mathematics Letters, 23,10(2010)1221-1225.

[1] S. Maset, Numerical solution of retarded functional differential equations as abstract Cauchy problems, J. Comput. Appl. Math. 161(2003)259-282.

[11] B.J.MacCartin, Exponential fitting of the delayed recruitment/renewal equation. J. Comput. Appl. Math., 136(2001)343-356.
[12] M.K. Kadalbajoo, K.K. Sharma, $\varepsilon-$uniform fitted mesh method for singularly perturbed differential difference equations with mixed type of shifts with layer behavior, Int. J. Comput. Math. 81 (2004)49-62.
[13] C.G. Lange, R.M. Miura, Singular perturbation analysis of boundary-value problems for differential difference equations, SIAM J. Appl. Math. 42(1982)502-531.
[14] H. Tian, The exponential asymptotic stability of singularly perturbed delay differential equations with a bounded lag, J. Math. Anal. Appl. 270(2002)143-149.
[15] E. R. Doolan, J.J.H. Miller, and W. H. A. Schilders, Uniform Numerical Methods for Problems with Initial and Boundary Layers, Boole, Press, Dublin, (1980).
[16] P.A. Farrell, A.F. Hegarty, J.J.H. Miller, E.O'Riordan and G.I.Shishkin, Robust Computational Techniques for Boundary Layers, Chapman-Hall/CRC, New York, (2000).
[17] H.G. Roos, M. Stynes and L. Tobiska, Numerical Methods for Singularly Perturbed Differential Equations, Convection-Diffusion and Flow Problems, Springer Verlag, Berlin, (1996).
[18] G.M. Amiraliyev and H. Duru, A note on a parameterized singular perturbation problem, J.Comput. Appl. Math. 182 (2005) 233-242.
[19] G.M. Amiraliyev, The convergence of a finite difference method on layer-adapted mesh for a singularly perturbed system, Applied Mathematics and Computation, 162,3(2005)1023-1034