Comparative Analysis of Simulation Methods of the fractional Brownian motion

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Abstract – This paper analyzes the statistical simulation algorithms of generalized Wiener process and increases of the generalized Wiener process. Models built with specified accuracy and reliability in space $L_2(T)$. To build statistical models we use various spectral representation of random processes – namely in the series and as integrals.

The advantages and disadvantages of each representations was compared.

Key words – Gaussian random processes, simulation, model accuracy, reliability models, generalized Wiener process, random process with independent increments, processes with stationary increments, spectral representation.

I. Basic concepts

Let (T, B, μ) - be some measurable space and $\mu(T) = 1$.

Definition 1. Generalized Wiener process (fractional Brownian motion) with Hurst index $\alpha \in (0,1)$ is the Gaussian random process $W_\alpha(t),\ t \in [0,T]$ with correlation function $R_\alpha(t,s) = \frac{1}{2}\left(|t|^{2\alpha} + |s|^{2\alpha} - |t-s|^{2\alpha}\right)$, and $W_\alpha(0) = 0$, $EW_\alpha(t) = 0$.

If $\alpha = \frac{1}{2}$ have a standard Wiener process.

Generalized Wiener process can be represented as a series[1]

$$W_\alpha(t) = \sum_{k=1}^{\infty}\left(a_k \sin(x_k t) X_k + b_k (1 - \cos(y_k t)) Y_k\right), \quad (1)$$

where $\{X_k, Y_k\}$- independent standard Gaussian random variables,

$\{x_k\}$- real zeros Bessel functions $J_{-\alpha}(x)$,

$\{y_k\}$- real zeros Bessel functions $J_{1-\alpha}(x)$,

$$a_k = \frac{\pi^\alpha \sqrt{2C}}{x_k^{\alpha+1} J_{1-\alpha}(x_k)}, \qquad b_k = \frac{\pi^\alpha \sqrt{2C}}{y_k^{\alpha+1} J_{-\alpha}(y_k)},$$

$$C = \frac{\Gamma(2\alpha+1)\sin(\pi\alpha)}{\pi^{2\alpha+1}}.$$

Zeros of Bessel functions can be calculated with the required accuracy [2]

$$x_n = \left(n + \frac{3}{4} - \frac{\alpha}{2}\right)\pi - \frac{4\alpha^2 - 1}{2\pi(4n + 3 - 2\alpha)} + \ldots$$

$$y_n = \left(n + \frac{5}{4} - \frac{\alpha}{2}\right)\pi - \frac{4(1-\alpha)^2 - 1}{2\pi(4n + 1 + 2\alpha)} + \ldots .$$

For Bessel functions will use the representation

$$J_{1-\alpha}^2(x_n) = \sqrt{\frac{2}{\pi x_n}}\left(\cos\left(x_n + \frac{2\alpha\pi - \pi}{4}\right) - \frac{4\alpha^2 - 1}{8x_n}\sin\left(x_n + \frac{2\alpha\pi - \pi}{4}\right)\right)$$

$$J_{-\alpha}^2(y_n) = \sqrt{\frac{2}{\pi y_n}}\left(\cos\left(y_n + \frac{2(1-\alpha)\pi - \pi}{4}\right) - \frac{4(1-\alpha)^2 - 1}{8y_n}\sin\left(y_n + \frac{2(1-\alpha)\pi - \pi}{4}\right)\right)$$

The model of random process can be obtained as

$$S_\alpha(t,M) = \sum_{k=1}^{M}\left(a_k \sin(x_k t) X_k + b_k (1 - \cos(y_k t)) Y_k\right),$$

where $\{X_k, Y_k\}$ - uncorrelated strictly sub-Gaussian random variables. Properties of strictly sub-Gaussian random variables and processes studied in [3].

Zeros Bessel functions just can not find, the ywill find with some accuracy. For a_k, b_k, x_k, y_k let approximate values $\breve{a}_k, \breve{b}_k, \breve{x}_k, \breve{y}_k$.

Let $|a_k - \breve{a}_k| \le h_k^a$, $|b_k - \breve{b}_k| \le h_k^b$,

$|x_k - \breve{x}_k| \le h_k^x$, $|y_k - \breve{y}_k| \le h_k^y$,

where $h_k^a, h_k^b, h_k^x, h_k^y$ - precision computation. The model of random process can be obtained as

$$\breve{S}_\alpha(t,M) = \sum_{k=1}^{M}\left(\breve{a}_k \sin(\breve{x}_k t) X_k + \breve{b}_k (1 - \cos(\breve{y}_k t)) Y_k\right). \quad (2)$$

Accuracy of simulation $\Delta(t)$ is $\Delta(t) = W_\alpha(t) - \breve{S}_\alpha(t,M)$.

In [2,4] studied estimations of accuracy and reliability of models in different functional spaces, namely, space $L_2(T)$, in Orlicz spaces, in spaces of continuous functions.

Standard Wiener process $W(t)$ is the process with independent increments. Generalized Wiener process $W_\alpha(t)$ is the process with stationary increments. Therefore, random process $w(t) = W_\alpha(t + \Delta) - W_\alpha(t)$ is stationary Gaussian random process with correlation function [5]

$$Ew(t+\tau)w(t) = \frac{1}{2}\left(|\tau + \Delta|^{2\alpha} + |\tau - \Delta|^{2\alpha} - 2|\tau|^{2\alpha}\right)$$

and spectral density $f(\lambda) = \frac{A^2}{\pi}\left(\frac{1 - \cos(\lambda\Delta)}{|\lambda|^{2\alpha+1}}\right)$,

$\lambda \in (-\infty, +\infty)$, where

$$A^2 = \left(\frac{2}{\pi}\int_0^\infty \frac{1 - \cos(\lambda)}{\lambda^{2\alpha+1}} d\lambda\right)^{-1} = \left(-\frac{2}{\pi}\Gamma(-2\alpha)\cos(\alpha\pi)\right)^{-1}.$$

Let $\xi(t)$ - be a real Gaussian stationary random process with $E\xi(t) = 0$, $R(\tau)$ – correlation function $\xi(t)$, $F(\lambda)$ - spectral function $\xi(t)$, $R(\tau) = \int_0^\infty \cos(\lambda t) dF(\lambda)$. Gaussian stationary random process can be represented as

$$\xi(t) = \int_0^\infty \cos(\lambda t) d\xi_1(\lambda) + \int_0^\infty \sin(\lambda t) d\xi_2(\lambda) =$$

$$= \int_0^\Lambda \cos(\lambda t) d\xi_1(\lambda) + \int_0^\Lambda \sin(\lambda t) d\xi_2(\lambda) +$$

$$+ \int_\Lambda^\infty \cos(\lambda t) d\xi_1(\lambda) + \int_\Lambda^\infty \sin(\lambda t) d\xi_2(\lambda),$$

where $\xi_1(t)$ and $\xi_2(t)$ - the centered and uncorrelated random processes with uncorrelated increments such as $0 \le \lambda_1 < \lambda_2$ and

$$E(\xi_1(\lambda_2) - \xi_1(\lambda_1))^2 = F(\lambda_2) - F(\lambda_1),$$

$$E(\xi_2(\lambda_2) - \xi_2(\lambda_1))^2 = F(\lambda_2) - F(\lambda_1).$$

Let D_Λ - be some partition of the interval $[0, \Lambda]$, $D_\Lambda : 0 = \lambda_0 < \lambda_1 < \ldots < \lambda_n = \Lambda$. The model of random process $\xi(t)$ can be obtained as $S_n(t,\Lambda) = \sum_{i=0}^{n-1} [\cos(\lambda_i t)\eta_{1i} + \sin(\lambda_i t)\eta_{2i}]$,

where $\{\eta_{1i}, \eta_{2i}\}$ - centered uncorrelated strictly sub-Gaussian random variables with $E(\eta_{1i})^2 = E(\eta_{2i})^2 = F(\lambda_{i+1}) - F(\lambda_i)$.

Therefore, random process $w(t)$ can be represented as

$$w(t) = \int_0^\infty \cos(\lambda t) d\xi_1(\lambda) + \int_0^\infty \sin(\lambda t) d\xi_2(\lambda). \quad (3)$$

And for partition $D_\Lambda : 0 = \lambda_0 < \lambda_1 < \ldots < \lambda_n = \Lambda$, the model of random process $w(t)$ can be obtained as

$$w_n(t,\Lambda) = \sum_{k=0}^{n-1} (\sin(\lambda_k t) X_k + \cos(\lambda_k t) Y_k), \quad (4)$$

where $\{X_k, Y_k\}$ - uncorrelated strictly sub-Gaussian random variables with $EX_k = EY_k = 0$ and $E(X_k)^2 = E(Y_k)^2 = \int_{\lambda_k}^{\lambda_{k+1}} f(\lambda) d\lambda$.

We study simulation of generalized Wiener process with representation (1), simulation of increments of generalized Wiener process with representation (3).

II. The accuracy and reliability of an model in $L_2(T)$

Let random process $X(t)$ and all $X_n(t,\Lambda)$ belongs to certain functional Banach space $A(T)$ with norm of $\|\cdot\|$. Let the two numbers be as follow $\delta > 0$ and $0 < \alpha < 1$. Model $X_n(t,\Lambda)$ approximates process $X(t)$ with reliability $1-\varepsilon$ and accuracy δ in the norm of space $A(T)$, if the following inequality holds $P\{\|X(t) - X_n(t,\Lambda)\| > \delta\} \le \varepsilon$.

Whereas, $W_\alpha(0) = 0$, then for all Δ the model of generalized Wiener process we constructed as

$$W_\alpha(t+\Delta) = W_\alpha(t) + w(t). \quad (5)$$

Simulation of fractional Brownian motion is reduced to simulation of stationary Gaussian random process. Methods of simulation of stationary Gaussian processes studied in [6-7].

There are theorems.

Theorem 1. Model $\breve{S}_\alpha(t,M)$ approximates process $W_\alpha(t)$ with accuracy $\delta > 0$ and reliability $1-\varepsilon$, $0 < \varepsilon < 1$ in the norm of space $L_2([0,T])$, if inequalities hold

$$\delta^2 > B1_M, \quad \text{and}$$

$$\exp\left\{\frac{1}{2}\right\} \frac{\delta}{\sqrt{B1_M}} \exp\left\{-\frac{\delta^2}{2B1_M}\right\} \le \varepsilon,$$

where $B1_M = T \sum_{k=M+1}^{\infty} (a_k^2 + 4b_k^2) +$.

$$+ T\sum_{k=1}^{M} \left((Ta_k h_k^x + h_k^a)^2 + (Tb_k h_k^y + 2h_k^b)^2 \right).$$

Theorem 2. Model $S_n(t,\Lambda)$ approximates process $\xi(t)$ with reliability $1-\varepsilon$ and accuracy δ in the norm of space $L_2(T)$, if for numbers Λ and n inequalities hold

$$B2_{n,\Lambda} < \delta^2 \text{ and } \exp\left\{\frac{1}{2}\right\} \frac{\delta}{\sqrt{B2_{n,\Lambda}}} \exp\left\{-\frac{\delta^2}{2B2_{n,\Lambda}}\right\} \le \varepsilon,$$

where $B2_{n,\Lambda} = \int_T E(\xi(t) - S_n(t,\Lambda))^2 d\mu(t)$.

Theorem 3. Model $w_n(t,\Lambda)$ approximates process $w(t)$ with reliability $1-\varepsilon$ and accuracy δ in the norm of space $L_2(T)$, if for numbers Λ and n inequalities hold

$B3_{n,\Lambda} < \delta^2$ and

$$\exp\left\{\frac{1}{2}\right\} \frac{\delta}{\sqrt{B3_{n,\Lambda}}} \exp\left\{-\frac{\delta^2}{2B3_{n,\Lambda}}\right\} \le \varepsilon,$$

where

$$B3_{n,\Lambda} = 2T \sum_{i=0}^{n} \int_{\lambda_i}^{\lambda_{i+1}} \left(1 - \frac{\sin(T(\lambda - \lambda_i))}{T(\lambda - \lambda_i)}\right) f(\lambda) d\lambda + .$$

$$+ T\left(\int_\Lambda^\infty f(\lambda) d\lambda\right).$$

Let for D_Λ implemented $T(\lambda_{i+1} - \lambda_i) \le 1$, then the corollary.

Corollary 1. Model $w_n(t,\Lambda)$ approximates process $w(t)$ with reliability $1-\varepsilon$ and accuracy δ in the norm of space $L_2(T)$, if for numbers Λ and n inequalities hold

$$G1_{n,\Lambda} < \delta^2 \text{ and } \exp\left\{\frac{1}{2}\right\} \frac{\delta}{\sqrt{G1_{n,\Lambda}}} \exp\left\{-\frac{\delta^2}{2G1_{n,\Lambda}}\right\} \le \varepsilon,$$

where $G1_{n,\Lambda} = \frac{T}{3} \sum_{i=0}^{n} \int_{\lambda_i}^{\lambda_{i+1}} (\lambda - \lambda_i)^2 f(\lambda) d\lambda + .$

$$+ T\left(\int_\Lambda^\infty f(\lambda) d\lambda\right).$$

Let for D_Λ implemented $\lambda_{i+1} - \lambda_i = \frac{\Lambda}{n}$ and $\frac{T\Lambda}{n} \le 1$, then the corollary.

Corollary 2. Model $w_n(t,\Lambda)$ approximates process $w(t)$ with reliability $1-\varepsilon$ and accuracy δ in the norm of space $L_2(T)$, if for numbers Λ and n inequalities hold

$$G2_{n,\Lambda} < \delta^2$$

and

$$\exp\left\{\frac{1}{2}\right\}\frac{\delta}{\sqrt{G2_{n,\Lambda}}}\exp\left\{-\frac{\delta^2}{2G2_{n,\Lambda}}\right\} \le \varepsilon ,$$

where

$$G2_{n,\Lambda} = \frac{T^3\Lambda^2}{3n^2}\int_0^{\Lambda} f(\lambda)d\lambda + T\left(\int_{\Lambda}^{\infty} f(\lambda)d\lambda\right).$$

III. Simulation

Using the model (2) requires significant computing resources. Table 1 shows the parameters of the model for different values of Hurst index, accuracy of simulation in space $L_2(T)$, for reliability $\varepsilon = 0.05$.

TABLE 1

PARAMETERS OF MODEL

δ	ε	n	h	α
0.01	0.05	170	0.0001	0.9
0.01	0.05	380	0.00002	0.8
0.01	0.05	1500	0.00002	0.7
0.01	0.05	2200000	0.0000005	0.4
0.05	0.05	26000	0.00001	0.4
0.1	0.05	5000	0.00005	0.4
0.05	0.05	1600000	0.000001	0.3
0.1	0.05	170000	0.00001	0.3

Realization of the generalized Wiener process for different values α and δ are presented in Fig. 1-3.

Table 2 shows the parameters of the model of stationary process for different values of Hurst index, accuracy of simulation in space $L_2(T)$, for reliability $\varepsilon = 0.05$.

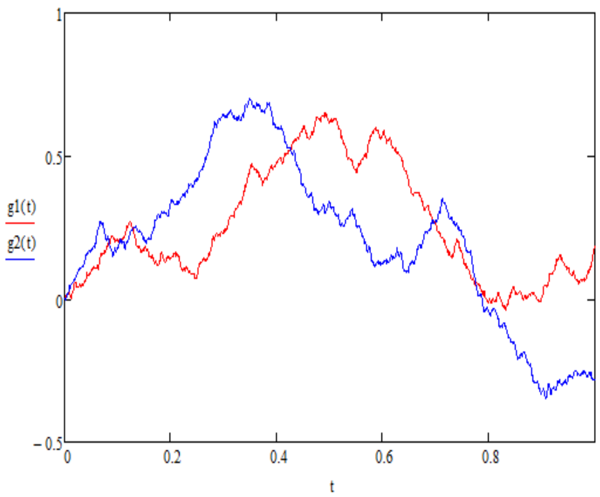

Fig. 1. Realization of generalized Wiener process with $\alpha = 0.7$ and $\delta = 0.01$

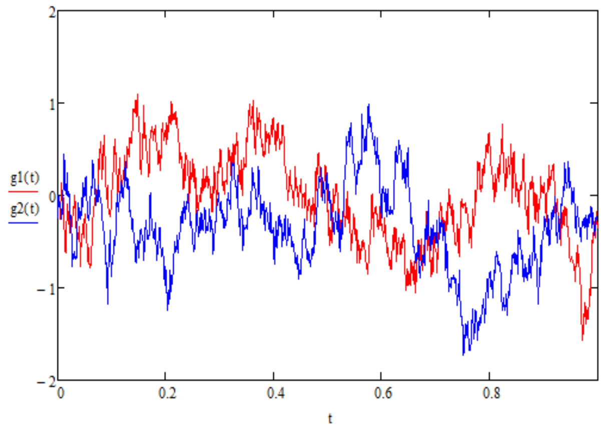

Fig .2. Realization of generalized Wiener process with $\alpha = 0.3$ and $\delta = 0.1$.

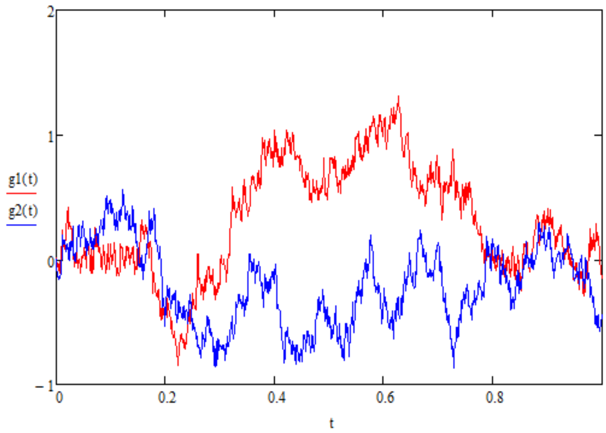

Fig. 3. Realization of generalized Wiener process with $\alpha = 0.4$ and $\delta = 0.1$.

TABLE 2

PARAMETERS OF MODEL

δ	ε	n	Λ	α
0.1	0.05	5 000 000	1 000 000	0.3
0.05	0.05	9 000 000	1 950 000	0.3
0.075	0.05	8 500 000	1 920 000	0.3
0.01	0.05	10000	1100	0.7
0.01	0.05	60	40	0.8

Realization of increments of the generalized Wiener process for $\alpha = 0.7$, $\delta = 0.01$ and $\varepsilon = 0.05$ are presented in Fig. 4.

Realization of the generalized Wiener process for $\varepsilon = 0.05$ and for different values α and δ are presented in Fig. 5-7.

Conclusions

Each of the algorithms which discussed above, requires a large number of terms.

The level of accuracy calculation by the first method consists of complexity of calculating zeros of Bessel functions and the functions itself.

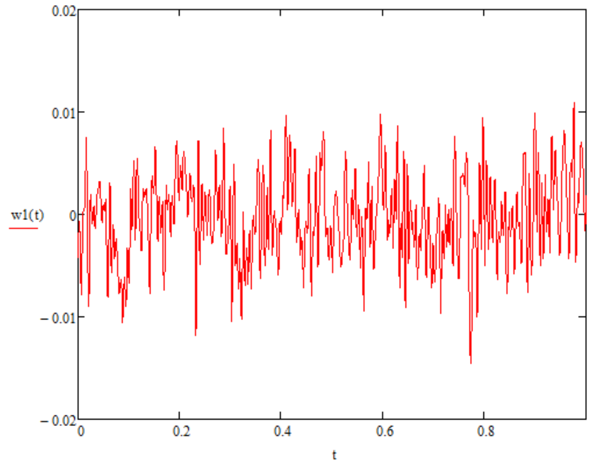

Fig. 4. Realizationofincrements of generalized Wiener process

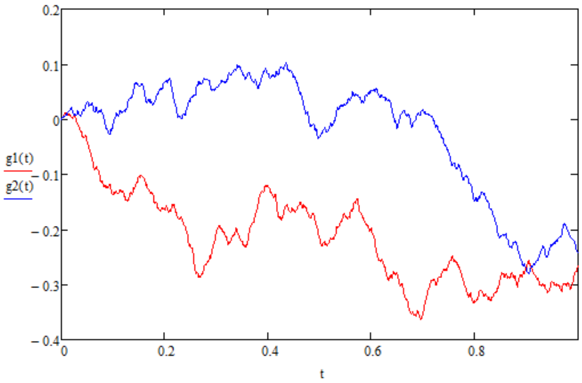

Fig. 5.Realizationofgeneralized Wiener process with $\alpha = 0.8$ and $\delta = 0.01$

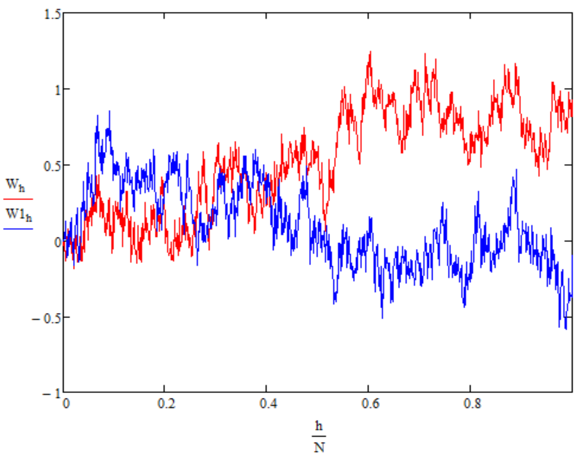

Fig. 6. Realizationofgeneralized Wiener process with $\alpha = 0.3$ and $\delta = 0.1$

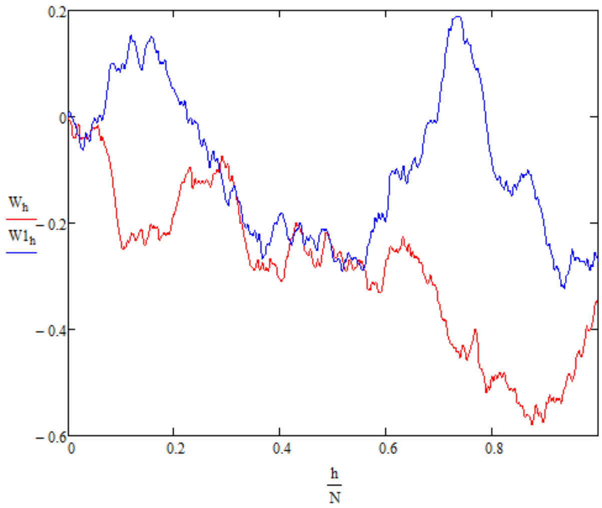

Fig. 7. Realization of generalized Wiener process with $\alpha = 0.7$ and $\delta = 0.01$

The second model from this point is more simple to implement.

To validate the quality of the simulation is possible to use the estimation of the stationary random process correlation function.

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