On the Interval Game-Theoretic Solutions and Their Axiomatizations

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Abstract – ***Natural questions for people or businesses that face interval uncertainty in their data when dealing with cooperation are: Which coalitions should form? How to distribute the collective gains or costs? The theory of cooperative interval games is a suitable tool for answering these questions. In this paper, we introduced and characterizated the new interval solutions concepts, i.e. interval CIS-value, interval ENSC-value and interval equal division solution by using cooperative interval games. Finally, we characterizd these interval solutions for two-player games.***

Key words – cooperative game theory, uncertainty, CIS-value, ENSC-value, ED-value.

I. Introduction

Uncertainty affects our decision making activities on a daily basis. There are many sources of uncertainty in the real world. The effect of uncertainty could be seen at noise in observation and experimental design, incomplete information and vagueness in preference structures. Managerial interest in theoretical explanations for organizational phenomena has acknowledged the role of uncertainty for a long time. In the sequel, the important issue of whether and why individuals and organizations choose to cooperate or not to cooperate when faced with uncertainty on outcomes or costs, has generated a productive line of research in recent years.

A situation, in which a finite set of players can obtain certain payoffs by cooperation can be described by a cooperative game with transferable utility, or simply a TU-game. A (point-valued) solution for TU-games assigns a payoff distribution to every TU-game. Cooperative game theory has been enriched by several models which provide decision making support in collaborative situations under uncertainty. These models are generalizations of the classical model regarding the type of coalition values. In classical cooperative game theory, the payoffs to coalitions of players are known with certainty, but when uncertainty is taken into consideration, the characteristic functions are not real-valued as in the classical framework.

In this paper, we examine a game theoretical solutions for TU-games that all have some egalitarian flavour in the sense that they assign to every player some initial payoff and distribute the remainder of the worth $v(N)$ of the grand coalition N equally among all players.

Examples of such solutions are the Centre-of-gravity of the Imputation-Set value, shortly denoted by CIS-value (see [5].), Egalitarian NonSeparable Contribution value, shortly denoted by ENSC-value and the equal division solution. The CIS-value assigns to every player its individual worth, and distributes the remainder of $v(N)$ equally among all players. The ENSC-value assigns to every game $< N, v >$ the CIS-value of its dual game. The equal division solution just distributes $v(N)$ equally among all players. In this study, we extend these solutions by using interval calculus, and we study some properties.

The paper is organized as follows. Section 2 contains preliminaries on TU-games and interval games. In Section 3, the notion of the interval game-theoretic solutions are introduced. Finally, we characterize these interval solutions for two-player games.

II. Preliminaries

An n-person game in characteristic function form is a pair $< N, v >$ where $N = \{1, 2, ..., n\}$ is the set of players and $v : 2^N \rightarrow \mathbb{R}$ is a characteristic function, such that $v(\varnothing) = 0$. Here, 2^N denotes the set of all coalitions of N. For each coalition S, the real number $v(S)$ is called the worth of S and represents the reward that coalition S can obtain if all its members act together. Since we take the player set fixed, we represent a TU-game $< N, v >$ by its characteristic function v. By G^N we denote the $2^n - 1$ dimensional vector space of all n person games.

A solution for TU-games is a function which assigns to every game $v \in G^N$, an n-dimensional vector which components are the payoffs of the players. One of the most famous solutions for TU-games is the Shapley value ([8]) given by

$$Sh(v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} m^{\pi}(v),$$

where $\Pi(N)$ is the set of permutations $\pi : N \rightarrow N$ of $N = \{1, 2, \ldots$, and for every $\pi \in \Pi(N)$, the corresponding marginal vector is given by $m_i^{\pi}(v) = v(P^{\pi}(i) \cup \{i\}) - v(P^{\pi}(i))$ for each $i \in N$, where $P^{\pi}(i) := \{r \in N \mid \pi^{-1}(r) < \pi^{-1}(i)\}$, and $\pi^{-1}(i)$ denotes the entrance number of player i.

We denote the size of a coalition $S \subset N$ by $|S|$. We recall that G^N is a $(2^{|N|}-1)$ dimensional linear space for which unanimity games form a basis. The unanimity game of $T \in 2^N \setminus \{\varnothing\}$, $u_T : 2^N \to R$ is defined by

$$u_T(S) = \begin{cases} 1, & \textit{if } T \subseteq S \\ 0, & \text{otherwise.} \end{cases}$$

The reader is referred to [4] and [6] for a survey on classical TU-games.

Next, we recall some preliminaries from interval calculus and the theory of cooperative interval games as discussed in [1, 2, 3].

A cooperative interval game is an ordered pair $< N, w >$ where $N = \{1, 2, \ldots$ is the set of players, and $w : 2^N \to I(R)$, , where $I(R)$ is the set of all nonempty, compact intervals in R, is the characteristic function such that $w(\varnothing) = [0,0]$. For each $S \in 2^N$, the worth set (or worth interval) $w(S)$ of coalition S in the interval game $< N, w >$ is of the form $w(S) = [\underline{w}(S), \overline{w}(S)]$, where $\underline{w}(S)$ is the minimal worth, which coalition S could receive on its own and $\overline{w}(S)$ is the maximal worth, which coalition S could get. The family of all interval games with player set N is denoted by IG^N.

Let $I, J \in I(R)$ with $I = \left[\underline{I}, \overline{I}\right]$, $J = \left[\underline{J}, \overline{J}\right]$. Then, $|I| = \overline{I} - \underline{I}$ is called the length of the interval I. Also,

1. $I + J = \left[\underline{I} + \underline{J}, \overline{I} + \overline{J}\right]$;
2. $\alpha I = \left[\alpha \underline{I}, \alpha \overline{I}\right]$, for $\alpha \in R_+$.

By (i) and (ii) we see that $I(R)$ has a cone structure.

In this paper we also need a subtraction operator. In the literature, several subtraction operators can be found. The subtraction operator of [7] is defined as follows. For intervals $I = [\underline{I}, \overline{I}]$ and $J = [\underline{J}, \overline{J}]$ this difference is defined as $I \,!\; J = [\underline{I} - \overline{J}, \overline{I} - \underline{J}]$.

An alternative subtraction operator that is used for interval games is that of [1]. This is defined for intervals $I = [\underline{I}, \overline{I}]$ and $J = [\underline{J}, \overline{J}]$ only if $|I| \geq |J|$, and for such intervals is defined by $I - J = \left[\underline{I} - \underline{J}, \overline{I} - \overline{J}\right]$. For the example above, $J - I = [2-6, 5-8] = [-4, -3]$, but $I - J$ is not defined. Therefore, this subtraction is defined only for so-called size-monotonic interval games, being interval games $< N, w >$ such that $< N, |w| >$ is a monotonic TU-game, i.e., $|w(S)| \leq |w(T)|$ for all $S \subseteq T \subseteq N$. [1] call I weakly better than J, which is denoted by $I \mu J$, if and only if $\underline{I} \geq \underline{J}$ and $\overline{I} \geq \overline{J}$.

So, we conclude that IG^N endowed with μ is a partially ordered set and has a cone structure with respect to addition and multiplication with non-negative scalars described above.

The model of interval cooperative games is an extension of the model of classical TU-games. Interval solutions are useful to solve reward/cost sharing problems with interval data using cooperative interval games as a tool. The components of interval payoff vectors belong to $I(R)$ and represent the interval payoffs of the players. We denote by $I(R)^N$ the set of all such interval payoff vectors.

III. A class of equal surplus sharing interval solutions

The interval CIS-value assing to every player its individual interval worth, and distributes the remainder of the interval worth of the grand coalition N equally among all players, i.e.

$$ICSI_i : SMIG^N \to I(R)^N$$

$$ICSI_i = v(\{i\}) + \tfrac{1}{|N|}\left(v(N) -_{j \in N} v(\{j\})\right)$$

for all $i \in N$. The dual game $v^* \in SMIG^N$ of interval game v is the game that assigns to each coalition $S \subseteq N$ the interval worth that is lost by the grand coalition N if coalition S leaves N, i.e.

$$v^*(S) = v(N) - v(N \setminus S)$$

for all $S \subseteq N$. The interval ENSC-value assigns to every game v the CIS-value of its dual game, i.e.

$$IENSC_i(v) = ICIS_i(v^*)$$

$$= -v(N \setminus \{i\}) + \tfrac{1}{|N|}\left(v(N) +_{j \in N} v(N \setminus \{j\})\right)$$

for all $i \in N$. Thus, the interval ENSC-value assigns to every playaer in a game its interval marginal contribution to the "grand coalition" and distributes the remainder equally among the players. Using these two interval solutions, we can define a class of interval solutions, by taking any convex combination of the two, i.e. for $\beta \in [0,1]$ we define

$$IENCIS^{\beta}(v) = \beta ICIS(v)$$
$$+ (1-\beta) IENSC(v)$$

such that

$$|v|(i) = |v|(N \setminus \{i\}) = |v|(j)$$

for all $i, j \in N$ with $i \neq j$. The interval solutions discussed above have some egalitarian flavour, in the sense that they equally split a surplus that is left after all players receive some individual payoff. Ignoring these individual payoffs, we obtain interval equal division solution given by

$$IED_i(v) = \frac{v(N)}{|N|}$$

for all $i \in N$.

Remark 1. We note that while interval CIS-value and interval ENSC-value are defined in $SMIG^N$, the interval ED-solution is defined in IG^N.

All interval solutions $IENCIS^{\beta}$, as defined in (1), are covariant. The only self-dual solution in this class is the average of the interval CIS-value and interval ENSC-value obtained by taking $\beta = \frac{1}{2}$. The interval ED-solution is self dual but not covariant.

In this study, we discuss the class of interval solutions that consists of all convex combinations of the interval ED-solution, the interval CIS-value and the interval ENSC-value, i.e. for $\alpha, \beta \in [0,1]$, we consider interval solutions $\phi^{\alpha,\beta}$ given by

$$\phi^{\alpha,\beta}(v) = \alpha IENCIS^{\beta}(v) + (1-\alpha) IED(v)$$

We denote the class of all interval solutions that are obtained in this way by $I\Phi := \{\phi^{\alpha,\beta} : \alpha, \beta \in [0,1]\}$. Clearly, the interesting solutions in this class are the interval CIS-value, which is obtained by taking $\alpha = \beta = 1$, the interval ENSC-value, which is obtained by taking $\alpha = 1$, $\beta = 0$ and the interval ED-solution, which is obtained by taking.

We can write $\phi^{\alpha,\beta}$ as We can write $\phi^{\alpha,\beta}$ as

$$\phi^{\alpha,\beta}(v) = \alpha\phi^{1,\beta}(v) + (1-\alpha)\phi^{0,1}(v)$$
$$= \alpha\beta\phi^{1,1}(v) + \alpha(1-\beta)\phi^{1,0}(v) + (1-\alpha)\phi^{0,1}(v)$$

for $\alpha, \beta \in [0,1]$.

Next, we provide an expression of the solutions $\phi^{\alpha,\beta}$ showing that they have some egalitarian flavour in the sense that they give each player i in a game v some value $\lambda_i^{\alpha,\beta}(v)$, and the remainder of $v(N)$ is equally split among all players.

Proposition 2. For every $v \in SMIG^N$ and $\alpha, \beta \in [0,1]$ it holds that

$$\phi_i^{\alpha,\beta}(v) = \lambda_i^{\alpha,\beta}(v) + \tfrac{1}{|N|}(v(N) -_{j \in N} \lambda_j^{\alpha,\beta}(v)),$$

where

$$\lambda_i^{\alpha,\beta}(v) = \alpha\left(\beta v(\{i\}) - (1-\beta)v(N \setminus \{i\})\right)$$

for $i \in N$ such that $|v|(i) = |v|(N \setminus \{i\}) = |v|(j)$ for all $i, j \in N$ with $i \neq j$.

Proposition 3. For every $\alpha, \beta \in [0,1]$ and $v \in SMIG^N$ it holds that $\phi_i^{\alpha,\beta}(v^*) = \phi^{\alpha,1-\beta}(v)$.

IV. On the characterization of equal surplus sharing interval solutions for two-player games

On the class of two-player games, the interval CIS-value and ENSC-value coincide. Thus, on this class we consider convex combinations of the interval CIS-value and the interval ED-solution. It is easy to prove the interval CIS-value satisfies standardness for two-player games as considered in, i.e.

$$\psi_i(v) = \tfrac{1}{2}v(\{i\}) - \tfrac{1}{2}(v\{j\})$$
$$= v(\{i\}) + \tfrac{1}{2}(v(N) - v(\{i\}) - v(\{j\}))$$

with $N = \{i, j\}$.

On the other hand, the interval ED-solution satisfies egalitarian standardness for two-player games: $\psi_i(v) = \frac{1}{2}v(N)$ for $i \in N$. We denote the class of two player games by

$$SMIG^2 = \{v \in SMIG^N : |N| = 2\}.$$

On this class, $I\Phi$ consists of all interval solutions for two-player games that assign to both players the same share in their individual interval worth, and distributes the remainder of interval worth of two-player coalition equally among the two players.

Definition 4. Let $\alpha \in [0,1]$. An interval solution ψ satisfies α-standardness for two-players if for every $v \in SMIG^N$ with $N = \{i, j\}$, $i \neq j$, it holds that

$$\psi_i(v) = \alpha v(i) +$$
$$\tfrac{1}{2}\left(v(N) - \alpha\left(v(\{i\}) + v(\{j\})\right)\right)$$

An interval solution ψ satisfies weak standardness for two-player games if there exists an $\alpha \in [0,1]$ such that ψ satisfies α-standardness for two player games.

Clearly, standardness for two-player games coincides with $\alpha = 1$, and egalitarian standardness coincides with $\alpha = 0$. Weak standardness is equivalent to requiring efficiency, symmetry and linearity on $SMIG^2$.

Proposition 5. A solution ψ on $SMIG^2$ satisfies weak standardness for two-player games if and only if it is efficient, symmetric and linear.

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