On Investigating the Differential Geometric Properties of the Knitted Fabrics Produced by Some Computer Aided Geometric Design Curves

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Abstract – In order to research the knitted fabric, some specific curves produced by computer aided geometric design are used in our paper, then some diferential geometric results of these curves are calculated and an aplication of this concept is given.

Key words – CAGD, computer curves, knitted fabric, geometry.

I. Introduction

The fabrics are built up from a number of yarns brought together to form a self supporting structure. In the late 1950s, hardware become available that allowed the machining of 3D shapes out of blocks of wood or steel. Bezier curves which represented by the formula $b^n(t)=\sum_{i=0}^{n-r} b_i^r(t)B_i^{n-r}(t)$ were independently developed by P. de Casteljau at Citroen and by P. Bezier at Renault Company in France. The theory of Bezier curves plays a central role in CAGD. They are numerically the most stable among polynomial bases currently used in CAD systems, was shown by Farauki and Rajon. Thus Bezier curves are the ideal geometric standart for the representation of piecewise polynomial curves. Also, Bezier curves lend themselves easily to a geometric understanding of many CAGD phenomena, [1-3]. Bezier curve segments are defined only by the position vectors of polygon vertices. Bezier curve segments are expressed as a convex combination of the polygon vertex position vectors which define the curve, and possess a variation diminishing property. Consequently the curve shape can be approximately anticipated from the polygon shape. That is to say, Bezier curves and surfaces are in a form that is easy for a person to control, [4-5]. Computer aided geometric design (CAGD) concerns itself with the mathematical description of shape for use in computer graphics, manufacturing or analysis, approximation theory, data structures and computer algebra. CAGD is a young field. The first work in this field began in the mid 1960s. The term computer aided geometric design was coined in 1974 by R.E. Barnhill and R.F. Riesenfeld in connection with a conference at the University of Utah, [6-8].

II. Preliminaries

A Bezier curve is defined by the equation $b^n(t)=\sum_{i=0}^{n-r} b_i^r(t)B_i^{n-r}(t)$. In this formula $B_i^n(t)=\binom{n}{i}t^i(1-t)^{n-i}$ are Bernstein polynomials and $b_i^r(t)$ are control points for $i\in\{0,...,n-r\}$. The special conditions of the Bernstein polynomials are $B_0^0(t)\equiv 1$ and $B_j^n(t)\equiv 1$ for $j\notin\{0,...,n\}$. Moreover, the sum of the coefficients of Bernstein polynomial is $\sum_{j=0}^{n} B_j^n(t)\equiv 1$. Alternatively, Bernstein polynomial may be written with the equation $B_i^n(t)=(1-t)B_i^{n-1}(t)+tB_{i-1}^{n-1}(t)$. The derivative of a Berstein polynomial B_i^n is obtained as

$$\frac{d}{dt}B_i^n(t)=n\left[B_{i-1}^{n-1}(t)-B_i^{n-1}(t)\right]$$

The r.th degree derivative of a Bezier curve is given by $\frac{d^r b^n}{dt^r}(t)=\frac{n!}{(n-r)!}.\sum_{i=0}^{n-r}\Delta b_i^r.B_i^{n-r}(t)$, here $\Delta^r b_i$ difference equation is in the form of $\Delta^r b_i=\sum_{i=0}^{r}\binom{r}{j}(-1)^{r-j}b_{i+j}$, in addition, $\Delta^r b_j=\Delta^{r-1}b_{j+1}-\Delta^{r-1}b_j$.

Theorem 3.1. Let $b_i\in E^3$ be the control points, the Serret-Frenet frame $\{T,N,B\}|_{t=0}$ at the $t=0$ start point is given with the equations

$$T|_{t=0}=\frac{\Delta b_0}{\|\Delta b_0\|}$$

$$N|_{t=0}=\frac{\Delta b_1}{\|\Delta b_1\|}\csc\phi-\frac{\Delta b_0}{\|\Delta b_0\|}\cot\phi$$

$$B|_{t=0}=\frac{\Delta b_0\wedge\Delta b_1}{\|\Delta b_0\wedge\Delta b_1\|},$$

see the details in [4].

Theorem: Let $b_0,b_1,...,b_n$ be control points for the n^{th} order Bezier curve $b^n(t)$. The curvatures at the starting and ending point, i.e. at $t=0$ and $t=1$

$$\kappa|_{t=0}=\frac{n-1}{n}\frac{\|\Delta b_1\|}{\|\Delta b_0\|^2}.\sin\theta \qquad (1)$$

$$\kappa|_{t=1}=\frac{n-1}{n}\frac{\|\Delta b_{n-2}\|}{\|\Delta b_{n-1}\|^2}.\sin\psi \qquad (2)$$

here the angle θ is an angle between Δb_1 and Δb_0, the angle ψ is an angle between Δb_{n-1} and Δb_{n-2}, [4].

III. Metarial and Methods

Theorem: The curvature radius of the Bezier curve at the starting point $t=0$ is

$$\rho(t)|_{t=0}=\frac{n}{n-1}\frac{\|\Delta b_0\|^2}{\|\Delta b_1\|}\frac{1}{\sin\theta}$$

and the curvature radius of the Bezier curve at the ending point $t=1$ is

$$\rho(t)|_{t=1}=\frac{n}{n-1}\frac{\|\Delta b_{n-1}\|^2}{\|\Delta b_{n-2}\|}\frac{1}{\sin\psi}.$$

Proof: By substuting the Equations (1) and (2) on the curvature radius formula $\rho(t)=\frac{1}{\kappa}$ for $t=0$ and $t=1$, we can get the above equations.

Theorem: Let $b_i\in E^3$ be control points of the Bezier curve $b^n(t)$ at $t=0$. The curvature center $m|_{t=0}$ can be calculated by the formula

$$m|_{t=0}=b_0+\frac{n}{n-1}\frac{\|\Delta b_0\|}{\|\Delta b_1\|}\left[\frac{\|\Delta b_0\|}{\|\Delta b_1\|}\Delta b_1-\Delta b_0\cos\theta\right]\cos ec^2\theta$$

Proof: By using the equation of curvature center $m|_{t=0}=\alpha(t_0)+\rho(t_0)N(t_0)$ at the starting point t=0, we get

$$m|_{t=0}=b_0+\frac{n}{n-1}\frac{\|\Delta b_0\|^2}{\|\Delta b_1\|}\frac{1}{\sin\theta}\left[\frac{\Delta b_1}{\|\Delta b_1\|}\cos ec\theta-\frac{\Delta b_0}{\|\Delta b_0\|}\cot\theta\right]$$

$$=b_0+\frac{n}{n-1}\frac{\|\Delta b_0\|}{\|\Delta b_1\|}\left[\frac{\|\Delta b_0\|}{\|\Delta b_1\|}\Delta b_1-\Delta b_0\cos\theta\right]\cos ec^2\theta$$

Theorem: For the Bezier curve $b^n(t)$ with the control points $b_i\in E^3$ at $t=0$, the curvature circle β is

$$\beta(\varphi)=b_0+\frac{n}{n-1}\frac{\|\Delta b_0\|^2}{\|\Delta b_1\|}(1-\cos\frac{\varphi}{\rho(t)})\cos ec^2\theta.\Delta b_1$$
$$+\frac{n}{n-1}\frac{\|\Delta b_0\|}{\|\Delta b_1\|}(-\cos\theta+\cos\frac{\varphi}{\rho(t)}\cos\theta+\sin\frac{\varphi}{\rho(t)}\sin\theta)\cos ec^2\theta.\Delta b_0$$

Proof:

$$\beta(\varphi)=\alpha(t_0)+\rho(t_0)N(t_0)-\rho(t_0)\cos\frac{\varphi}{\rho(t_0)}N(t_0)+\rho(t_0)\sin\frac{\varphi}{\rho(t_0)}T(t)$$

$$=b_0+\frac{n}{n-1}\frac{\|\Delta b_0\|^2}{\|\Delta b_1\|}\frac{1}{\sin\theta}[\frac{\Delta b_1}{\|\Delta b_1\|}\cos ec\theta-\frac{\Delta b_0}{\|\Delta b_0\|}\cot\theta]$$

$$-\frac{n}{n-1}\frac{\|\Delta b_0\|^2}{\|\Delta b_1\|}\frac{1}{\sin\theta}\cos\frac{\varphi}{\rho(t)}[\frac{\Delta b_1}{\|\Delta b_1\|}\cos ec\theta-\frac{\Delta b_0}{\|\Delta b_0\|}\cot\theta]$$

$$+\frac{n}{n-1}\frac{\|\Delta b_0\|^2}{\|\Delta b_1\|}\frac{1}{\sin\theta}\sin\frac{\varphi}{\rho(t)}\frac{\Delta b_0}{\|\Delta b_0\|}$$

$$=b_0+\frac{n}{n-1}\frac{\|\Delta b_0\|^2}{\|\Delta b_1\|}\frac{1}{\sin\theta}\frac{\Delta b_1}{\|\Delta b_1\|}\frac{1}{\sin\theta}-\frac{n}{n-1}\frac{\|\Delta b_0\|^2}{\|\Delta b_1\|}\frac{1}{\sin\theta}\frac{\Delta b_0}{\|\Delta b_0\|}\frac{\cos\theta}{\sin\theta}$$

$$-\frac{n}{n-1}\frac{\|\Delta b_0\|^2}{\|\Delta b_1\|}\frac{1}{\sin\theta}\cos\frac{\varphi}{\rho(t)}\frac{\Delta b_1}{\|\Delta b_1\|}\frac{1}{\sin\theta}+\frac{n}{n-1}\frac{\|\Delta b_0\|^2}{\|\Delta b_1\|}\frac{1}{\sin\theta}\cos\frac{\varphi}{\rho(t)}\frac{\Delta b_0}{\|\Delta b_0\|}\frac{\cos\theta}{\sin\theta}$$

$$+\frac{n}{n-1}\frac{\|\Delta b_0\|^2}{\|\Delta b_1\|}\frac{1}{\sin\theta}\sin\frac{\varphi}{\rho(t)}\frac{\Delta b_0}{\|\Delta b_0\|}$$

$$=b_0+\frac{n}{n-1}\frac{\|\Delta b_0\|^2}{\|\Delta b_1\|}(1-\cos\frac{\varphi}{\rho(t)})\cos ec^2\theta.\Delta b_1$$

$$+\frac{n}{n-1}\frac{\|\Delta b_0\|}{\|\Delta b_1\|}(-\cos\theta+\cos\frac{\varphi}{\rho(t)}\cos\theta+\sin\frac{\varphi}{\rho(t)}\sin\theta)\cos ec^2\theta.\Delta b_0$$

Result: Let $b_i(t)\in E^3$ be Bezier curve and $B(t)|_{t=0}$ be a binormal of the Bezier curve at the starting point $t=0$. The curvature axis at the starting point can be obtain from the equation $d(\mu)|_{t=0}=m+\mu B(t)$ for $\mu\in IR$. Therefore we get the equation of the curvature axis by

$$d(\mu)|_{t=0}=b_0+\frac{n}{n-1}\frac{\|\Delta b_0\|}{\|\Delta b_1\|}\cos ec^2\theta\left[\frac{\|\Delta b_0\|}{\|\Delta b_1\|}\Delta b_1-\Delta b_0\cos\theta\right]$$
$$-\mu\frac{\Delta b_0\wedge\Delta b_1}{\|\Delta b_0\wedge\Delta b_1\|}$$

Theorem: Let $b_i\in E^3$ be control points of $b^n(t)$ at the ending point $t=1$. If the curvature center at the point $t=1$ represented by $m|_{t=1}$, then its formula can be found by

$$m|_{t=1}=b_n+\frac{n}{n-1}\frac{\|\Delta b_{n-1}\|}{\|\Delta b_{n-2}\|}\left[\cos\psi.\Delta b_{n-1}-\frac{\|\Delta b_{n-1}\|}{\|\Delta b_{n-2}\|}\Delta b_{n-2}\right]\cos ec^2\psi$$

Proof: The curvature center of the nonunit speed Bezier curve is given by the formula $m|_{t=1}=\alpha(t_0)+\rho(t_0)N(t_0)$. From this formula we can obtain

$$m|_{t=1}=b_n+\frac{n}{n-1}\frac{\|\Delta b_{n-1}\|^2}{\|\Delta b_{n-2}\|}\frac{1}{\sin\psi}\left[\frac{\Delta b_{n-1}}{\|\Delta b_{n-1}\|}\cot\psi-\frac{\Delta b_{n-2}}{\|\Delta b_{n-2}\|}\cos ec\psi\right]$$

$$=b_n+\frac{n}{n-1}\frac{\|\Delta b_{n-1}\|^2}{\|\Delta b_{n-2}\|}\left[\frac{\Delta b_{n-1}}{\|\Delta b_{n-1}\|}\cos\psi-\frac{\Delta b_{n-2}}{\|\Delta b_{n-2}\|}\right]\cos ec^2\psi$$

Theorem: Let $b_i\in E^3$ be the control points of the Bezier curve $b^n(t)$ and the curvature circle $\beta(\varphi)$ at $t=1$ is obtained by the equation

$$\beta(\varphi)=b_n+\frac{n}{n-1}\frac{\|\Delta b_{n-1}\|}{\|\Delta b_{n-2}\|}(\cos\psi-\cos\frac{\varphi}{\rho(t)}\cos\psi+\sin\frac{\varphi}{\rho(t)}\sin\psi).\Delta b_{n-1}$$
$$+\frac{n}{n-1}\frac{\|\Delta b_{n-1}\|^2}{\|\Delta b_{n-2}\|^2}(\cos\frac{\varphi}{\rho(t)}\cos\theta-1)\Delta b_{n-2}$$

Proof:

$$\beta(\varphi)=\alpha(t_0)+\rho(t_0)N(t_0)-\rho(t_0)\cos\frac{\varphi}{\rho(t_0)}N(t_0)+\rho(t_0)\sin\frac{\varphi}{\rho(t_0)}T(t)$$

$$=b_n+\frac{n}{n-1}\frac{\|\Delta b_{n-1}\|^2}{\|\Delta b_{n-2}\|}\frac{1}{\sin\psi}\left[\frac{\Delta b_{n-1}}{\|\Delta b_{n-1}\|}\cot\psi-\frac{\Delta b_{n-2}}{\|\Delta b_{n-2}\|}\cos ec\psi\right]$$

$$-\frac{n}{n-1}\frac{\|\Delta b_{n-1}\|^2}{\|\Delta b_{n-2}\|}\frac{1}{\sin\psi}\cos\frac{\varphi}{\rho(t)}\left[\frac{\Delta b_{n-1}}{\|\Delta b_{n-1}\|}\cot\psi-\frac{\Delta b_{n-2}}{\|\Delta b_{n-2}\|}\cos ec\psi\right]$$

$$+\frac{n}{n-1}\frac{\|\Delta b_{n-1}\|^2}{\|\Delta b_{n-2}\|}\frac{1}{\sin\psi}\sin\frac{\varphi}{\rho(t)}\frac{\Delta b_{n-1}}{\|\Delta b_{n-1}\|}$$

$$=b_n+\frac{n}{n-1}\frac{\|\Delta b_{n-1}\|^2}{\|\Delta b_{n-2}\|}\frac{1}{\sin\psi}\frac{\Delta b_{n-1}}{\|\Delta b_{n-1}\|}\frac{\cos\psi}{\sin\psi}-\frac{n}{n-1}\frac{\|\Delta b_{n-1}\|^2}{\|\Delta b_{n-2}\|}\frac{1}{\sin\psi}\frac{\Delta b_{n-2}}{\|\Delta b_{n-2}\|}\frac{1}{\sin\psi}$$

$$-\frac{n}{n-1}\frac{\|\Delta b_{n-1}\|^2}{\|\Delta b_{n-2}\|}\frac{1}{\sin\psi}\cos\frac{\varphi}{\rho(t)}\frac{\Delta b_{n-1}}{\|\Delta b_{n-1}\|}\frac{\cos\psi}{\sin\psi}+\frac{n}{n-1}\frac{\|\Delta b_{n-1}\|^2}{\|\Delta b_{n-2}\|}\frac{1}{\sin\psi}\cos\frac{\varphi}{\rho(t)}\frac{\Delta b_{n-2}}{\|\Delta b_{n-2}\|}\frac{1}{\sin\psi}$$

$$+\frac{n}{n-1}\frac{\|\Delta b_{n-1}\|^2}{\|\Delta b_{n-2}\|}\frac{1}{\sin\psi}\sin\frac{\varphi}{\rho(t)}\frac{\Delta b_{n-1}}{\|\Delta b_{n-1}\|}$$

$$=b_n+\frac{n}{n-1}\frac{\|\Delta b_{n-1}\|}{\|\Delta b_{n-2}\|}(\cos\psi+\cos\frac{\varphi}{\rho(t)}\cos\psi+\sin\frac{\varphi}{\rho(t)}\sin\psi)\cos ec^2\psi.\Delta b_{n-1}$$

$$+\frac{n}{n-1}\frac{\|\Delta b_{n-1}\|^2}{\|\Delta b_{n-2}\|^2}(\cos\frac{\varphi}{\rho(t)}-1)\cos ec^2\psi.\Delta b_{n-1}$$

Result: The curvature axis of the Bezier curve $b_i(t)\in E^3$ at the ending point is

$$d(\mu)|_{t=1}=m+\mu B(t)$$

here $\mu\in IR$ and $B(t)|_{t=1}$ is the Binormial vector of the curve. Therefore the result can be obtained by

$$d(\mu)|_{t=1}=b_1+\frac{n}{n-1}\frac{\|\Delta b_{n-1}\|}{\|\Delta b_{n-2}\|}\cos ec^2\psi\left[\cos\theta.\Delta b_{n-1}-\frac{\|\Delta b_{n-1}\|}{\|\Delta b_{n-2}\|}\Delta b_{n-2}\right]-\mu\frac{\Delta b_{n-1}\wedge\Delta b_{n-2}}{\|\Delta b_{n-1}\wedge\Delta b_{n-2}\|}.$$

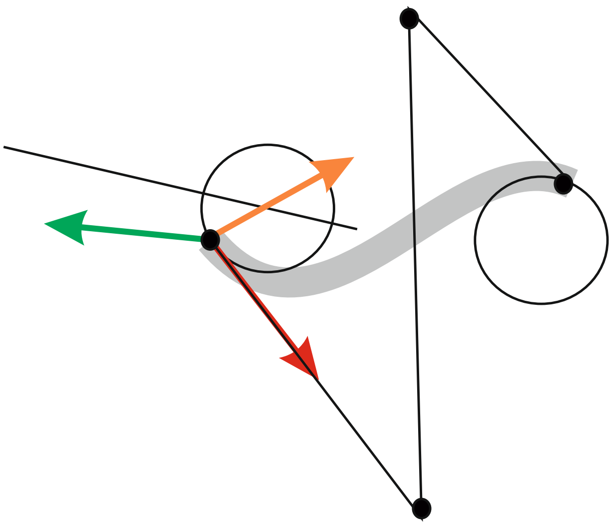
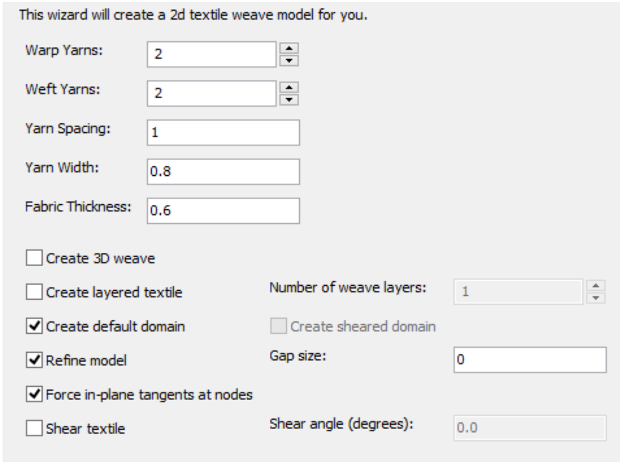


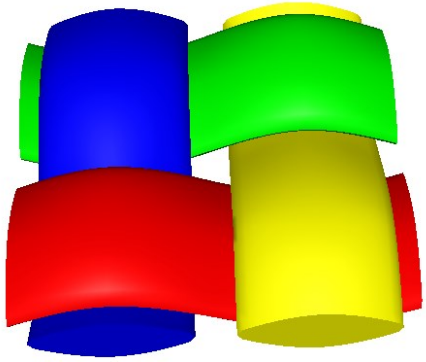
Fig. 1. The curvature circle and axis of the third order Bezier curve

An Application of Yarn Computer Modelling

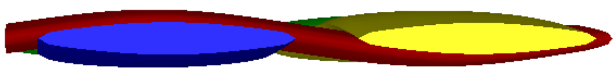
Thus the curvature raidus and curvature circle of the yarn has been found. The design studies for this example is given following. Here we will give some properties of the yarn in the textile program named with TexGen forexample the warp yarns, weft yarns, spacing, width, thickness.



The yarn surface can be modelled by the program TEXgen as following:



The side view of this knitting surface has been observed in the following figure profile.



Let us take the yarn as a third degree Bezier curve with four control points $\mathbf{b}_0,\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3$. Then the Bezier curve is defined by $b^3(t)=\sum_{i=0}^{3}\mathbf{b}_i^r(t)B_i^{3-r}(t)$, for another representation, we can write the Bezier curve as $b^3(t)=(1-t)^3\mathbf{b}_0+3(1-t)^2t\mathbf{b}_1+3(1-t)t^2\mathbf{b}_2+t^3\mathbf{b}_3$.

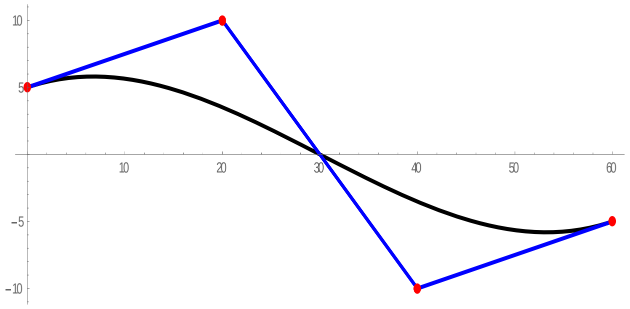
For the design of a Bezier curve model we will get the control points as

$\mathbf{b}_0=(0,5)$, $\mathbf{b}_1=(20,10)$, $\mathbf{b}_2=(40,-10)$, $\mathbf{b}_3=(60,-5)$

and the output of the Bezier curve in the program MATHEMATICA as following:

Show[Graphics[{Thickness[0.005],PointSize[Large],BezierCurve[{{0,5},{20,10},{40,−10},{60,−5}}],RGBColor[

0,0,1],Line[{{0,5},{20,10},{40,−10},{60,−5}}],RGBColor[1,0,0],Point[{{0,5},{20,10},{40,−10},{60,−5}}]}],Axes→True]



Thus the difference operator named control convex hull vectors are

$$\Delta\mathbf{b}_0 = \mathbf{b}_1 - \mathbf{b}_0 = (20,5)$$

$$\Delta\mathbf{b}_1 = \mathbf{b}_2 - \mathbf{b}_1 = (20,-20)$$

$$\Delta\mathbf{b}_2 = \mathbf{b}_3 - \mathbf{b}_2 = (20,5)$$

and the norms of them are

$$\|\Delta\mathbf{b}_0\| = \sqrt{20^2 + 5^2} \cong 20.616$$

$$\|\Delta\mathbf{b}_1\| = \sqrt{20^2 + (-20)^2} \cong 28.284$$

$$\|\Delta\mathbf{b}_2\| = \sqrt{(20)^2 + 5^2} \cong 20.616$$

The angle θ is an angle between the vectors $\Delta\mathbf{b}_0$ and $\Delta\mathbf{b}_1$ at the beginning point, i.e. θ: [illegible]$_1)$. Therefore we can obtain the angle $\cos\theta = \frac{<\Delta\mathbf{b}_0, \Delta\mathbf{b}_1>}{\|\Delta\mathbf{b}_0\|.\|\Delta\mathbf{b}_1\|} \cong 0.515$. Then the angle is found as $\theta = \arccos(0.515) \cong 0.870$. The angle ψ is an angle between the vectors $\Delta\mathbf{b}_1$ and $\Delta\mathbf{b}_2$, i.e. ψ: [illegible]$_2)$. Thus the angle ψ can be calculated from the equation $\cos\psi = \frac{<\Delta\mathbf{b}_1, \Delta\mathbf{b}_2>}{\|\Delta\mathbf{b}_1\|.\|\Delta\mathbf{b}_2\|} \cong 0.515$. As a result the angle is obtained as $\psi = \arccos(0.515) \cong 0.870$. The curvature of the yarn at the beginning point is $\kappa|_{t=0} = \frac{n-1}{n}\frac{\|\Delta b_1\|}{\|\Delta b_0\|^2}.\sin\theta \cong 0.034$. The curvature radius is $\rho(t)|_{t=0} = \frac{1}{\kappa|_{t=0}} \cong 29.412$. Also for the control points $b_i \in E^3$ of the Bezier curve $b^n(t)$ at the beginning point $t=0$, the curvature center $m|_{t=0}$ of the yarn is $m|_{t=0} \cong (8.012, 37.114)$.

Conclusion

In this paper we have researched some diferential geometric properties of computer aided geometric design curves and we have given an application for a knitted fabric.

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