Reduction Principle of the Mathematical Model

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***Abstract* - The detection and elimination principle of redundant elements in the mathematical model is proposed in this paper. The efficiency of the proposed approach has been analyzed based on the neural network model of economic system and differential equations model. Results prove the 16 times increasing in the model accuracy.**

Keywords - mathematical model, reduction, neural network, incorrectness.

I. Introduction

The identification of the mathematical model is reversed problem, and therefore incorrect. Errors often appear due to the model redundancy that results in more parameters for this identification problem [1].

In this paper we propose the new reduction principle of the mathematical model. This principle is a simplified form has long been used for the reduction of mathematical models in a representation of ordinary differential equations [2]. The novelty of our principle is in simpler identifying and removing of redundant parameters (elements) within the model. This allows to increase the correctness of the mathematical model identification as shown for the models in the forms of a neural network and differential equations.

II. Reduction Principle

Suppose that for the simulated object exist the exact mathematical model of parameters vector $p = (p_1,\ldots,p_n)$ is priori known. The parameters of the model are computed by some identification procedure. Let zero value of parameter means the element absence and the solution of the identification problem – the vector p – is found by continuous dependence on the conditions in a environs of these conditions.

We complicate the mathematical model by introducing the excess parameters in the vector $p = (p_1,\ldots,p_n, p_{n+1},\ldots,p_m)$. In this case, for redundant parameters $p_{n+1},\ldots,p_m$ the identification procedure will calculate almost zero values (within the accuracy of calculations):

$$p_i \approx 0;\ \ i = n+1,\ldots,m. \qquad (1)$$

However only the property (1) can not identify and remove the excess parameters, because some values of the parameters $p_1,\ldots,p_n$ are not redundant even though they are nearly zero.

Therefore, we introduce the small random perturbations named *disturb* to the identification problem within the vicinity of the continuity solution. The identification procedure will calculate the parameter vector p' for perturbed problem, different from the vector p for unperturbed problem. For each parameter we compute the modules of relative deviations as following:

$$\delta_i = abs((\, p'_i - p_i) \,/\, p'_i\,);\ \ i=1,\ldots,m. \qquad (2)$$

For the necessary parameters absolute deviations $(p'_i - p_i)$ tend to zero while perturbations tend to zero due to continuous relation between parameters and perturbations. The same situation is for relative deviations:

$$\delta_i \to 0;\ i = 1,\ldots,n;\ \text{if}\ disturb \to 0. \qquad (3)$$

Contrary for the unnecessary parameters relative deviation values (2) are close to one within the interval of model continuity due to (1):

$$\delta_i \approx 1;\ i = n+1,\ldots,m;\ \text{if}\ disturb \neq 0. \qquad (4)$$

Criteria (3) and (4) are received for the model precision. We extend these criteria to the arbitrary mathematical models.

In general, unnecessary parameters can be found by much larger relative deviations (2) compared to the necessary parameters. The consistent elimination of the unnecessary elements of the mathematical model improves the accuracy and stability of the identification problem. Numerous examples confirm this conclusion [1] including the example shown in this article.

Studying of a problem has shown that in reduced model the relative deviations values are assembled in a compact group relatively close to zero.

There are many different applications of the proposed reduction principle.

Using the reduction principle we can effectively extend the structure of the mathematical model by checking each novel element and removing unnecessary elements.

But alternatively we can apply the reduction principle in the algorithm described below:

Step 1. Identification of the model parameters p_i .

Step 2. Identification of the slightly perturbed models with parameters p'_i .

Step 3. Calculation the modules of relative deviations of parameters as following: $\delta_i = abs((p'_i - p_i) / p'_i)$.

Step 4. In case of no relative deviations, which is much larger than the average value the reduction algorithm is finished.

Step 5. Removing the model element with the biggest δ_i and return to step 1.

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III. Neural network reduction

Reduction method has been widely used for a long time to regulate mathematical models in a representation of ordinary differential equations [1]. We develop an application of the reduction principle to models based on neural networks.

There were a lot of researches to simplify the structure of neural networks [3,4]. To simplify the models using first and second derivatives of the objective function. But our proposed solution for this problem is simple, universal and does not depend on the method of network training.

We consider the example of neural network application for model reduction that approximates the economic system of stocks profits, bonds and interest rates on deposits based on time series of the macroeconomic indicators S^{ec} and indicators of stock market S^{fin}.

The model input are 8 economic indicators: the consumer price index *isc*; the money supply *mf*; the household income *dn*; the public spending *dv*; the gross domestic product *dgp*; the average rate on deposits *dep*; the index of bonds *bond*; the trading system index of the first stock *pfts*.

The output data is the average rate of the profit of deposits *dep*, index of bonds *bond*, and the trading system index of the first stock *pfts*.

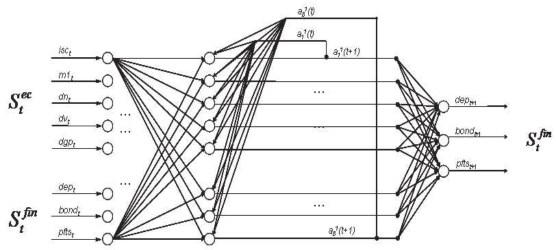
Input and output data were acquired quarterly during the period from 2002 to 2013, and represented by 44 samples.

According to acquired data we train the three-layer recursive neural network with 12 neurons in the hidden layer by using back propagation method.

The activation function is represented as the sigmoid with parameter α=0.5. There are 276 variable parameters of transfers between neurons.

The structure of the neural network is shown in Fig. 1.

The method of identification is the "back propagation" with the approximation criterion RMS (Root Mean Square) error of the output data reproduction.



$$\mathbf{a}^1(t+1) = f(\mathbf{IW}\cdot(\mathbf{S}_t^{ec}, \mathbf{S}_t^{fin}) + \mathbf{DW}\cdot\mathbf{a}^1(t) + \mathbf{b}^1) \quad t = \overline{0, T-1}, \quad \mathbf{a}^1(0) = 0.$$

$$\mathbf{S}_{t+1}^{fin} = g(\mathbf{LW}\cdot\mathbf{a}^1(t+1) + \mathbf{b}^2),$$

Fig. 1. The structure of the neural network and it's equations; IW – the transmission matrix of input to hidden layer; DW – the recurrent matrix; LW – the transmission matrix of hidden to output layer; $a^1(t)$ – the output vector of hidden layer.

The iterative network reduction is a sequential removing of the connections according to the above algorithm, but without stopping reduction. The graph of mean square error of approximation of one output variable depending on the number of iteration is shown in Fig. 2.

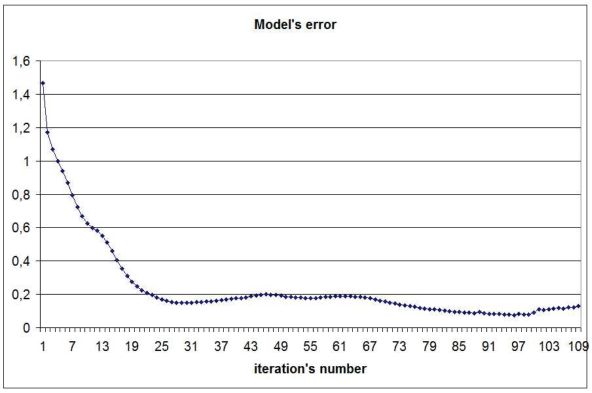


Fig. 2. The error reducing from the reduction of neural network.

Fig. 3 and Fig. 4 show the reduction in the range of the relative errors δ_i values. The relative deviations values in reduced network are assembled in a compact group near zero. This is a sign of the end of the reduction process.

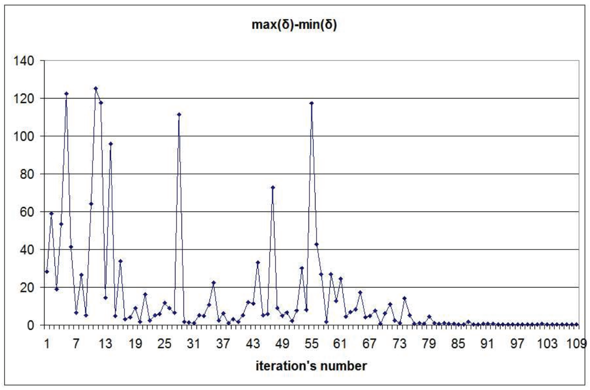


Fig. 3. The reducing in the range of values of relative errors δ_i from the reduction of neural network.

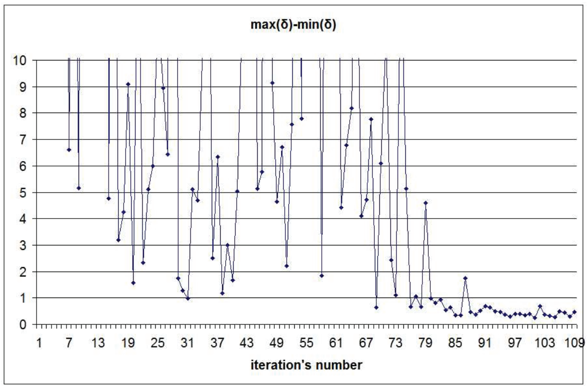


Fig. 4. The graph fig.3 on a larger scale.

We found that the smallest error value is 0.09 on the 99-th iteration, while 35% of connections is removed.

The reduction of the network reduced the inaccuracy 16 times.

IV. Differential equations reduction

This example shows a reduction for relatively simple models of electrical object.

You need to build a simple model of oscillator, shown in Figure 5. In this figure except generator are two differentiating link to compute derivatives.

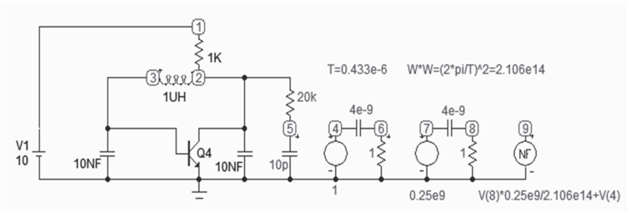


Fig. 5. Oscillator circuit in MisroCap 9

The generator output signal y is a voltage of node 5. This voltage is differentiated twice by RC-links. Controlled voltage source at node 9 defines an internal signal of model v.

On the output signal y found a oscillation's period T = 0.433•10-6. The corresponding value of the square of the frequency ω2 = 2.106•1014 is the only option for the linear conservative subsystem of second order:

$$\ddot{y} + \omega^2 y = \omega^2 v \tag{5}$$

Hammerstein model (Fig. 6) is chosen with the linear subsystem (5) and nonlinearity in the form of polynomial.

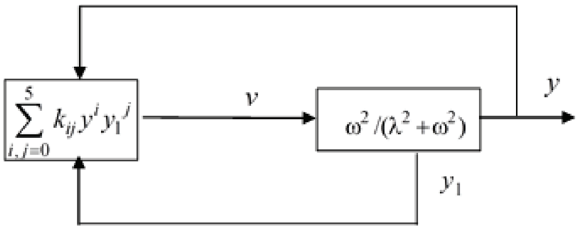


Fig. 6. Hammerstein model.

The corresponding differential equations of the oscillator model

$$\begin{aligned} &dy/dt = y_1, \\ &dy_1/dt = -\omega^2 y + \omega^2 v, \\ &v = \sum_{i,j=0}^{5} k_{ij} y^i y_1^j, \quad i + j \le 5 \end{aligned} \tag{6}$$

consist an equations of a linear conservative subsystem of second order (ω^2=2.106•10^{14}) and nonlinear polynomial function of fifth degree in two arguments.

Note that model is of second order, while the equation of oscillator circuit shown in Fig. 5 is of sixth order.

Identification of this simple mathematical model is difficult because incorrect.

By successive differentiation with a constant time $\tau = 4 \cdot 10^{-9}$ the second derivative is found. Then the internal signal of the model v is calculated by equation (5).

The analysis of the circuit in Fig. 5 in the MicroCap9 give the 300 discrete values y_m, y_{1m}, v_m, (m=1,…,M), of the necessary functions y, y_1, v.

The 21 coefficient of non-linear function calculates the identification procedure

$$\min_{k} \sum_{m=1}^{300} \left(v_m - \sum_{i,j=0}^{5} k_{ij} y_m^i y_{1m}^j \right)^2 . \tag{7}$$

But because of its incorrectness, the solution of the system (6) with the calculated coefficients is completely different from the output of the oscillator.

The coefficients of nonlinear function can be calculated with Tihonov's regularization [2]:

$$\min_{k} \left(\sum_{m=1}^{300} \left(v_m - \sum_{i,j=0}^{5} k_{ij} y_m^i y_{1m}^j \right)^2 + \alpha \sum_{i,j=0}^{5} k_{ij}^2 \right). \tag{8}$$

But by selecting the regularization factor α can not find a model with an adequate output signal.

The desired result was obtained only after the reduction of the polynomial $\sum_{i,j=0}^{5} k_{ij} y_m^i y_{1m}^j$.

The reduction procedure described above reduces the number of coefficients of the nonlinear function from 21 to 5. For $\alpha = 0.05$ the corresponding polynomial is

v=0.45437+0.43143•y^4–1.9343•10^{-13}•y^3•y_1^2+ +1.0573•10^{-19}•y^2•y_1^3–5.4322•10^{-33}•y_1^5.

In Fig. 7 shows the transient processes in the generator and its model. First, the projection of the phase portrait of the generator onto the plane (V(4), V(7)). Below is the phase portrait of the model. Even lower the time dependences of the output signals of the generator and its model.

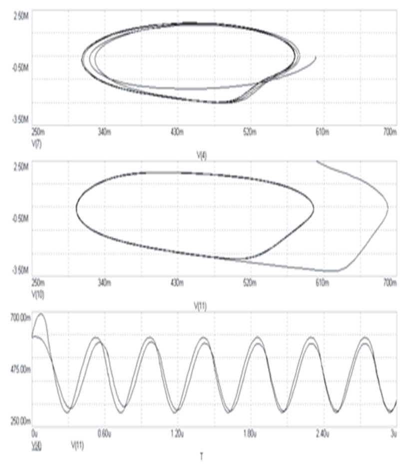


Fig. 7. Output signals of generator and model

The model repeats the established oscillations with a maximum relative error of 6%. An error in transient processes can be explained by essentially different orders of the generator and the model. The scheme of the generator is of the sixth order, and the model is only of the second order.

V. Conclusion

We propose simple principle of mathematical model reduction. This principle works for mathematical models of any nature. It is only necessary to the procedure of model parameters identification and the continuous dependence of parameters on the identification conditions.

We showed that application for the neural network model and the differential equations model. Provided example shows effective increasing the accuracy of both mathematical models.

In further research we will conduct more study on the efficiency of our proposed approach comparing to existing methods.

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