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ELABORATION OF EQUIPOTENTIAL SURFACES OF PLANETS USING BIORTHOGONAL EXPANSIONS

Purpose. Using known and fixed Earth potential, presented asthe biorthogonal expansion, to culculate the geoid surface, which describes the actual shape of the planet. The external gravitational field is generally described by the series of spherical functions. Since the geoid is determined with the help of such functions, a question arises converning the identity to define the shape, moreover its several points does not belong to the region of convergence. **Methodology and results**. We consider representation of potential by convergent series everywhere, which makes it possible to find the geoid without specifying the location of points on the surface, although the geoid heights calculation is carried out by various relations. According to the known function of the mass distribution of the Earth, represented by the second degree polynomial, internal and external potential of elliptical planet are defined and the equipotential surfaces are found. Calculated values via these formulas and their degree of coincidence was analyzed. Defined in two ways surfaces do not coincide with each other because the difference in the values of the radius-vector amouts up to ten meters. So, when applying biorthogonal expansions of higher orders in constructing equipotential surfaces based on information about the external gravitational field it is necessary to take into account characteristics of expansion. **Originality.** Method of determining the shape of the Earth using the biorthogonal expansions of mass distribution function is proposed. This representation is characterized by a convergence for considered series and gives the opportunity to build digital models of the geoid (volumetric or as an isolines map). **Practical significance.** The results of numerical experiments, described in the article, led to the conclusion about the possibility of determining the equipotential surfaces that adequately describe the physical surface of the planet not only of the second but higher orders using biorthogonal expansions only with additional investigations. Calculation of geoid heights with high accuracy opens the way to observe many regional and local geodynamic phenomena, such as the movement of tectonic plates, and high accuracy leveling using GPS technology can solve a number of geodetic problems.

Key words: potential, spherical functions, level surface, convergence of series, ellipsoid.

Introduction

External gravitational field of planets is presented in sufficient detail by series of spherical functions [Pavlis, 2008; Yung, 2001; Konopliv, 1999; Konopliv, 2001]. Several works are devoted to the study of convergence of series in spherical functions in any unspecified region [Kraup, 1969; Marchenko, 1983; Marchenko, 1998; Pellynen, 1978, Moritz, 1983]. Besides, practical research on representation of potential by other functions is also actual [Meshcheryakov, 1991; Antonov, 1988; Zahrebyn, 1976; Marchenko, 1982, Balmino, 1975]. Other approach is description of three-dimensional part of the function of the mass distribution through biorthogonal series, which further could represented the potential (internal and external) of the planet. Such representation is represented by a universally convergent series [Chernyaha, 2014], thus makes it possible to determine the the Earth surface levelwith more confidence.

Purpose

Definition of equipotential surfaces by known fixed potential (internal and external) determines the shape of relativity surface (geoid for the Earth). As a rule, the external gravitational field is described by series of spherical functions. Since some points of the Earth form don't belong to convergence region of such series, the question of adequate description of this shape arises.

Methodology

Internal and external potential of three dimensional part of mass distribution of planet is described by expression:

$$U = f\int_{\tau}\frac{\delta}{r}d\tau = \sum_{m+n+k=0}^{N} b_{mnk}U_{mnk}(x_1,x_2,x_3), \quad (1)$$

where

$$\delta(x_1,x_2,x_3) = \sum_{m+n+k=0}^{N} b_{mnk}W_{mnk}(x_1,x_2,x_3), \quad (2)$$

$$U_{mnk}(x_1,x_2,x_3) = f\int_{\tau}\frac{W_{mnk}(\xi,\eta,\varsigma)}{r(\xi,\eta,\varsigma,x_1,x_2,x_3)}d\tau. \quad (3)$$

By the known coefficients b_{mnk} and fixed potential U_0 (sea level potential) expression:

$$U_0 = \sum_{m+n+k=0}^{N} b_{mnk}U_{mnk}(x_1,x_2,x_3) \quad (4)$$

defines the surface where it is constant (geoid – for the Earth, selenoid – for the Moon, areoid – for the Mars) and series (4) is convergent. This is a consequence of average convergence expansion (2) [Fys, 1983] and uniform convergence of (1). Therefore, the main problem in proposed approach is determining of coefficients b_{mnk}. Some of them may be defined according to data of gravitational field of the planet (Earth), for example, Stokes constants [Fys, 1997; Fys, 2006]. The research in [Chernyaha, 2014] showed incomplete representation for some directions, so there is a need for additional studies considering the form of linear combinations U_{mnk}. However, for $N \le 2$ expansion coefficients are uniquely determined by the formulas [Fys, 2008]

$$b_{000} = \delta_C,$$
$$b_{002} = \frac{7}{2}\left[5\left(\frac{-C_{20}}{2H}+C_{20}\right)\delta_C - \gamma^2 b_{000}\right],$$
$$b_{200} = \frac{7}{2}\left[5\left(2C_{22}-\frac{C_{20}}{2H}\right)\delta_C - \alpha^2 b_{000}\right], \quad (5)$$
$$b_{020} = \frac{7}{2}\left[5\left(-\frac{C_{20}}{2H}-2C_{22}\right)\delta_C - \beta^2 b_{000}\right].$$
$$b_{101} = b_{011} =, b_{110} = 0,$$

where C_{20}, C_{22} – Stokes constants, that are given to the principal axes of inertia.

To simplify calculations we take data accepted in GRS – 80 [Moritz, 1979], namely:

$$U_0 = 62636860.85 м^2c^{-2},$$
$$fM = (398600.50 \pm 0.05) км^3c^{-2},$$

and the Stocks constants per the model of the gravitational field GEM-10 [Lerch, 1979].

In this case the equation of the surface with constant potential is as follows:

$$\begin{aligned} u_0 &= b_{000}U_{000}(x_1,x_2,x_3) + \\ &+ b_{200}U_{200}(x_1,x_2,x_3) + b_{002}U_{002}(x_1,x_2,x_3) + \\ &+ b_{020}U_{020}(x_1,x_2,x_3) + b_{110}U_{110}(x_1,x_2,x_3) + \\ &+ b_{101}U_{101}(x_1,x_2,x_3) + b_{011}U_{011}(x_1,x_2,x_3), \end{aligned} \quad (6)$$

where $u_0 = \frac{U_0 R}{fM}$.

Let us write in detail each item U_{mnk}, taking the shape of the planet as sphere (partial case of ellipsoid) with radius $R = 6371 km$.

$$U_{000} = -\frac{3fM}{2R}\left(\frac{1}{3}\rho^2 - 1\right),$$
$$U_{200} = -\frac{3fM}{8R^3}\left(\frac{1}{7}\rho^4\left(1+4\sin^2\theta\cos^2\lambda\right) - \frac{2}{5}\rho^2\left(1+2\sin^2\theta\cos^2\lambda\right)+\frac{1}{3}\right),$$
$$U_{020} = -\frac{3fM}{8R^3}\left(\frac{1}{7}\rho^4\left(1+4\sin^2\theta\sin^2\lambda\right) - \frac{2}{5}\rho^2\left(1+2\sin^2\theta\sin^2\lambda\right)+\frac{1}{3}\right),$$
$$U_{002} = -\frac{3fM}{8R^3}\left(\frac{1}{7}\rho^4\left(1+4\cos^2\theta\right) - \frac{2}{5}\rho^2\left(1+2\cos^2\theta)\right)+\frac{1}{3}\right),$$
$$U_{110} = -\frac{3fM}{R^3}\sin^2\theta\sin2\lambda\left(\frac{1}{7}\rho^4 - \frac{1}{5}\rho^2\right),$$
$$U_{101} = -\frac{3fM}{R^3}\sin2\theta\cos\lambda\left(\frac{1}{7}\rho^4 - \frac{1}{5}\rho^2\right),$$
$$U_{011} = -\frac{3fM}{R^3}\sin2\theta\sin\lambda\left(\frac{1}{7}\rho^4 - \frac{1}{5}\rho^2\right) \quad (7)$$

From equation (4) at fixed values of latitude, longitude and following designations:

$$\begin{aligned} a_0 = \frac{3}{56}\Big[& b_{200}\left(1+4\sin^2\theta\cos^2\lambda\right) + \\ &+ b_{020}\left(1+4\sin^2\theta\sin^2\lambda\right) + \\ &+ b_{020}\left(1+4\cos^2\theta\right) + 4b_{110}\sin^2\theta\sin2\lambda + \\ &+ 4b_{101}\sin2\theta\cos\lambda + 4b_{011}\sin2\theta\sin\lambda\ \Big], \end{aligned} \quad (8)$$

$$\begin{aligned} a_1 = -\frac{3}{20}\Big[& b_{200}\left(1+2\sin^2\theta\cos^2\lambda\right) + \\ &+ b_{020}\left(1+2\sin^2\theta\sin^2\lambda\right) + b_{002}\left(1+2\cos^2\theta\right) + \\ &+ \left(4b_{110}\sin^2\theta\sin2\lambda + 4b_{101}\sin2\theta\cos\lambda + \right. \\ &\left. + 4b_{011}\sin2\theta\sin\lambda\right)\Big] - \frac{1}{2}, \end{aligned} \quad (9)$$

$$a_2 = -\frac{(b_{200} + b_{020} + b_{002})}{8} - u_0 + \frac{3}{2},$$

$$U_2 = a_0\rho^4 + a_1\rho^2 + a_2 \qquad (10)$$

we obtain the formula

$$\rho^2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0a_2}}{2a_0}. \qquad (11)$$

Determining of value for the radius-vector using the formula shown above does not yield reliable results. Therefore further we define it using approximate methods, such as method of half-division.

According to this formulas we make calculations the results of which are shown in Table 1.

Similarly we receive the relation for external potential V, which elements are represented as

$$U_{000} = \frac{fM}{r}, \quad U_{200} = -\frac{fM}{35r^3}\left(1 - 3\sin^2\theta\cos^2\lambda\right),$$

$$U_{020} = -\frac{fM}{35r^3}\left(1 - 3\sin^2\theta\sin^2\lambda\right),$$

$$U_{002} = -\frac{fM}{35r^3}\left(1 - 3\cos^2\theta\right), \qquad (12)$$

$$U_{110} = \frac{fM}{35r^3}\sin^2\theta\cos 2\lambda, \quad U_{101} = \frac{fM}{35r^3}\sin 2\theta\cos\lambda,$$

$$U_{011} = \frac{fM}{35r^3}\sin 2\theta\sin\lambda.$$

The equation for calculating the radius-vector ρ at given values is as follows:

$$V_2 = -\frac{1}{35\rho^3}\Big[\left(1 - 3\sin^2\theta\cos^2\lambda\right)b_{2000} +$$

$$+\left(1 - 3\sin^2\theta\sin^2\lambda\right)b_{020} + \left(1 - 3\cos^2\theta\right)b_{002} -$$

$$-6b_{110}\sin^2\theta\cos 2\lambda - 6b_{101}\sin 2\theta\cos\lambda -$$

$$-6b_{011}\sin 2\theta\sin\lambda\Big] + \frac{1}{\rho}. \qquad (13)$$

Values U_2, V_2 calculated per formulas (10) and (13) respectively, match each other on the surface of the sphere (R = 6371 *km*).

Similarly, we determine the value of the radius-vector using approximation methods; the results of calculations are shown in a Table 2.

Table 1

Values of relative radius-vectors of the geoid, calculated per the inner potential U_2 in meters

Longitude θ (º)	Latitude λ (º)					
	0	30	60	90	120	150
0	6362420.173	6362420.173	6362420.173	6362420.173	6362420.173	6362420.173
30	6362891.463	6362888.739	6362890.093	6362891.459	6362888.737	6362890.095
60	6363833.619	6363825.456	6363829.519	6363833.615	6363825.454	6363829.521
90	6364304.486	6364293.606	6364299.025	6364304.486	6364293.606	6364299.025
120	6363833.615	6363825.454	6363829.521	6363833.619	6363825.456	6363829.519
150	6362891.459	6362888.737	6362890.095	6362891.463	6362888.739	6362890.093

Table 2

Values of relative radius-vectors of the geoid, calculated per the inner potential V_2

Longitude θ (º)	Latitude λ (º)					
	0	30	60	90	120	150
0	6362420.173	6362420.173	6362420.173	6362420.173	6362420.173	6362420.173
30	6362891.464	6362890.087	6362888.742	6362891.464	6362890.087	6362888.742
60	6363833.628	6363829.498	6363825.466	6363833.628	6363829.498	6363825.466
90	6364304.501	6364298.995	6364293.621	6364304.501	6364298.995	6364293.621
120	6363833.628	6363829.498	6363825.466	6363833.628	6363829.498	6363825.466
150	6362891.464	6362890.087	6362888.742	6362891.464	6362890.087	6362888.742

The comparison of two tables shows a discrepancy to tens of meters of radius-vector values for surfaces, determined by different methods. Using biorthogonal expansion higher than the second order is possible only with the use of harmonic functions like in (12), since not all expansion coefficients *bvnk* can be defined according to data on the external gravity field (only their linear combinations are established). In this regard, to obtain radius-vectors of equal potential surfacesusing formulas like in (11) more research is needed.

Originality

Method of determining the shape of the Earth using the biorthogonal expansions of mass distribution function is proposed. This representation is characterized by convergence for considered series and gives the opportunity to build digital models of the geoid (volumetric or as an isolines map). Obtained formulas make it possible to compare the numerical values of the geoid surfaces determined according to various expressions.

Practical significance

The results of numerical experiments, described in the article, led to the conclusion about the possibility of determining the equipotential surfaces that adequately describe the physical surface of the planet. Calculation of geoid heights with high accuracy opens the way to study plenty of regional and local geodynamic phenomena, such as movement of tectonic plates. High accuracy leveling using GPS technology can solve a number of geodetic problems. The resulting surface, being close to the physical one, makes it possible to determine the components of the normal vector to level surface of the Earth.

Conclusions

As a result of numerical experiments we arrive to the following conclusions:

1. Equipotential surfaces obtained per the internal and external gravitational field for the density of the second order inclusively are not identical to each other (the difference reaches ten meters);
2. Potentials, determined by means of biorthogonal expansions to the second order inclusive (formula (10) and (13)), are equal on the sphere surface;
3. Determination of geoid by coefficients defined via Stokes constants of higher orders will be subject to further investigations.

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ДОВИЗНАЧЕННЯ ЕКВІПОТЕНЦІАЛЬНИХ ПОВЕРХОНЬ ПЛАНЕТ З ВИКОРИСТАННЯМ БІОРТОГОНАЛЬНИХ РОЗКЛАДІВ

Мета. За відомим фіксованим потенціалом Землі, поданим за допомогою біортогонального розкладу, як одного з варіантів його представлення, знайти поверхню геоїда, яка описує реальну фігуру планети. Зовнішнє гравітаційне поле описується, як правило, рядами за кульовими функціями. Оскільки геоїд визначають з їх використанням, тому виникає питання ідентичності визначення фігури, тим паче, що частина її точок не належить області збіжності. **Методика і результати роботи**. У роботі розглянуто представлення потенціалу всюди збіжними рядами, що дає можливість знаходити геоїд без уточнення розміщення точок на його поверхні, хоча обчислення висот геоїда здійснюється за різними співвідношеннями. За відомою функцією розподілу мас надр Землі, представленою многочленом другого степеня, визначено внутрішній та зовнішній потенціал еліптичної планети, за яким знайдено еквіпотенціальні поверхні. Проаналізовано обчислені значення за цими формулами та степінь їх співпадання. Визначені двома способами поверхні рівня не співпадають між собою, бо різниця в значеннях радіус-векторів досягає десятків метрів. Тому застосувати біортогональні розклади вищих степенів під час побудови еквіпотенціальних поверхонь на основі інформації про зовнішнє гравітаційне поле необхідно з урахуванням особливостей розкладу. **Наукова новизна.** Запропонований метод визначення фігури Землі з використанням біортогональних розкладів функції розподілу мас. Таке представлення характеризується збіжністю для розглянутих рядів та дає можливість будувати цифрові моделі геоїда (об'ємні, або у вигляді карт ізоліній). **Практична значущість.** Результати числових експериментів, наведених у статті, дали змогу зробити висновок про можливість визначення еквіпотенціальних поверхонь, які адекватно описують фізичну поверхню планети, не тільки другого, а і вищих порядків з використанням біортогональних розкладів лише за додаткових досліджень. Обчислення висот геоїда з високою точністю відкриває шлях до дослідження багатьох регіональних та локальних геодинамічних явищ, наприклад, руху тектонічних плит, а високоточне нівелювання за допомогою GPS-технологій дає змогу розв'язувати низку геодезичних задач.

Ключові слова: потенціал, кульові та сферичні функції, поверхні рівня, збіжність рядів, еліпсоїд.

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