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MODIFIED PROBABILITY MEASURE FUNCTOR OF IN THE COARSE CATEGORY

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It is known that the probability measure functor does not preserve the class of locally compact metric spaces. Therefore, it does not have its straightforward counterpart in the coarse category. We define a modified functor of probability measures on the category of proper metric spaces and coarse maps.

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I. Introduction

The coarse category (i.e. the category of coarse spaces and coarse maps) was introduced by Roe in [9]. This theory turned out to be an appropriate universe for studying asymptotic properties of structures more general then metric spaces.

I. Protasov classified some recent results concerning algebraic structures in the coarse category as those belonging to the asymptotic algebra.

In particular, in [4] the hyperspace functor acting in the category of coarse topological spaces was considered. It was proved in [4] that the hyperspace functor determines a monad in the coarse category and the natural problem arises whether another monads in the category of compact Hausdorff spaces have their counterparts in the coarse category.

In this paper we consider the case of probability measure monad. It was noted in [13] that the functor of probability measures has no natural extension to the category of coarse topological spaces and the main reason of it is that this functor fails to preserve the class of locally compact spaces.

We introduce a modified functor of probability measures. To be more precise, a probability measure in our sense is a pair $(\mu, \operatorname{supp}\mu)$, where, as usual, $\operatorname{supp}\mu$ denotes the support of μ. Then convergence means the weak* convergence of measures together with convergence of their supports. The obtained spaces have some applications in the mathematical economics. Note also that these spaces are tightly connected to the mm-spaces in the sense of Gromov [5].

In this note we show that the modified probability measure functor determines a monad in the coarse category of proper metric spaces. We also consider the notion of tensor product related to this monad.

II. The coarse category of metric spaces

We start with some necessary definitions (see, e.g. [1]).

A metric space is said to be *proper* if every its closed ball is compact.

A map $f\colon X \to Y$ between metric spaces is said to be *coarse* [9] if:

1) f is *coarsely uniform*, i.e. for every $\epsilon > 0$ there exists $\eta > 0$ such that $d(x, y) < \epsilon$ implies $d(f(x), f(y)) < \eta$ for every $x, y \in X$;

2) f is *coarsely proper* in the sense that the preimage of every bounded set is bounded.

The proper metric spaces and coarse maps form a category, which we denote by CMS.

A Spaces of probability measures

Given a metric space X, we denote by $P(X)$ the space of probability measures in X with compact support and by $\exp X$ the hyperspace of X. The space $P(X)$ is endowed with the weak* topology and the space $\exp X$ with the Vietoris topology (see, e.g., [3] for details). Note that, for $x \in X$, by δ_x we denote the Dirac measure concentrated in x. For a metric space (X, d), the Vietoris topology is generated by the Hausdorff metric, d_H,

$$d_H(A, B) = \inf\{\varepsilon > 0 \mid A \subset O_\varepsilon(B),\ B \subset O_\varepsilon(A)\}.$$

Let $\mathcal{P}(X) = \{(\mu, A) \in P(X) \times \exp X \mid \text{supp } \mu \subset A\}$.

If (X, d) is a proper metric space, we endow $\mathcal{P}(X)$ with the metric $\widehat{d} = d_{\mathcal{P}(X)}$ defined as follows:

$$d_{\mathcal{P}(X)}((\mu_1, A_1), (\mu_2, A_2)) = d_{KR}(\mu_1, \mu_2) + d_H(A_1, A_2).$$

Here d_{KR} denotes the Kantorovich-Rubinshtein metric on the space of probability measures of a metric space

(X,d) (see [7]). Given $\mu,\nu\in P(X)$, we let

$$d_{KR}(\mu,\nu)=\inf\left\{\int_{X\times X} d\mathrm{d}\lambda \mid \lambda\in P(X\times X)\right.$$
$$\left.\text{is a measure with marginals } \mu,\ \nu\right\}.$$

Proposition 1. If (X,d) is a proper metric space, then so is $(\mathcal{P}(X),\widehat{d})$.

$\square$ *Proof.* It is sufficient to prove that any closed ball centered at $(\delta_{x_0},\{x_0\})$, for some base point $x_0\in\mathcal{P}(X)$ with $\widehat{d}((\mu,A),(\delta_{x_0},\{x_0\}))\le r$ we have

$$\operatorname{supp}\mu\subset A\subset B=\{y\in X|d(y,x_0)\le r\}.$$

Therefore, the closed ball of radius r centered at $(\delta_{x_0},\{x_0\})$ in $\mathcal{P}(X)$ is a closed subset of $P(B)\times\exp B$. Since $P(B)\times\exp B$ is compact, we are done. ■

The set A is said to be the support of $(\mu,A)\in\mathcal{P}(X)$. The motivation of this term lies in the theory of normal functors in the category of compact Hausdorff spaces developed by E.V. Shchepin [10].

Proposition 2. The set $\mathcal{P}_\omega(X)$ of points with finite supports is dense in $\mathcal{P}(X)$.

$\square$ *Proof.* This follows from the general results of the theory of normal functors (see [10]). ■

III. Modified probability measure monad

Let (X,d) be an object of the category CMS.

Let us define the map $\eta_X\colon X\to\mathcal{P}(X)$ as follows: $\eta_X(x)=(\delta_x,\{x\})$.

Proposition 1. The map η_X is coarsely uniform and metric proper.

$\square$ *Proof.* Choose an arbitrary $\epsilon>0$ and such points $x,y\in X$ that $d_X(x,y)<\epsilon$. To show that η_X is coarsely uniform we consider

$$d_{\mathcal{P}(X)}(\eta_X(x),\eta_X(y))=d_{\mathcal{P}(X)}((\delta_x,\{x\}),(\delta_y,\{y\}))$$
$$=d_{KR}(\delta_x,\delta_y)+d_H(\{x\},\{y\})=d_X(x,y)+d_X(x,y)=$$
$$=2d_X(x,y)<2\epsilon.$$

Thus, one can take $\delta=2\epsilon$ in the definition of coarse uniformity.

Now we need to show that the map η_X is metric proper. Fix an arbitrary $\epsilon>0$ and $A\subset\mathcal{P}(X)$ such that $\operatorname{diam}A<\epsilon$.

Then

$$\operatorname{diam}\eta_X^{-1}(A)=\operatorname{diam}\{x\in X|\eta_X(x)=(\delta_x,\{x\})\in A\}$$
$$=\inf\{d_X(x,y)|x,y\in\eta_X^{-1}(A)\}.$$

Since $\operatorname{diam}A<\epsilon$, we see that

$$\inf\{d_{\mathcal{P}(X)}((\nu_1,B_1),(\nu_2,B_2))|(\nu_i,B_i)\in A\}<\epsilon.$$

That $(\nu_i,B_i)\in A$ means there exist $x_1,x_2\in X$ such that $\delta_{x_1}=\nu_1,\delta_{x_2}=\nu_2$ and $B_1=\{x_1\},B_2=\{x_2\}$. $(*)$

Then for arbitrary $x_1,x_2\in\eta_X^{-1}(A)$ we can find $(\nu_i,B_i)\in A$ that $(*)$ holds.

Since

$$d_{\mathcal{P}(X)}((\delta_{x_1},\{x_1\}),(\delta_{x_2},\{x_2\}))$$
$$=d_{KR}(\delta_{x_1},\delta_{x_2})+d_H(\{x_1\},\{x_2\})=2d(x_1,x_2)<\epsilon,$$

that is why $d(x_1,x_2)<\frac{\epsilon}{2}$ and therefore $\operatorname{diam}\eta_X^{-1}(A)<$ $<\frac{\epsilon}{2}$. ■

Proposition 2. The class of maps $\eta=(\eta_X)$ is a natural transformation of the identity functor into the functor $\mathcal{P}$.

$\square$ *Proof.* We have to show that the diagram

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & \mathcal{P}(X) \arrow[d, "\mathcal{P}(f)"] \\ Y \arrow[r, "\eta_Y"] & \mathcal{P}(Y) \end{tikzcd}$$

commutes. Here the map $\mathcal{P}f\colon\mathcal{P}(X)\to\mathcal{P}(Y)$ acts by the formula

$$\mathcal{P}f(\mu,A)=(\sum_{i=1}^k\alpha_i\delta_{f(x_i)},f(A)).$$

Let $x\in X$, then $\eta_Y(f(x))=(\delta_{f(x)},\{f(x)\})$ and

$$\mathcal{P}(f)(\eta_X(x))=\mathcal{P}(f)(\delta_x,\{x\})=(\delta_{f(x)},\{f(x)\}).$$

Therefore we are done. ■

For the space $\mathcal{P}(X)$, one can define the space $\mathcal{P}^2(X)$. Obviously, the set $\mathcal{P}^2_\omega(X)$ of elements of the form $(M,\mathcal{A})$, where $M=\sum\limits_{i=1}^k\alpha_i\delta_{(\mu_i,A_i)}$, $\mathcal{A}=\{(\mu_i,A_i)|i=1,\dots,k^{'}\}$, for some $k^{'}\ge k$, $\mu_i=\sum\limits_{j=1}^{l_i}\beta_{ij}\delta_{x_{ij}}$, $A_i=\{x_{ij}|j=1,\dots,l_i^{'},\}$ $l_i^{'}\ge l_i$, is dense in $\mathcal{P}^2(X)$.

Define the map $\psi_X\colon\mathcal{P}^2_\omega(X)\to\mathcal{P}_\omega(X)$ by the formula

$$\psi_X(M,\mathcal{A})=\left(\sum_{i=1}^k\sum_{j=1}^{l_i}\alpha_i\beta_{ij}\delta_{x_{ij}},\bigcup_{i=1}^{k^{'}}A_i\right),$$

where

$$\mu_i=\sum_{j=1}^{l_i}\beta_{ij}\delta_{x_{ij}},\quad A_i=\{x_{ij}|j=1,\dots,l_i^{'}\},\ l_i^{'}\ge l_i.$$

Proposition 3. The map ψ_X is coarsely uniform and metric proper.

□ *Proof.* We denote by $\psi'_X\colon P^2(X)\to P(X)$ and $u_X\colon \exp^2 X\to\exp X$ the monad multiplications for for the probability measure monad and the hyperspace monad respectively. Note that both of them preserve the distances. Since $\mathcal{P}^2_\omega(X)$ is dense in $\mathcal{P}^2(X)$, in order to show that the map $\psi_X\colon\mathcal{P}^2(X)\to\mathcal{P}(X)$ is coarsely uniform, we have to show that so is $\psi_X\colon\mathcal{P}^2_\omega(X)\to\mathcal{P}_\omega(X)$.

Let $M=\sum\limits_{i=1}^{k_1}\alpha_i^1\delta_{(\mu_i,A_i)}$, $\mathcal{A}=\{(\mu_i,A_i)|i=1,\dots,k_1^{'}\}$,

$N=\sum\limits_{s=1}^{k_2}\alpha_s^1\delta_{(\nu_s,B_s)}$, $\mathcal{B}=\{(\nu_s,B_s)|s=1,\dots,k_2^{'}\}$,

where

$\mu_i=\sum\limits_{j=1}^{l_i^1}\beta_{ij}^1\delta_{x_{ij}}$, $A_i=\{x_{ij}|j=1,\dots,{l_i}^{'1}\}$,

$\nu_s=\sum\limits_{t=1}^{l_s^2}\beta_{st}^2\delta_{y_{st}}$, $B_s=\{y_{st}|t=1,\dots,{l_s}^{'2}\}$.

Now, estimate the distance between $\psi_X(M,\mathcal{A})$, $\psi_X(N,\mathcal{B})$ in the space $\mathcal{P}_\omega(X)$. By the definition,

$$d_{\mathcal{P}^2_\omega(X)}\left(\psi_X(M,\mathcal{A}),\psi_X(N,\mathcal{B})\right)=$$

$$=d_{\mathcal{P}_\omega(X)}\left(\sum_{i=1}^{k_1}\sum_{j=1}^{l_i^1}\alpha_i^1\beta_{ij}^1\delta_{x_{ij}},\right.$$

$$\left.\bigcup_{i=1}^{k_1^{'}}A_i,\sum_{s=1}^{k_2}\sum_{t=1}^{l_s^2}\alpha_s^2\beta_{st}^2\delta_{y_{st}},\bigcup_{s=1}^{k_2^{'}}B_s\right)=$$

$$=d_{KR}\left(\sum_{i=1}^{k_1}\sum_{j=1}^{l_i^1}\alpha_i^1\beta_{ij}^1\delta_{x_{ij}},\right.$$

$$\left.\sum_{s=1}^{k_2}\sum_{t=1}^{l_s^2}\alpha_s^2\beta_{st}^2\delta_{y_{st}}\right)+d_H\left(\bigcup_{i=1}^{k_1^{'}}A_i,\bigcup_{s=1}^{k_2^{'}}B_s\right)$$

$$=\sum_{i,j,s,t=1}^{k_1,l_i^1,k_2,l_t^2}\eta_{ijst}d(x_{ij},y_{st})+d_H\left(\bigcup_{i=1}^{k_1^{'}}A_i,\bigcup_{s=1}^{k_2^{'}}B_s\right).\qquad(*)$$

Thus,

$$d_{\mathcal{P}^2_\omega(X)}\left((M,\mathcal{A}),(N,\mathcal{B})\right)=d_{KR}\left(\sum_{i=1}^{k_1}\alpha_i^1\delta_{(\mu_i,A_i)},\right.$$

$$\left.\sum_{s=1}^{k_2}\alpha_s^1\delta_{(\nu_s,B_s)}\right)+d_H(\mathcal{A},\mathcal{B})=$$

$$=\sum_{i,s=1}^{k_1,k_2}\gamma_{is}\sum_{j,t=1}^{l_i^1,l_s^2}\gamma_{jt}d(x_{ij},y_{st})+\sum_{i,s=1}^{k_1,k_2}\gamma_{is}d_H(A_i,B_s)$$

$$+d_H\left(\{(\mu_i,A_i)|i=1,\dots,k_1^{'}\},\right.$$

$$\left.\{(\nu_s,B_s)|s=1,\dots,k_2^{'}\}\right)<\delta$$

and we conclude that $(*)$ does not exceed 3δ. Therefore, the map ψ_X is coarsely uniform.

In order to show that ψ_X is metric proper, we choose an arbitrary set $U\subset\mathcal{P}(X)$ with $\operatorname{diam}U<\varepsilon$. Let $(M,\mathcal{A}),(N,\mathcal{B})\in\psi_X^{-1}(U)$ and consider $d_{\mathcal{P}^2_\omega(X)}((M,\mathcal{A}),(N,\mathcal{B}))$. Applying the reverse considerations we can make the conclusion that $\operatorname{diam}\psi_X^{-1}(U)<$ $<8\varepsilon=\delta$. Therefore, ψ_X is metric proper and we are done. ■

Proposition 4. The class of maps $\psi=\psi(X)$ is a natural transformation of the functor $\mathcal{P}^2$ into the functor $\mathcal{P}$.

□ *Proof.* Let $(M,\mathcal{A})\in\mathcal{P}^2(X)$ and , then

$$M=\sum_{i=1}^{k}\alpha_i\beta_{(\mu_i,A_i)},\mathcal{A}=\{(\mu_i,A_i)|i=1,\dots,k^{'}\},k^{'}\geq k,$$

$$\text{where }\mu_i=\sum_{j=1}^{l_i}\beta_{ij}\delta_{x_{ij}},\quad A_i=\{x_{ij}|j=1,\dots,l_i^{'}\},l_i^{'}\geq l_i.$$

Let us consider

$$\psi_X(M,\mathcal{A})=\left(\sum_{i=1}^{k}\sum_{j=1}^{l_i}\alpha_i\beta_{ij}\delta_{x_{ij}},\bigcup_{i=1}^{k^{'}}A_i\right),$$

then

$$\mathcal{P}f(M,\mathcal{A})=\mathcal{P}f\left(\sum_{i=1}^{k}\sum_{j=1}^{l_i}\alpha_i\beta_{ij}\delta_{x_{ij}},\bigcup_{i=1}^{k^{'}}A_i\right)=$$

$$=\left(\sum_{i=1}^{k}\sum_{j=1}^{l_i}\alpha_i\beta_{ij}\delta_{f(x_{ij})},f\left(\bigcup_{i=1}^{k^{'}}A_i\right)\right)$$

and, on the other hand,

$$\mathcal{P}^2f(M,\mathcal{A})=\left(\sum_{i=1}^{k}\alpha_i\delta_{\mathcal{P}f(\mu_i,A_i)},\mathcal{P}f(\mathcal{A})\right)$$

$$=\left(\sum_{i=1}^{k}\alpha_i\delta_{\left(\sum\limits_{j=1}^{l_i}\beta_{ij}\delta_{f(x_{ij})},f(A_i)\right)},\mathcal{P}f(\{(\mu_i,A_i)\})\right)$$

$$=\left(\sum_{i=1}^{k}\alpha_i\delta_{\left(\sum\limits_{j=1}^{l_i}\beta_{ij}\delta_{f(x_{ij})},f(A_i)\right)},\right.$$

$$\left\{\left(\sum_{j=1}^{l_1}\beta_{1j}\delta_{f(x_{1j})},f(A_1)\right),\dots,\right.$$

$$\left.\left.\left(\sum_{j=1}^{l_{k'}}\beta_{k'j}\delta_{f(x_{k'j})},f(A_{k'})\right)\right\}\right).$$

That is why

$$\psi_X(\mathcal{P}^2f(M,\mathcal{A}))=\left(\sum_{i=1}^{k}\alpha_i\sum_{j=1}^{l_i}\beta_{ij}\delta_{f(x_{ij})},\bigcup_{i=1}^{k^{'}}f(A_i)\right)$$

$$=\left(\sum_{i=1}^{k}\sum_{j=1}^{l_i}\alpha_i\beta_{ij}\delta_{f(x_{ij})},f\left(\bigcup_{i=1}^{k^{'}}A_i\right)\right).$$

■

Proposition 5. The map $\mathcal{P}f\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ is coarsely uniform and metric proper if so is $f\colon X \to Y$.

$\square$ *Proof.* Let $(\mu, A) \in \mathcal{P}(X)$, then $\mathcal{P}f(\mu, A) = \left(\sum\limits_{i=1}^{k} \alpha_i \delta_{f(x_i)}, f(A)\right)$, where $f(A) =$
$= \{f(x_i) | i = 1, \ldots, k'\}$ and $(\nu, B) \in \mathcal{P}(X)$, then

$$\mathcal{P}f(\nu, B) = \left(\sum_{j=1}^{l} \beta_j \delta_{f(y_j)}, f(B)\right),$$

where $f(B) = \{f(y_j) | j = 1, \ldots, l\,'\}$.

Note that

$$d_{\mathcal{P}(X)}((\mu, A), (\nu, B)) = d_{KR}(\mu, \nu) + d_H(A, B) =$$

$$= d_{KR}\left(\sum_{i=1}^{k} \alpha_i \delta_{x_i}, \sum_{j=1}^{l} \beta_j \delta_{y_j}\right) + d_H(A, B) < \epsilon.$$

Consider

$$\widehat{d}(\mathcal{P}f(\mu, A), \mathcal{P}f(\nu, B)) = d_{\mathcal{P}(Y)}\left(\left(\left(\sum_{i=1}^{k} \alpha_i \delta_{f(x_i)},\right.\right.\right.$$

$$\left.\left.\left. f(A)\right), \left(\sum_{j=1}^{l} \beta_j \delta_{f(y_j)}, f(B)\right)\right)\right) =$$

$$= d_{KR}\left(\sum_{i=1}^{k} \alpha_i \delta_{f(x_i)}, \sum_{j=1}^{l} \beta_j \delta_{f(y_j)}\right) + d_H(f(A), f(B)).$$

First,

$$d_{KR}\left(\sum_{i=1}^{k} \alpha_i \delta_{x_i}, \sum_{j=1}^{l} \beta_j \delta_{y_j}\right) = \sum_{i=1}^{k}\sum_{j=1}^{l} \gamma_{ij} d_Y(f(x_i), f(y_j)),$$

where γ_{ij} are the constants defined from the definition of the Kantorovich-Rubinstein metric (see [7] for details).

The map f is coarsely uniform, which means that for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$d_X(x, y) < \epsilon \Rightarrow d_Y(f(x), f(y)) < \delta.$$

That is why

$$d_{KR}\left(\sum_{i=1}^{k} \alpha_i \delta_{x_i}, \sum_{j=1}^{l} \beta_j \delta_{y_j}\right) \le \delta \sum_{i=1}^{k}\sum_{j=1}^{l} \gamma_{ij} = \delta.$$

It is not difficult to show that $d_H(f(A), f(B)) < \delta$. Therefore, $d_{\mathcal{P}(Y)}(\mathcal{P}f(\mu, A), \mathcal{P}f(\nu, B)) < 2\delta$ and as a result the map $\mathcal{P}f$ is coarsely uniform.

To prove the map $\mathcal{P}f$ is metric proper we consider an arbitrary subset $\mathcal{A} \subset \mathcal{P}(Y)$ with $\operatorname{diam} \mathcal{A} < \epsilon$ and show that $\operatorname{diam} \mathcal{P}f^{-1}(\mathcal{A}) < \delta$.

Let $(\nu_1, B_1), (\nu_2, B_2) \in \mathcal{P}f^{-1}(\mathcal{A})$ then there exist such $(\mu_1, A_1), (\mu_2, A_2) \in \mathcal{A}$ that $\mathcal{P}f(\nu_i, B_i) = (\mu_i, A_i), \quad i = 1, 2$. We have

$$d_{\mathcal{P}(X)}(\nu_1, B_1), (\nu_2, B_2)) = d_{KR}(\nu_1, \nu_2) + d_H(B_1, B_2)$$

$$= d_{KR}\left(\sum_{i_1=1}^{k_1} \alpha^1_{i_1} \delta_{x^1_{i_1}}, \sum_{i_2=1}^{k_2} \alpha^2_{i_2} \delta_{x^2_{i_2}}\right) + D_H(B_1, B_2).$$

From the properness of f it follows

$$d_{KR}\left(\sum_{i_1=1}^{k_1} \alpha^1_{i_1} \delta_{x^1_{i_1}}, \sum_{i_2=1}^{k_2} \alpha^2_{i_2} \delta_{x^2_{i_2}}\right)$$

$$= \sum_{i_1=1}^{k_1}\sum_{i_2=1}^{k_2} \gamma_{i_1 i_2} d_X(x^1_{i_1}, x^2_{i_2}) < \delta,$$

$$d_H(B_1, B_2) < \delta.$$

Similarly as in the proof of Proposition 4.1 we see that $\mathcal{P}(f)$ is a metric proper map. ∎

Recall that a monad on a category $\mathcal{C}$ is a triple $\mathbb{T} = (T, \eta, \mu)$ consisting of an endofunctor $T\colon \mathcal{C} \to \mathcal{C}$ and natural transformations $\eta\colon 1_{\mathcal{C}} \to T$ (*unit*) and $\mu\colon T^2 \to T$ (*multiplication*) such that $\mu \circ T\eta = \mu \circ \eta T = 1$ and $\mu \circ T\mu = \mu \circ \mu T$.

Theorem 1. *The triple $\mathbb{P} = (\mathcal{P}, \eta, \psi)$ is a monad on the category CMS.*

$\square$ *Proof.* We are going to show that the diagrams

$$\begin{tikzcd}
\mathcal{P}(X) \arrow[r, "\eta_{\mathcal{P}(X)}"] \arrow[d, "P\eta_X"'] \arrow[dr, "1_{\mathcal{P}(X)}"] & \mathcal{P}^2(X) \arrow[d, "\psi_X"] \\
\mathcal{P}^2(X) \arrow[r, "\psi_X"] & \mathcal{P}(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}^3(X) \arrow[r, "\mathcal{P}(\psi_X)"] \arrow[d, "\psi_{\mathcal{P}(X)}"'] & \mathcal{P}^2(X) \arrow[d, "\psi_X"] \\
\mathcal{P}^2(X) \arrow[r, "\psi_X"] & \mathcal{P}(X)
\end{tikzcd}$$

are commutative.

We first start with $\mathcal{P}_\omega(X)$ and $\mathcal{P}^2_\omega(X)$. It is easy to see that the first diagram is commutative, because

$$\psi_X(\eta_{\mathcal{P}_\omega(X)}(\mu, A)) = \psi_X(\delta_{(\mu, A)}, \{(\mu, A)\})$$

$$= \psi_X\left(\delta_{\left(\sum\limits_{i=1}^{k} \alpha_i \delta_{x_i}, \{x_1, \ldots, x_{k'}\}\right)}, \{(\mu, A)\}\right)$$

$$= \left(\sum_{i=1}^{k} \alpha_i \delta_{x_i}, \bigcup_{i=1}^{k'} \{x_i\}\right)$$

and

$$\psi_X(\mathcal{P}_\omega(\eta_X)(\mu, A)) = \psi_X\left(\sum_{i=1}^{k} \alpha_i \delta_{(\delta_{x_i}, \{x_i\})}, \{(\delta_{x_1}, \{x_1\}),\right.$$

$$\left.\ldots, (\delta_{x_{k'}}, \{x_{k'}\})\}\right) =$$

$$= \left(\sum_{i=1}^{k} \alpha_i \delta_{x_i}, \bigcup_{i=1}^{k'} \{x_i\}\right).$$

Since the set $\mathcal{P}_\omega(X)$ is dense in $\mathcal{P}(X)$ and therefore this set is coarsely dense in $\mathcal{P}(X)$. We conclude that the diagram is commutative for each $(\mu, A) \in \mathcal{P}(X)$ (see [4, Proposition 2.3]).

Now consider a pair $(\mathbb{M},\mathbb{A})\in\mathcal{P}^3_\omega(X)$, where

$$\mathbb{M}=\sum_{s=1}^{n}\eta_s\delta_{(M_s,\mathcal{A}_s)},\ \mathbb{A}=\{(M_s,\mathcal{A}_s)|i=1,\dots,n^{'}\}$$

with $n^{'}\geq n$, $M_s=\sum\limits_{i=1}^{k_s}\alpha_i^s\delta_{(\mu_i^s,A_i^s)}$, $\mathcal{A}_s=\{(\mu_i^s,A_i^s)|i=1,\dots,k_s^{'}\}$,

$$\mu_i^s=\sum_{j=1}^{l_i^s}\beta_{ij}^s\delta_{x_{ij}^s},\qquad A_i^s=\{x_{i1}^s,\dots,x_{il_i^{s'}}^s\},$$

Then we obtain

$$\mathcal{P}_\omega(\psi_X)(\mathbb{M},\mathbb{A})=\left(\sum_{s=1}^{n}\eta_s\delta_{\psi_X(M_s,\mathcal{A}_s)},\psi_X(\{M_s,\mathcal{A}_s\})\right),$$

where

$$\psi_X(\{M_s,\mathcal{A}_s\})=\left(\sum_{i=1}^{k_s}\sum_{j=1}^{l_i^s}\alpha_i^s\beta_{ij}^s\delta_{x_{ij}^s},\bigcup_{i=1}^{k_s^{'}}(\mu_i^s,A_i^s)\right).$$

Therefore

$$\psi_X(\mathcal{P}_\omega(\psi_X)(\mathbb{M},\mathbb{A}))$$
$$=\left(\sum_{s=1}^{n}\sum_{i=1}^{k_s}\sum_{j=1}^{l_i^s}\eta_s\alpha_i^s\beta_{ij}^s\delta_{x_{ij}^s},\bigcup_{i=1}^{k_s^{'}}\bigcup_{s=1}^{n^{'}}A_i^s\right).$$

Also,

$$\psi_{\mathcal{P}_\omega(X)}(\mathbb{M},\mathbb{A})=\left(\sum_{s=1}^{n}\sum_{i=1}^{k_s}\eta_s\alpha_i^s\delta_{(\mu_i^s,A_i^s)},\bigcup_{s=1}^{n^{'}}\mathcal{A}^s\right)$$

and

$$\psi_X(\psi_{\mathcal{P}_\omega(X)}(\mathbb{M},\mathbb{A}))=\left(\sum_{s=1}^{n}\sum_{i=1}^{k_s}\sum_{j=1}^{l_i^s}\eta_s\alpha_i^s\beta_{ij}^s\delta_{x_{ij}^s},\bigcup_{i=1}^{k_s^{'}}\bigcup_{s=1}^{n^{'}}A_i^s\right)$$
$$=\psi_X(\mathcal{P}_\omega(\psi_X)(\mathbb{M},\mathbb{A})).$$

Similarly as above, we note that the set $\mathcal{P}^3_\omega(X)$ is dense in $\mathcal{P}^3(X)$ and the restriction of the diagram on $\mathcal{P}^3_\omega(X)$ is commutative (see again [4, Proposition 2.3]).

This means that the second diagram is commutative as well. ■

IV. Tensor products

We are going to define two tensor products $\otimes$ and $\widetilde{\otimes}$. The tensor products in the category of compact Hausdorff spaces are investigated in [12]. Let $a\in\mathcal{P}(X),b\in\mathcal{P}(Y)$. For every $x\in X$ we denote by $i_X\colon Y\to X\times Y$ the map acting by the formula:

$$i_X(y)=(x,y),\ y\in Y$$

and the map $f_b\colon X\to\mathcal{P}(X\times Y)$ in the following way

$$f_b(x)=\mathcal{P}i_X(b),\ x\in X.$$

Definition 1. The element

$$a\otimes b=\psi_{X\times Y}\circ\mathcal{P}f_b(a)\in\mathcal{P}(X\times Y),$$

where $a\in\mathcal{P}(X),b\in\mathcal{P}(Y)$ is said to be the *tensor product* $\otimes$ of a and b.

Similarly, we define a variation of this tensor product. For every $y\in Y$ we denote by

$$j_y\colon X\to X\times Y,\ j_y(x)=(x,y),\ x\in X$$

and the map

$$g_a\colon Y\to\mathcal{P}(X\times Y),\ g_a(y)=\mathcal{P}j_y(a),y\in Y.$$

Definition 2. The element

$$a\widetilde{\otimes}b=\psi_{X\times Y}\circ\mathcal{P}g_a(b)\in\mathcal{P}(X\times Y),$$

where $a\in\mathcal{P}(X),b\in\mathcal{P}(Y)$ is said to be the *tensor product* $\widetilde{\otimes}$ of a and b.

It is easy to see that the operations $\otimes$ and $\widetilde{\otimes}$ coincide for the modified probability measure monad.

Proposition 1. The tensor product $\otimes\colon\mathcal{P}(X)\times\mathcal{P}(Y)\to\mathcal{P}(X\times Y)$ is a coarse map (respectively $\widetilde{\otimes}$).

□ *Proof.* First, we are going to show that $\otimes$ is a coarsely uniform map, which means that for every $\varepsilon>0$ there exists $\delta>0$ such that

$$d_{\mathcal{P}(X)\times\mathcal{P}(Y)}((a,b),(c,d))<\delta\Rightarrow d_{\mathcal{P}(X\times Y)}(a\otimes b,c\otimes d)<\varepsilon.$$

From $(a,b)\in\mathcal{P}(X)\times\mathcal{P}(Y)$ we have

$$a=(\mu_1,A_1)=\left(\sum_{i=1}^{k_1}\alpha_i^1\delta_{x_i^1},A_1=\{x_1^1,\dots,x_{k_1^{'}}^1\}\right),$$
$$b=(\nu_1,B_1)=\left(\sum_{j=1}^{l_1}\beta_j^1\delta_{y_j^1},B_1=\{y_1^1,\dots,y_{l_1^{'}}^1\}\right),$$

and from $(c,d)\in\mathcal{P}(X)\times\mathcal{P}(Y)$ we have

$$c=(\mu_2,A_2)=\left(\sum_{l=1}^{k_2}\alpha_l^2\delta_{x_l^2},A_2=\{x_1^2,\dots,x_{k_2^{'}}^2\}\right),$$
$$d=(\nu_2,B_2)=\left(\sum_{s=1}^{l_2}\beta_s^2\delta_{y_s^2},B_2=\{y_1^2,\dots,y_{l_2^{'}}^2\}\right).$$

Since

$$a\otimes b=\left(\sum_{i=1}^{k_1}\sum_{j=1}^{l_1}\alpha_i^1\beta_j^1\delta_{(x_i^1,y_j^1)},A_1\times B_1\right),$$
$$c\otimes d=\left(\sum_{l=1}^{k_2}\sum_{s=1}^{l_2}\alpha_l^2\beta_s^2\delta_{(x_l^2,y_s^2)},A_2\times B_2\right),$$

we can obtain the distance between the $a\otimes b$ and $c\otimes d$:

$$d_{\mathcal{P}(X\times Y)}(a\otimes b,c\otimes d)$$

$$= d_{\mathcal{P}(X\times Y)}\left(\left(\left(\sum_{i=1}^{k_1}\sum_{j=1}^{l_1}\alpha_i^1\beta_j^1\delta_{(x_i^1,y_j^1)}, A_1\times B_1\right),\right.\right.$$

$$\left.\left.\left(\sum_{l=1}^{k_2}\sum_{s=1}^{l_2}\alpha_l^2\beta_s^2\delta_{(x_l^2,y_s^2)}, A_2\times B_2\right)\right)\right)$$

$$= d_{KR}\left(\sum_{i=1}^{k_1}\sum_{j=1}^{l_1}\alpha_i^1\beta_j^1\delta_{(x_i^1,y_j^1)}, \sum_{l=1}^{k_2}\sum_{s=1}^{l_2}\alpha_l^2\beta_s^2\delta_{(x_l^2,y_s^2)}\right)$$

$$+d_H(A_1\times B_1, A_2\times B_2)$$

$$= \sum_{i=1}^{k_1}\sum_{j=1}^{l_1}\sum_{l=1}^{k_2}\sum_{s=1}^{l_2}\gamma_{ijls}d\left((x_i^1,y_j^1),(x_l^2,y_s^2)\right)$$

$$+d_H(A_1\times B_1, A_2\times B_2), \qquad (*)$$

where γ_{ijls} are the constants defined from the Kantorovich-Rubinshtein metric with the property $\sum\limits_{i=1}^{k_1}\sum\limits_{j=1}^{l_1}\sum\limits_{l=1}^{k_2}\sum\limits_{s=1}^{l_2}\gamma_{ijls} = 1$.

On the other hand,

$$d_{\mathcal{P}(X)\times\mathcal{P}(Y)}((a,b),(c,d))$$

$$= \max\left\{\begin{array}{l} d_{\mathcal{P}(X)}\left(\left(\sum\limits_{i=1}^{k_1}\alpha_i^1\delta_{x_i^1}, A_1\right), \left(\sum\limits_{l=1}^{k_2}\alpha_l^2\delta_{x_l^2}, A_2\right)\right) \\ d_{\mathcal{P}(Y)}\left(\left(\sum\limits_{j=1}^{l_1}\beta_j^1\delta_{y_j^1}, B_1\right), \left(\sum\limits_{s=1}^{l_2}\beta_s^2\delta_{y_s^2}, B_2\right)\right)\end{array}\right.$$

$$= \max\left\{\begin{array}{l} d_{KR}\left(\sum\limits_{i=1}^{k_1}\alpha_i^1\delta_{x_i^1}, \sum\limits_{l=1}^{k_2}\alpha_l^2\delta_{x_l^2}\right) + d_H(A_1,A_2) \\ d_{KR}\left(\sum\limits_{j=1}^{l_1}\beta_j^1\delta_{y_j^1}, \sum\limits_{s=1}^{l_2}\beta_s^2\delta_{y_s^2}\right) + d_H(B_1,B_2)\end{array}\right.$$

$$= \max\left\{\begin{array}{l} \sum\limits_{i=1}^{k_1}\sum\limits_{l=1}^{k_2}\gamma_{il}^1 d_X(x_i^1,x_l^2) + d_H(A_1,A_2) \\ \sum\limits_{j=1}^{l_1}\sum\limits_{s=1}^{l_2}\gamma_{js}^2 d_Y(y_j^1,y_s^2) + d_H(B_1,B_2)\end{array}\right. < \varepsilon,$$

so we are able to estimate each of (1) in the following way

$$\sum_{i=1}^{k_1}\sum_{j=1}^{l_1}\sum_{l=1}^{k_2}\sum_{s=1}^{l_2}\gamma_{ijls}d\left((x_i^1,y_j^1),(x_l^2,y_s^2)\right) < 2\varepsilon$$

and

$$d_H(A_1\times B_1, A_2\times B_2) < 2\varepsilon.$$

Fixing $\delta = 4\varepsilon$ we are done with coarse uniformity.

In order to show the metric properness, consider $a\otimes b\in\mathcal{P}(X\times Y)$. Let $(\delta_{(x_0,y_0)}, \{(x_0,y_0)\})$ be a base point of $\mathcal{P}(X\times Y)$. There exist compact neighborhoods A and B of x_0 and y_0 in X and Y respectively such that $a\otimes b\in\mathcal{P}(A\times B)$.

Then $\otimes^{-1}(\mathcal{P}(A\times B))\subset\mathcal{P}(A)\times\mathcal{P}(B)$. Since the latter set is compact, the map $\otimes$ is proper. ■

V. Remarks

One can also obtain similar results for the following metrization of the space $\mathcal{P}(X)$:

$$d_{\mathcal{P}(X)}((\mu_1,A_1),(\mu_2,A_2)) = \max\{d_{KR}(\mu_1,\mu_2), d_H(A_1,A_2)\}.$$

We leave as an open question that of extension of the above results onto the coarse categories (see, e.g. [8, 6, 9, 11]).

One can also furmulate the following question: is there a counterpart of the modified probability measure monad in the asymptotic category of Dranishnikov [1]?

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МОДИФІКОВАНИЙ ФУНКТОР ЙМОВІРНІСНИХ МІР У ГРУБІЙ КАТЕГОРІЇ

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Добре відомо, що функтор ймовірнісних мір не зберігає класу локально компактних метричних просторів. Звідси випливає, що цей функтор не має безпосереднього аналога в грубій категорії. Ми означуємо модифікований функтор ймовірнісних мір в категорії власних метричних просторів і грубих відображень.

Ключові слова: ймовірнісна міра, груба категорія, монада

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