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ANALYSIS OF LINEAR AND NONLINEAR EQUATION FOR OSCILLATING MOVEMENT

Моделювання процесу вібрації машин вимагає знання характеру диференціального рівняння вібраційного руху. Проаналізовано лінійне та нелінійне диференціальні рівняння, які моделюють процес виникнення стійких та нестійких вібрацій у вібраційних системах.

Modelling of oscillating processes requires one to know the character and properties of a differential equation for oscillating movement. This paper presents an analysis of the solutions for linear and nonlinear equations of oscillating movement aimed at solving oscillations in technological systems of vibrating machines.

Problem definition. The application of a nonlinear oscillating system is demonstrated to be suitable for the analysis of vibrating machines for technological processing of mechanical parts. For the realisation it is necessary to assemble a mathematical apparatus suitable for modelling of this process. We attempt for this purpose a solution of a set of linear and nonlinear differential equations for an oscillating system.

Analysis of solutions of equation $\ddot{x} + \omega^2 x = \varepsilon f(t, x, u_x)$, where $f(t, x, u_x) = F_0 \cos(\omega t)$

A simple example of oscillating motion is provided by a solid object with weight m which is deviated from its steady state position. According to Hooke's Law, there is a force *F* (elastic) which acts on the deviated object which is proportional to the deviation *x* in the form of $F = -k^2 x$, where $k, k > 0$ is a characteristic constant. The object will move (oscillate) under the influence of this force.

The equation in the form

$$m\ddot{x} = -k^2 x \tag{1}$$

where m is the weight of the object and $x = x(t)$, $\dot{x} = \frac{dx}{dt}$, $\ddot{x} = \frac{d^2 x}{dt^2}$, expresses the oscillatory motion on assumption that the gravitation forces and resistance of the environment are disregarded.

When oscillating object is moving in the environment, which presents the moving object with resistance and the resistance is proportional to speed $\dot{x}$, thus the resistance is $-q\dot{x}$, where $q > 0$ is the constant of this proportion, Equation (1) expressing the movement of an oscillating object becomes

$$m\ddot{x} + q\dot{x} + k^2 x = 0 \tag{2}$$

Equation (2) above represents motion which is called an independent oscillation.

After rearranging Equation (2) becomes

$$\ddot{x} + \frac{q}{m}\dot{x} + \frac{k^2}{m}x = 0$$

Let's denote $\frac{q}{m} = 2p$ and $\frac{k^2}{m} = \omega^2$. Differential equation (2) can be then rewritten as

$$\ddot{x} + 2p\dot{x} + \omega^2 x = 0 \tag{3}$$

Equation (3) is a linear differential equation with constant coefficients and with the right side equal to zero.
Its characteristic equation is

$$\lambda^2 + 2p\lambda + \omega^2 = 0 \tag{4}$$

and its roots are $\lambda_1 = -p + \sqrt{p^2 - \omega^2}$, $\lambda_2 = -p - \sqrt{p^2 - \omega^2}$.

Three scenarios are possible here

a) The roots of Equation (4) are complex. Let's denote $p^2 - \omega^2 = -\Omega^2$, where $\Omega > 0$.

Equation (4) will have complex roots when

$$p^2 - \omega^2 < 0$$

which is when

$$q < 2k\sqrt{m}$$

In this scenario the roots of Equation (4) are

$$\lambda_1 = -p + i\Omega, \qquad \lambda_2 = -p - i\Omega$$

Then a general solution of differential Equation (2) is the function

$$x = c_1 e^{-pt} \cos\left(\left(p^2 - \omega^2\right)t\right) + c_2 e^{-pt} \sin\left(\left(p^2 - \omega^2\right)t\right)$$

or in the form of

$$x = c_1 e^{-pt} \cos(\Omega t) + c_2 e^{-pt} \sin(\Omega t), \tag{5}$$

where c_1, c_2 are real constants.

Function (5) represents the position of oscillating point *x* in time. For the movement to occur, it is necessary that for constants c_1, c_2 it holds that $c_i \neq 0$ at least for one $i = 1,2$. It is proven that the movement which occurs in this case has the following characteristic property:

The oscillating object crosses the steady state position (null position) indefinite number of times over equal time interval $T/2$. This is called a 'half-period of oscillation', while time *T* is called a 'period of oscillation'.

Let's substitute the following for constants c_i, $i = 1,2$

$$c_1 = r\sin\alpha$$

$$c_2 = r\cos\alpha,$$

where $r > 0$. Then it becomes possible to rearrange Function (5) as follows

$$x = e^{-pt}\left(r\sin\alpha\cos(\Omega t) + r\cos\alpha\sin(\Omega t)\right),$$

from which

$$x = e^{-pt} r\sin(\Omega t + \alpha) \tag{6}$$

A graphical representation of this function for one particular example is shown in Fig. 1.

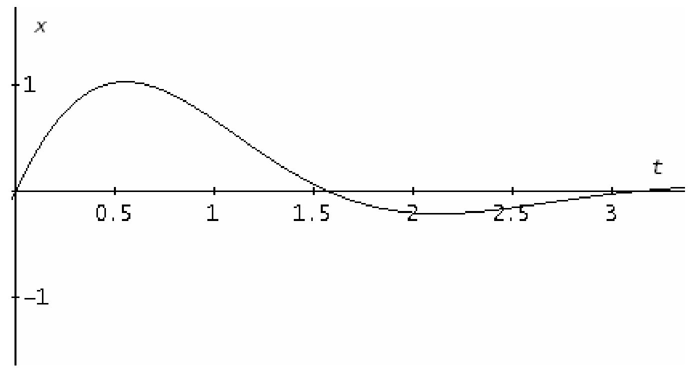

Fig. 1. Graphical representation of function $x = 2e^{-t}\sin 2t$

Function (6) has a trivial solution of $x = 0$ when $\Omega t + \alpha$ is an integer multiple of π, i.e. $\Omega t_k + \alpha = k\pi$, where $k = 0,1,2,\ldots$ This means that oscillating object crosses the steady position infinite number of times in time t_k. The time period between two subsequent crossings of the steady state position can be derived from the following equations

$$\Omega t_{k+1} + \alpha = (k+1)\pi$$

$$\Omega t_k + \alpha = n\pi$$

from which we can derive

$$\Omega(t_{k+1} - t_k) = \pi$$

Hence for a half period of oscillation

$$\frac{T}{2} = t_{k+1} - t_k = \frac{\pi}{\Omega}$$

As

$$\Omega^2 = \omega^2 - p^2,\ p = \frac{q}{2m},$$

then

$$T = \frac{2\pi}{\sqrt{\omega^2 - p^2}} = \frac{2\pi m}{\sqrt{4m^2\omega^2 - q^2}}.$$

Parameter Ω represents circular frequency of the observed motion. Such a motion, for which the object crosses the steady state position at least twice, we call 'oscillatory motion', otherwise we refer to 'non-oscillatory' or 'non-vibrating' motion.

b) The second scenario for the characteristic Equation (4), when the roots are both real and different, provides for a generic solution of differential Equation (2) in the form

$$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t},\ c_1, c_2 \in R,$$

$$\lambda_1 = -p + \sqrt{p^2 - \omega^2},\ \lambda_2 = -p - \sqrt{p^2 - \omega^2},$$

where

$$p^2 - \omega^2 > 0,$$

as $p = \frac{q}{2m}$, then $q^2 > 4\omega^2 m^2$ and $q > 2\omega m$.

In this case the object crosses the steady state position at most once, when

$$x = 0,$$

i.e. equation

$$c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} = 0,$$

when $c_i \neq 0, i = 1,2$, has at most one solution. In this case the type of motion is non-oscillatory (non-vibrating). The corresponding graphical representation of this function for one particular example is shown in Fig. 2.

c) The third scenario for the roots of characteristic Equation (2), when $\lambda_1 = \lambda_2 = -p$, where

$$p^2 - \omega^2 = 0,\ \text{i.e. } p^2 = \omega^2,\ |p| = |\omega|,\ \omega > 0,\ p > 0.$$

A generic solution of differential Equation (2) is a function

$$x = c_1 e^{-pt} + c_2 t e^{-pt},$$

$$x = e^{-pt}(c_1 + tc_2) \text{ or } x = e^{-\omega t}(c_1 + tc_2),$$

thus the motion is non-oscillatory. Its graphical representation for one particular example is shown in Fig. 3.

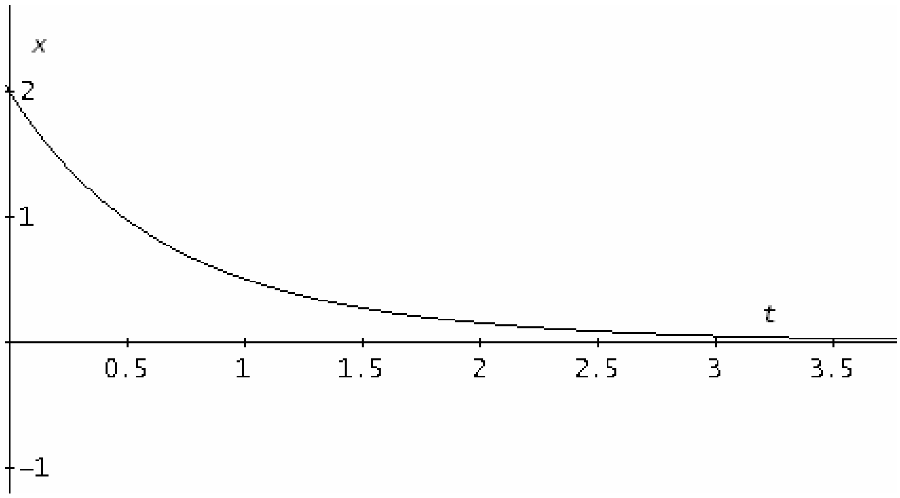

Fig. 2. Graphical representation of function $x = e^{-2t} + e^{-t}$

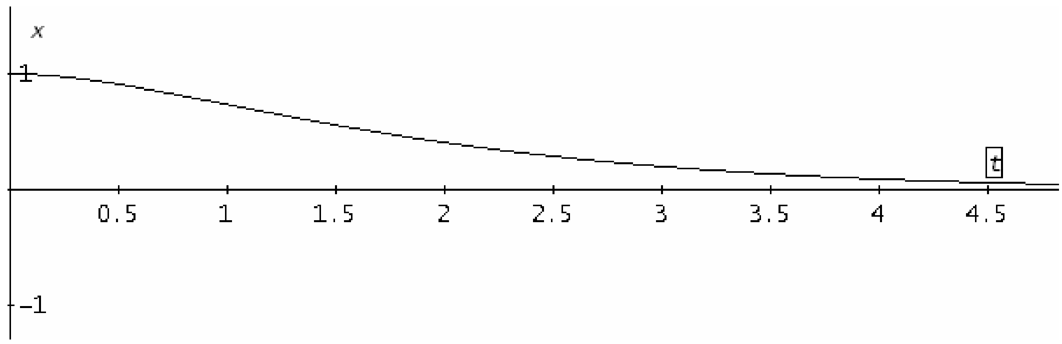

Fig. 3. Graphical representation of function $x = e^{-t} + te^{-t}$

By using the limit of $\lim_{t \to \infty} x$ we will show that for all three scenarios with increasing $t \to \infty$ the deviation of x converges to 0, i.e. function $x = x(t)$ represents damped movement (damped vibration, damped non-oscillatory movement).

If $p > 0$, i.e. $q > 0$ the following relations hold:

a) $\lim_{t \to \infty} x(t) = \lim_{t \to \infty} e^{-pt}\left(c_1 \cos(\Omega t) + c_2 \sin(\Omega t)\right) = 0$,

b) $\lim_{t \to \infty} x(t) = \lim_{t \to \infty}\left(c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\right) = \lim_{t \to \infty}\left(c_1 e^{\left(-p+\sqrt{p^2-\omega^2}\right)t} + c_2 e^{\left(-p-\sqrt{p^2-\omega^2}\right)t}\right) = 0$,

where $-p + \sqrt{p^2 - \omega^2} < 0, \quad -p - \sqrt{p^2 - \omega^2} < 0$,

c) $\lim_{t \to \infty} x(t) = \lim_{t \to \infty}\left(c_1 e^{-pt} + c_2 te^{-pt}\right) = 0$.

If there is no resistance from environment, i.e. when $q = 0$, thus $p = 0$. The movement can be described by the equation

$$m\ddot{x} = -\omega^2 x$$

or

$$m\ddot{x} + \omega^2 x = 0,$$

$$\ddot{x} + \frac{\omega^2}{m} x = 0.$$

If we consider a unit weight ($m = 1$) then we will analyse only a simplified equation

$$\ddot{x} + \omega^2 x = 0 .$$

If there is an external force $P(t)$, which is only a function of time t and which acts on a moving system, thus the resulting motion is described by the following differential equation

$$m\ddot{x} + q\dot{x} + k^2 x = P(t) \tag{7}$$

and this is an equation of internal oscillation.

One of the simplest forms of this function is $P(t) = F_0 \cos(\omega t), \omega > 0$, i.e. function $P(t)$ is periodic.

Non-linear second order differential equation. Let's consider the solution properties of the following differential equation

$$M(t)x'' + B(x')x' + K(x)x = u + d(t, x, x', x'', u), \tag{8}$$

where $M(t) \neq 0$, $x = x(t)$, $u = u(t)$, $P(t) = \varepsilon f(t, x,) = u$ are continuous functions.

It is possible to express Equation (8) using a system of differential equations

$$\begin{gathered} x_1'(t) = x_2(t) \\ x_2'(t) = -\frac{K(x_1(t))}{M(t)} x_1(t) - \frac{B(x_2(t))}{M(t)} x_2(t) + \frac{u}{M(t)}, \\ x_1 = x_1(t) = x,\ x_2 = x_2(t) = x' . \end{gathered} \tag{9}$$

Let $\bar{x}(t) = (x_1(t), x_2(t))^T$ is a general solution of System (9). For each solution $\bar{x}(t)$, $x_1(t_0) = x_1^0$, $x_2(t_0) = x_2^0$, $t_0 \in J$ we assume that it exists within interval J. Let's denote the right boundary position of interval J as $h > t_0 > 0$, thus $J_0 = \langle t_0, h)$.

Let's in Equation (8) set $B(x_2) = 0$, $u(t) = 0$, then differential Equation (8) is expressed by the following system

$$\begin{gathered} x_1' = x_2 \\ x_2' = -\frac{K(x_1)}{M(t)} x_1 , \end{gathered} \tag{10}$$

where $-\dfrac{K(x_1)}{M(t)} \in C_0(D, R) = C_0(D \equiv J \times R^2, R)$.

a_1) Let's assume System (10) such that $K(x_1) = 0$, then all solutions of $x(t), t \in J_0$ of System (10) are so that $x_2(t) = c$, $c \in R$ is a constant and $x_1(t) = c(t - t_0), c \in R$. For $c = 0$, $x(t)$ becomes 2-trivial solution and 1-constant solution.

a_2) Let's assume System (10) such that $K(x_1) \neq 0$.

Let for each non-trivial solution $x(t), t \in J_0$ of System (10) exist a function $r(t) > 0$ and function $u(t)$ such that $u(t) \in C_1(J, R), u(t) \neq 0$, where $C_1(J, R)$ is a space of real functions which can be derived and these functions are of one real variable t defined over the interval J. Let the following hold for solutions $x_i(t), t \in J_0, i = 1,2$

$$\begin{gathered} x_1(t) = r(t)\cos v(t) \\ x_2(t) = r(t)\sin v(t) \end{gathered} \tag{11}$$

Function $r(t)$ is called a polar function and function $v(t)$ is called an angular function. System (10) is expressed using equations (11) in the form

$$r'(t)\cos v(t) - r(t)\sin v(t)v'(t) = r(t)\sin v(t)$$

$$r'(t)\sin v(t)+r(t)\cos v(t)v'(t)=-\frac{K(t,r(t)\cos v(t))}{M(t)}r(t)\cos v(t) \quad (12)$$

After rearranging System (12) we derive equations

$$\frac{r'(t)}{r(t)}= \sin v(t)\cos v(t)-\frac{K(t,r(t)\cos v(t))}{M(t)}\cos v(t)\sin v(t) \quad (13)$$

$$v'(t)=-(\sin v(t))^2-\frac{K(t,r(t)\cos v(t))}{M(t)}(\cos v(t))^2,\ v(t)\neq(2k+1)\frac{\pi}{2},\ k\in Z \text{ is an integer.}$$

For reasons of brevity we denote $I_i,\ i=1,2$ integrals as follows

$$I_1=\int_{t_0}^{h}\left(\sin y(t)\cos y(t)-\frac{K(t,r(t)\cos y(t))}{M(t)}\sin y(t)\cos y(t)\right)dt$$

$$I_2=\int_{t_0}^{h}\left(-(\sin y(t))^2-\frac{K(t,r(t)\cos y(t))}{M(t)}(\cos y(t))^2\right)dt,\ v(t)\neq(2k+1)\frac{\pi}{2},\ k\in Z, \quad (14)$$

where $y(t)$, $t\in J_0$ denotes a continuous function.

Let for all continuous functions $y(t), t\in J_0$ exist integrals I_1, I_2 as defined in the following statements a) to f). Then all non-trivial solutions $x(t), t\in J_0$ of System (9) are:

a) unbounded, when $I_1=\infty, I_2=\pm\infty$,

b) unbounded, when $I_1=\infty, I_2=K$, where $K\in R$ is a constant, $K+u(t_0)\neq\frac{k\pi}{2}$, $k\in Z$ is an integer,

c) bounded so that $x_1(t)\to 0$, when $I_1=-\infty, I_2=\pm\infty$,

d) bounded so that $x_1(t)\to 0$, when $I_1=-\infty, I_2=K$, where $K\in R$ is a constant, $K+u(t_0)\neq\frac{k\pi}{2}$, $k\in Z$ is an integer,

e) bounded, when $I_1=L,\ L>0,\ I_2=\pm\infty$,

f) bounded, when $I_1=L,\ L>0,\ I_2=K$, $K\in R$ is a constant, $K+u(t_0)\neq\frac{k\pi}{2}$, $k\in Z$ is an integer.

Integrating System (13) over the interval $\langle t_0,h\rangle$ yields

$$r(h)=r(t_0)\exp\int_{t_0}^{h}\left(\sin y(t)\cos y(t)-\frac{K(t,r(t)\cos y(t))}{M(t)}\sin y(t)\cos y(t)\right)dt$$

$$v(h)=v(t_0)+\int_{t_0}^{h}\left(-(\sin y(t))^2-\frac{K(t,r(t)\cos y(t))}{M(t)}(\cos y(t))^2\right)dt \quad (15)$$

Considering the assumption that a certain value of $r(h)$ exists such that $r(h)>0$ $(r(h)\to\infty)$, i.e. if a polar function $r(t)$ is bounded (unbounded), then each non-trivial solution $x(t), t\in J_0$ of System (9) is bounded (unbounded). The situation when $x_1(t)\to 0$, $x_2(t)\to 0$ occurs exactly when $I_1=-\infty$ or if $r(h)=0$.

In our future work we will consider Equation (8) in the following scenarios:

1. when function $P(t)$ has values $P(t)=\varepsilon f(t,x,u_x)$;

2. when there is no resistance from environment, i.e. for $q = 0$, thus $p = 0$, i.e. the movement is described by equation $m\ddot{x} = -\omega^2 x$ or

$$m\ddot{x} + \omega^2 x = 0$$

$$\ddot{x} + \frac{\omega^2}{m} x = 0 .$$

If we consider a unit weight (*m = 1*) then we will analyse only a simplified equation

$$\ddot{x} + \omega^2 x = 0 .$$

Conclusions. In the first part of this paper, we have analysed a linear differential equation of an oscillating movement. In the following paper we will continue explore the solutions of its non-linear alternative.

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