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ELIMINATION OF FINITE EIGENVALUES OF STRONGLY SINGULAR SYSTEMS BY FEEDBACKS IN LINEAR SYSTEMS

A new problem of decreasing of the degree of closed-loop characteristic polynomial by suitable choice of state feedbacks for strongly singular linear systems is formulated and solved. Necessary and sufficient conditions are established under which it is possible to choose state feedbacks such that the nonzero closed-loop characteristic polynomial has zero degree. A procedure for computation of the feedback gain matrices is proposed.

Introduction. Dai has shown [1, 2] that for singular (descriptor) linear systems $Ex_{i+1} = Ax_i + Bu_i$, $E, A \in R^{n\times n}, B \in R^{n\times m}$, $\det E = 0$ it is possible to choose a matrix $K \in R^{m\times n}$ of the state-feedback $u = Kx$ such that the nonzero closed-loop characteristic polynomial $\det[Ez - (A + BK)]$ has zero degree. It is easy to show that for standard systems $(E = I)$ does not exist such state-feedbacks.

Main subject of this note is to established necessary and sufficient conditions for strongly singular linear systems under which it is possible to choose state feedbaks such that the nonzero closed-loop characteristic polynomial has zero degree.

This procedure of decreasing of the degree of closed-loop characteristic polynomial by feedbacks will be called the elimination of finite eigenvalues of matrices by feedbacks since the closed-loop has no finite eigenvalues (poles).

This type of problem arises in designing of the perfect observers for linear standard systems [1, 2, 4]. To the best knowledge of the author this elimination of finite eigenvalues of matrices by feedbacks in linear systems has not been considered yet.

Problem formulation. Let $R^{n\times m}$ be the set of $n\times m$ real matrices and $R^n := R^{n\times 1}$. Consider the linear continuous-time system

$$E\dot{x} = Ax + Bu \tag{1}$$

where $x = x(t) \in R^n$ and $u = u(t) \in R^m$ are the state and input vectors, respectively and $E, A \in R^{n\times n}, B \in R^{n\times m}$.

It is assumed that $\det E = 0, rank\, B = m$,

$$rank\,[Es - A, B] = n \quad \text{for all} \quad s \in \mathbf{C} \ \text{(the field of complex numbers)} \tag{2}$$

and the pencil (E, A) is not regular, i.e.

$$\det[Es - A] = 0 \quad \text{for all} \quad s \in \mathbf{C}\,. \tag{3}$$

We are looking for a gain matrix $K \in R^{m\times n}$ of the state-feedback

$$u = v + Kx \tag{4}$$

such that

$$\det[Es - (A + BK)] = \alpha \neq 0 \tag{5}$$

where α is a real number independent of s.

The problem can be stated as follows. Given E, A, B and α, find K satisfying (5).

We shall establish necessary and sufficient conditions for the existence of a solution to the problem and we shall give a procedure for computation of the gain matrix K.

Problem solution. If $m = n$ the problem can be solved as follows. We can always choose A_c so that $\det[Es - A_c] = \alpha$. The assumptions $rank\, B = m$ and $m = n$ implies $\det B \neq 0$ and from $A_c = A + BK$ we obtain $K = B^{-1}[A_c - A]$.

Thus, we assume than $m < n$.

Theorem. There exists K satisfying (5) if and only if the conditions (2) is satisfied.

Proof. Necessity. From the equality

$$[Es - (A + BK)] = [Es - A, B]\begin{bmatrix} I \\ -K \end{bmatrix} \tag{6}$$

it follows that (5) implies (2).

Sufficiency. Let $F(i_1, i_2, ..., i_n)$ be the $n \times n$ minor composed of the $i_1, i_2, ..., i_n$ columns of the matrix $[Es - A, B]$ and $G(i_1, i_2, ..., i_n)$ be the $n \times n$ minor composed of the $i_1, i_2, ..., i_n$ rows of the matrix $\begin{bmatrix} I \\ -K \end{bmatrix}$.

Then from the Binett – Cauchy formula [3] we obtain

$$det[\,Es - (\,A + BK\,)\,] = \sum_{1 \le i_1 < i_2 < \cdots < i_n \le n+m} F(i_1, i_2, ..., i_n)G(i_1, i_2, ..., i_n). \tag{7}$$

If (2) holds then there exists at least one nonzero minor $F(k_1, k_2, ..., k_n)$ which is independent of s. From the structure of $\begin{bmatrix} I \\ -K \end{bmatrix}$ it follows that it is always possible to choose the entries of K so that the minor $G(k_1, k_2, ..., k_n)$ is nonzero and all remaining minors $G(i_1, i_2, ..., i_n)$ corresponding to nonzero minors $F(i_1, i_2, ..., i_n)$ are zero. In this case from (7) and (5) we obtain

$$\det[Es - (A + BK)] = F(k_1, k_2, ..., k_n)G(k_1, k_2, ..., k_n) = \alpha\,. \tag{8}$$

Hence, it is always possible to choose K so that

$$G(k_1, k_2, ..., k_n) = \frac{\alpha}{F(k_1, k_2, ..., k_n)}\,. \tag{9}$$

The choice in general case is not unique.

To simplify the choice of K the following procedure based on elementary operations is recommended. The following elementary row and column operations will be used:

1) Multiplication of the ith row (column) by scalar c. This elementary row (column) operation will be denoted by $L[i \times c]$ $(R[i \times c])$.

2) Addition the jth row (column) multiplied by a polynomial $b = b(s)$ to the ith row (column). This elementary row (column) operation will be denoted by $L[i + j \times b]$ $(R[i + j \times b])$.

3) Interchange the ith and jth rows (columns). This elementary row (column) operation will be denoted by $L[i, j]$ $(R[i, j])$.

If the condition (2) is satisfied then there exists an unimodular matrix $U = (s)$ ($\det U \neq 0$ is independent of s) of elementary columns operations such that

$$[Es - A, B]U = [0 \;\; I_n]. \tag{10}$$

From (6) and (10) we have

$$[Es - (A + BK)] = [Es - A, B]UU^{-1}\begin{bmatrix} I \\ -K \end{bmatrix} = \begin{bmatrix} 0 & I_n \end{bmatrix}\begin{bmatrix} G_1 \\ \cdots \\ G \end{bmatrix} = G \tag{11}$$

and

$$[Es - (A + BK)] = \det G \tag{12}$$

where

$$\begin{bmatrix} G_1 \\ G \end{bmatrix} = U^{-1}\begin{bmatrix} I \\ -K \end{bmatrix}, \; G = G(K) \in R^{n \times n}. \tag{13}$$

To find G using elementary columns operations we perform the reduction (10) and we perform simultaneously corresponding elementary row operations of the matrix $\begin{bmatrix} I \\ -K \end{bmatrix}$ taking into account the following correspondence [4]: $R[i \times c] \to L[i \times \frac{1}{c}], R[i + j \times b] \to L[j - i \times b]$, $R[i, j] \to L[i, j]$. The entries of K are chosen so that $\det G = \alpha$.

If the condition (2) is satisfied then the matrix K can be found by the use of the following procedure

Procedure

Step 1. Using elementary column operations perform the reduction (10) and performing simultaneously corresponding elementary row operations defined by U^{-1} on the matrix $\begin{bmatrix} I \\ -K \end{bmatrix}$ find the matrix G.

Step 2. Choose the entries of K so that $\det G = \alpha$

Example 1. Consider the system (1) with

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & -2 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{14}$$

Find $K = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \end{bmatrix}$ such that (5) is satisfied for $\alpha = 1$. In this case

$$n = 3, m = 2, rank[E, B] = rank\, E = rank\, B = 2 \text{ and } \det[Es - A] = \begin{vmatrix} s & -1 & 0 \\ 0 & s & 0 \\ 1 & 2 & 0 \end{vmatrix} = 0 \text{ for all } s \in \mathbf{C}.$$

The condition (2) is satisfied since

$$rank[Es - A, B] = rank\left[\begin{array}{ccc:cc} s & -1 & 0 & 1 & 0 \\ 0 & s & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 & 0 \end{array}\right] = 3 \text{ for all } s \in \mathbf{C}.$$

Hence problem has a solution and using the procedure we obtain

Step 1. To reduce the matrix

$$[Es-A,B]=\begin{bmatrix} s & -1 & 0 & 1 & 0 \\ 0 & s & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 & 0 \end{bmatrix}$$

to the form

$$[0 \quad I_n]=\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{16}$$

we perform the following elementary column operations $R[1+4\times(-s)], R[2+5\times(-s)]$, $R[2+4], R[2+1\times(-2)], R[1,3], R[3,4], R[4,5]$ or equivalently we postmultiply the matrix (15) by the unimodular matrix

$$U=\begin{bmatrix} -2 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1+2s & 0 & 1 & 0 & -s \\ -s & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{17}$$

On the matrix

$$\begin{bmatrix} I \\ -K \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_{11} & -k_{12} & -k_{13} \\ -k_{21} & -k_{22} & -k_{23} \end{bmatrix} \tag{18}$$

we perform simultaneously the following elementary row operations $L[4-1\times s], L[5+2\times s]$, $L[4+2], L[1+2\times 2], L[1,3], L[3,4], L[4,5]$ or equivalently we premultiply the matrix (18) by the unimodular matrix

$$U^{-1}=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ s & -1 & 0 & 1 & 0 \\ 0 & s & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 & 0 \end{bmatrix} \tag{19}$$

and we obtain

$$U^{-1}\begin{bmatrix} I \\ -K \end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ s-k_{11} & -1-k_{12} & -k_{13} \\ -k_{21} & s-k_{22} & -k_{23} \\ 1 & 2 & 0 \end{bmatrix}$$

and

$$G = \begin{bmatrix} s - k_{11} & -1 - k_{12} & -k_{13} \\ -k_{21} & s - k_{22} & -k_{23} \\ 1 & 2 & 0 \end{bmatrix}. \tag{20}$$

Step 2. From (20) we have

$$\det G = (k_{13} + 2k_{23})s - k_{13}k_{22} + k_{23}(1 + k_{12}) + 2(k_{21}k_{13} - k_{23}k_{11}). \tag{21}$$

We choose, for example, $k_{23} = 1, k_{13} = -2, k_{12} = -1, k_{21} = k_{11} = 0$. Then $\det G = 2k_{22}$ and for $k_{22} = \frac{1}{2}$ we obtain (5) for $\alpha = 1$. The desired matrix K has the form

$$K = \begin{bmatrix} 0 & -1 & -2 \\ 0 & \frac{1}{2} & 1 \end{bmatrix}.$$

Concluding remarks. A new problem of decreasing of the degree of closed-loop characteristic polynomial by suitable choice of state feedbacks for standard linear systems has been formulated and solved. Conditions have been established under which it is possible to choose state feedbacks (4) for strongly singular system (1) such that (5) holds. It has been shown that the problem has a solution if and only if the condition (2) is satisfied. A procedure for computation of the matrix K of (4) has been proposed and illustrated by a numerical example. The considerations presented for standard continuous-time linear systems are also valid with slight modifications for strongly singular discrete-time linear systems.
An extension of this problem for linear two-dimensional systems [4, 5] is also possible.

1 Dai. L. Observers for discrete Singular Systems // IEEE Trans. Autom. Contr., AC-33. – Febr. 1988. – No 2. P. 187–191. 2. Dai L., Singular Control Systems, Springer Verlag. –Berlin – Tokyo, 1989. 3. Gantmacher F.R. Theory of Matrices, vols. 1 and 2. – Chelsea, New York, 1959. 4. Kaczorek T. Linear Control Systems, vol. 1 and 2, Research Studies Press and J. – Wiley, New York, 1993. 5. Kaczorek T. Perfect observers for singular 1D and 2D linear systems // Proc. Polish – German Symposium "Science Research Education", 28–29 Sept. 2000. Zielona Góra, Poland. – P. 9–16.