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A variational method of homogeneous solutions for axisymmetric elasticity problems for cylinder

Chekurin V. F., Postolaki L. I.

Pidstryhach Institute for Applied Problems of Mechanics and Mathematics
National Academy of Sciences of Ukraine
3-b, Naukova Str., 79060, Lviv, Ukraine

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A variational method of homogeneous solutions for axisymmetric elasticity problems for semiinfinite and finite cylinders with loaded end faces and free lateral surface has been developed. As examples of application of the proposed approach the problem of bending of the thick disk by concentrated forces applied to its end faces have been considered.

Keywords: *cylindrical body, axisymmetric elasticity problem, Love's function, biharmonic equation, variational method of homogeneous solutions*

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1. Introduction

The method of homogeneous solutions was applied, in particular, to plane elasticity problems for rectangular domains [1,2]. According to this method the solution is represented as a series expansion in the eigenfunctions of some homogeneous problem (so called Papkovich's homogeneous solutions [2]). The homogeneous solutions do not form an orthonormal functional basis. That substantially complicates the numerical realization of the method. Just for mixed problems, when the normal displacements and tangential tractions or normal traction and tangential displacements are given, it is becomes possible to obtain analytical relations expressing the coefficients of the solution via the functions of boundary conditions.

Variational method of homogeneous solutions for plane problems was suggested in papers [3,4]. Under this method the solution in form of the eigenfunction expansion is subordinated to the boundary conditions due to the quadratic functional norm. This reduces the problem to an infinite system of algebraic equations for the expansion coefficients, which have being solved by the reduction method [3]. Four types of boundary problems have been considered with the use of this method: two ones with boundary conditions in stresses or displacements prescribed on the opposite sides of the rectangle, and two other with mixed boundary conditions, given on these sides. The problems for piecewise homogeneous strip have been also considered [5]. The convergence of the method was studied for some specific cases in [3,4].

A system of homogeneous solutions for an axisymmetric elasticity problem for a cylinder with homogeneous conditions in stresses on the lateral cylindrical surface is presented in the monograph [6]. System is obtained with the use the Papkovich-Neuber's representation. An example of application of the method for solving the axisymmetric problem for a semi-infinite cylinder with traction-free lateral surface and loaded end face is considered in [6]. The expansion coefficients for this problem were determined by minimization of a quadratic functional. The functional specifies the deviation of the sought-for solution from the given functions of boundary conditions in stresses prescribed on the cylinder's end face. In the presented numeric examples the solution containing two expansion terms had been used.

In this paper the solution of the axisymmetric elasticity problem for cylinder is represented with the use of Love function. That reduces the problem to biharmonic equation. Basing on this representation, the systems of homogeneous solutions in cylindrical coordinates have been obtained. These systems have been used for variational formulation and solving of axisymmetric elasticity problems for semi-infinite and finite cylinders with traction-free lateral surface and different types of boundary conditions given on their end faces.

2. Systems of homogeneous solutions of the biharmonic equation in cylindrical coordinates

Consider the class of axisymmetric elasticity problems for a semi-infinite cylinder $0 \leqslant r \leqslant a$, $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant z \leqslant \infty$ (r, θ, z — cylindrical coordinates). Let the lateral surface of the cylinder $r = a$ be free of traction:

$$\sigma_{rr}|_{\xi=1} = 0, \qquad \sigma_{rz}|_{\xi=1} = 0 \tag{1}$$

and on the end surface $z = 0$ one pair of boundary conditions (2)–(5) are given (problems I–IV correspondingly):

$$\sigma_{zz}|_{\zeta=0} = \sigma(\xi), \qquad \sigma_{rz}|_{\zeta=0} = \tau(\xi), \tag{2}$$

$$u_z|_{\zeta=0} = u(\xi), \qquad u_r|_{\zeta=0} = v(\xi), \tag{3}$$

$$\sigma_{zz}|_{\zeta=0} = \sigma(\xi), \qquad u_r|_{\zeta=0} = v(\xi), \tag{4}$$

$$u_z|_{\zeta=0} = u(\xi), \qquad \sigma_{rz}|_{\zeta=0} = \tau(\xi). \tag{5}$$

Here, $\xi \equiv r/a$, $\zeta \equiv z/a$ are dimensionless coordinates, $\sigma(\xi)$, $\tau(\xi)$, $u(\xi)$, $v(\xi)$ are given functions. Function $\sigma(\xi)$ satisfies the condition $\int\limits_0^1 \sigma(\xi)\xi d\xi = 0$.

With the use the Love function χ the axisymmetric elasticity problem can be reduced to biharmonic equation [7]:

$$\nabla^2\nabla^2\chi = 0, \tag{6}$$

where $\nabla^2 = \frac{\partial^2}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial}{\partial\xi} + \frac{\partial^2}{\partial\zeta^2}$ is the axisymmetric Laplace operator.

The components of the stress tensor σ_{zz}, σ_{rr}, $\sigma_{\theta\theta}$, σ_{rz} and displacement vector u_r, u_z can be expressed via the function χ as follows:

$$\frac{1}{2\mu}\sigma_{zz} = \frac{\partial}{\partial\zeta}\left((2-\nu)\nabla^2\chi - \frac{\partial^2\chi}{\partial\zeta^2}\right), \qquad \frac{1}{2\mu}\sigma_{rr} = \frac{\partial}{\partial\zeta}\left(\nu\nabla^2\chi - \frac{\partial^2\chi}{\partial\xi^2}\right),$$

$$\frac{1}{2\mu}\sigma_{\theta\theta} = \frac{\partial}{\partial\zeta}\left(\nu\nabla^2\chi - \frac{1}{\xi}\frac{\partial\chi}{\partial\xi}\right), \qquad \frac{1}{2\mu}\sigma_{rz} = \frac{\partial}{\partial\xi}\left((1-\nu)\nabla^2\chi - \frac{\partial^2\chi}{\partial\zeta^2}\right),$$

$$u_r = -\frac{\partial^2}{\partial\xi\partial\zeta}, \qquad u_z = \frac{\partial^2}{\partial\zeta^2} + 2(1-\nu)\nabla^2\chi.$$

Here, ν and μ stand for Poisson and shear modulus.

Representing the solution of the equation (6) in the form

$$\chi = \exp(-\gamma\zeta)f(\xi),$$

where γ is a unknown constant, we come to the following equation for the unknown function $f(\xi)$

$$f^{IV}(\xi) + \frac{2}{\xi}f'''(\xi) + \left(2\gamma^2 - \frac{1}{\xi^2}\right)f''(\xi) + \left(2\gamma^2 + \frac{1}{\xi^2}\right)\frac{1}{\xi}f'(\xi) + \gamma^4 f(\xi) = 0.$$

Its general solution is

$$f(\xi) = \xi(AJ_1(\gamma\xi) + BY_1(\gamma\xi) + CY_0(\gamma\xi)\left(J_0(\gamma\xi)Y_1(\gamma\xi) - J_1(\gamma\xi)Y_0(\gamma\xi)\right))+ \\ + D\xi\left(Y_1(\gamma\xi)(J_0^2(\gamma\xi) - 1) - J_0(\gamma\xi)J_1(\gamma\xi)Y_0(\gamma\xi)\right). \quad (7)$$

Here J_0, J_1, Y_0, Y_1 are zero- and first-order Bessel and Neumann functions correspondingly, A, B, C, D are unknown constants.

Function $f(\xi)$ is singular at the point $\xi = 0$, so presentation of the solution in form (7) can not be used for solid cylinder. To avoid that, the solution are presented in [8,9] as a Fourier-Bessel expansion in function J_0.

Unlike that, to ensure finiteness of the solution $f(\xi)$ at the point $\xi = 0$, we put in the formula (7) $C = 0$ and $D = B$. With this we obtain the presentation for the solution

$$f(\xi) = \xi J_1(\gamma\xi)A - \frac{2}{\pi\gamma}J_0(\gamma\xi)B, \quad (8)$$

which is finite in the point $\xi = 0$. Function (8) is depending on the two unknown constants A and B, will what provides the possibility to subordinate the solution to two boundary conditions given on the cylinder's face end.

Substituting (8) into boundary conditions (1), we come to the following homogeneous system of equations with respect the unknown constants A and B:

$$\begin{gathered} ((1-2\nu)J_0(\gamma) - \gamma J_1(\gamma))A + \frac{2}{\pi\gamma}(\gamma J_0(\gamma) - J_1(\gamma))B = 0, \\ ((2\nu-2)J_1(\gamma) - \gamma J_0(\gamma))A - \frac{2}{\pi}J_1(\gamma)B = 0. \end{gathered} \quad (9)$$

Applying the compatibility condition to the system (9) we obtain the following transcendental equation for parameter γ:

$$\gamma^2\left(J_0^2(\gamma) + J_1^2(\gamma)\right) + 2(\nu-1)J_1^2(\gamma) = 0. \quad (10)$$

Equation (10) is equivalent to those obtained with the use of the Papkovich-Neuber representation for solution of axisymmetric elasticity problem [10].

Equation (10) has the only real root $\gamma = 0$, hence instead we should can consider the infinite sequences of complex roots $\gamma_k = \alpha_k + i\beta_k$, $-\gamma_k$, $\overline{\gamma}_k = \alpha_k - i\beta_k$ and $-\overline{\gamma}_k$ $(k = 1, \ldots, \infty)$, where i stands for imaginary unit, α_k, β_k are real constants. To guarantee decaying the solution as ζ approaches to infinity, we use only two sequences γ_k and $\overline{\gamma}_k$ with positive real parts.

The equation has being solved numerically by residual minimization. The values of the first 25 roots rounded to 5 decimal digits, calculated at $\nu = 0.25$, are presented in the Table 1.

Table 1. Roots of the transcendental equation.

k	α_k	β_k	k	α_k	β_k	k	α_k	β_k
1	2.69765	1.36735	6	18.75905	2.16604	11	34.50379	2.46622
2	6.05122	1.63814	7	21.91184	2.24211	12	37.64928	2.50949
3	9.26127	1.82853	8	25.06203	2.30817	13	40.79422	2.54932
4	12.43844	1.96742	9	28.21044	2.36656	14	43.93871	2.58623
5	15.60220	2.07642	10	31.35758	2.41886	15	47.08284	2.62059

So, the system (9) has infinity number of solutions A_k, B_k, such that $A_k = \kappa_k B_k$, where

$$\kappa_k = \frac{2J_1(\gamma_k)}{\pi\left((2\nu-2)J_1(\gamma_k) - \gamma_k J_0(\gamma_k)\right)}.$$

As a result, we obtain two complete infinite systems of homogeneous complex solutions decaying at infinity: the first one is

$$\sigma_{kzz}(\xi,\zeta)\equiv\sigma_{kzz}(\xi)\exp(-\gamma_k\zeta),\qquad \sigma_{krr}(\xi,\zeta)\equiv\sigma_{krr}(\xi)\exp(-\gamma_k\zeta),$$
$$\sigma_{k\theta\theta}(\xi,\zeta)\equiv\sigma_{k\theta\theta}(\xi)\exp(-\gamma_k\zeta),\qquad \sigma_{krz}(\xi,\zeta)\equiv\sigma_{krz}(\xi)\exp(-\gamma_k\zeta),$$
$$u_{kr}(\xi,\zeta)\equiv u_{kr}(\xi)\exp(-\gamma_k\zeta),\qquad u_{kz}(\xi,\zeta)\equiv u_{kz}(\xi)\exp(-\gamma_k\zeta)$$

and the second one is

$$\overline{\sigma}_{kzz}(\xi,\zeta)\equiv\overline{\sigma}_{kzz}(\xi)\exp(-\overline{\gamma}_k\zeta),\qquad \overline{\sigma}_{krr}(\xi,\zeta)\equiv\overline{\sigma}_{krr}(\xi)\exp(-\overline{\gamma}_k\zeta),$$
$$\overline{\sigma}_{k\theta\theta}(\xi,\zeta)\equiv\overline{\sigma}_{k\theta\theta}(\xi)\exp(-\overline{\gamma}_k\zeta),\qquad \overline{\sigma}_{krz}(\xi,\zeta)\equiv\overline{\sigma}_{krz}(\xi)\exp(-\overline{\gamma}_k\zeta),$$
$$\overline{u}_{kr}(\xi,\zeta)\equiv\overline{u}_{kr}(\xi)\exp(-\overline{\gamma}_k\zeta),\qquad \overline{u}_{kz}(\xi,\zeta)\equiv\overline{u}_{kz}(\xi)\exp(-\overline{\gamma}_k\zeta).$$

Here

$$\sigma_{kzz}(\xi)=2\mu\gamma_k^2\left(\kappa_k\left(2(\nu-2)J_0(\gamma_k\xi)+\gamma_k\xi J_1(\gamma_k\xi)\right)-\frac{2}{\pi}J_0(\gamma_k\xi)\right),$$
$$\sigma_{krr}(\xi)=2\mu\gamma_k^2\left(\kappa_k\left((1-2\nu)J_0(\gamma_k\xi)-\gamma_k\xi J_1(\gamma_k\xi)\right)+\frac{2\gamma_k\xi J_0(\gamma_k\xi)-J_1(\gamma_k\xi)}{\pi\gamma_k\xi}\right),$$
$$\sigma_{k\theta\theta}(\xi)=2\mu\gamma_k\left((1-2\nu)\gamma_k\kappa_k J_0(\gamma_k\xi)+\frac{2}{\pi\xi}J_1(\gamma_k\xi)\right),$$
$$\sigma_{krz}(\xi)=2\mu\gamma_k^2\left(\kappa_k((2\nu-2)J_1(\gamma_k\xi)-\gamma_k\xi J_0(\gamma_k\xi))-\frac{2}{\pi}J_1(\gamma_k\xi)\right),$$
$$u_{kr}(\xi)=\gamma_k\left(\kappa_k\gamma_k\xi J_0(\gamma_k\xi)+\frac{2}{\pi}J_1(\gamma_k\xi)\right),$$
$$u_{kz}(\xi)=\kappa_k\gamma_k\left(\gamma_k\xi J_1(\gamma_k\xi)+4(1-\nu)J_0(\gamma_k\xi)\right)-\frac{2}{\pi}\gamma_k J_0(\gamma_k\xi).$$

Following the approach proposed in [3,4], we represent the general solution of the problems I–IV in the real domain as the linear combination of the homogeneous solutions:

$$\sigma_{zz}(\xi,\zeta)=\frac{1}{2}\sum_{k=1}^{\infty}\left(B_k\sigma_{kzz}(\xi,\zeta)+\overline{B}_k\overline{\sigma}_{kzz}(\xi,\zeta)\right),\tag{11}$$

$$\sigma_{rr}(\xi,\zeta)=\frac{1}{2}\sum_{k=1}^{\infty}\left(B_k\sigma_{krr}(\xi,\zeta)+\overline{B}_k\overline{\sigma}_{krr}(\xi,\zeta)\right),\tag{12}$$

$$\sigma_{\theta\theta}(\xi,\zeta)=\frac{1}{2}\sum_{k=1}^{\infty}\left(B_k\sigma_{k\theta\theta}(\xi,\zeta)+\overline{B}_k\overline{\sigma}_{k\theta\theta}(\xi,\zeta)\right),\tag{13}$$

$$\sigma_{rz}(\xi,\zeta)=\frac{1}{2}\sum_{k=1}^{\infty}\left(B_k\sigma_{krz}(\xi,\zeta)+\overline{B}_k\overline{\sigma}_{krz}(\xi,\zeta)\right),\tag{14}$$

$$u_r(\xi,\zeta)=\frac{1}{2}\sum_{k=1}^{\infty}\left(B_k u_{kr}(\xi,\zeta)+\overline{B}_k\overline{u}_{kr}(\xi,\zeta)\right),\tag{15}$$

$$u_z(\xi,\zeta)=\frac{1}{2}\sum_{k=1}^{\infty}\left(B_k u_{kz}(\xi,\zeta)+\overline{B}_k\overline{u}_{kz}(\xi,\zeta)\right)+C.\tag{16}$$

The real constant C has been introduced to take into account the displacement of the cylinder as a rigid body. The general solution (11)–(16) depends on the infinite sequence of undetermined complex constants B_k. We will determine them to subordinating the solution to the boundary conditions imposed on the end faces of the cylinder applying the variational approach.

3. Variational method for a semiinfinite cylinder

We use the quadratic norm to subordinate of the solution to the boundary conditions (2)–(5) [3,4,10]. To do this we introduce for each problem I–IV the corresponding quadratic functional:

$$F_I = \int\limits_0^1 \left[\left(\sigma_{zz}|_{\zeta=0} - \sigma(\xi)\right)^2 + \left(\sigma_{rz}|_{\zeta=0} - \tau(\xi)\right)^2 \right] \xi d\xi, \tag{17}$$

$$F_{II} = \int\limits_0^1 \left[\left(u_z|_{\zeta=0} - u(\xi)\right)^2 + \left(u_r|_{\zeta=0} - v(\xi)\right)^2 \right] \xi d\xi, \tag{18}$$

$$F_{III} = \int\limits_0^1 \left[\left(\sigma_{zz}|_{\zeta=0} - \sigma(\xi)\right)^2 + \left(u_r|_{\zeta=0} - v(\xi)\right)^2 \right] \xi d\xi, \tag{19}$$

$$F_{IV} = \int\limits_0^1 \left[\left(u_z|_{\zeta=0} - u(\xi)\right)^2 + \left(\sigma_{rz}|_{\zeta=0} - \tau(\xi)\right)^2 \right] \xi d\xi. \tag{20}$$

Substituting the representation (11)–(16) into the functionals (17)–(20), and applying the necessary minimum conditions

$$\frac{\partial F_j}{\partial B_m} = 0, \qquad \frac{\partial F_j}{\partial \overline{B}_m} = 0, \qquad \frac{\partial F_{II}}{\partial C} = 0, \qquad j = I, II, III, IV \qquad m = 1, 2, \ldots.$$

we come to the infinite system of linear algebraic equations

$$\sum_{k=1}^{\infty} \sum_{p=1}^{2} C_{mk}^{sp} B_k^p = K_m^s. \tag{21}$$

The coefficients C_{mk}^{sp}, K_m^s $(s, p = 1, 2;\ m = 1, 2, \ldots)$ of system (21) for problems I–IV are defined by the formulas (22)–(29) correspondingly:

$$C_{mk}^{sp} = \frac{1}{2} \int\limits_0^1 \left(\sigma_{mzz}^s \sigma_{kzz}^p + \sigma_{mrz}^s \sigma_{krz}^p \right) \xi d\xi, \tag{22}$$

$$K_m^s = \int\limits_0^1 \left(\sigma(\xi) \sigma_{mzz}^s + \tau(\xi) \sigma_{mrz}^s \right) \xi d\xi, \tag{23}$$

$$C_{mk}^{sp} = \frac{1}{2} \int\limits_0^1 \left(u_{mz}^s u_{kz}^p + u_{mr}^s u_{kr}^p \right) \xi d\xi - \int\limits_0^1 u_{mz}^s \xi d\xi \int\limits_0^1 u_{kz}^p \xi d\xi, \tag{24}$$

$$K_m^s = \int\limits_0^1 \left(u(\xi) u_{mz}^s + v(\xi) u_{mr}^s \right) \xi d\xi - 2 \int\limits_0^1 u(\xi) \xi d\xi \int\limits_0^1 u_{mz}^s \xi d\xi, \tag{25}$$

$$C_{mk}^{sp} = \frac{1}{2}\int_0^1 \left(\sigma_{mzz}^s \sigma_{kzz}^p + u_{mr}^s u_{kr}^p\right) \xi d\xi, \tag{26}$$

$$K_m^s = \int_0^1 \left(\sigma(\xi)\sigma_{mzz}^s + v(\xi)u_{mr}^s\right) \xi d\xi, \tag{27}$$

$$C_{mk}^{sp} = \frac{1}{2}\int_0^1 \left(u_{mz}^s u_{kz}^p + \sigma_{mrz}^s \sigma_{krz}^p\right) \xi d\xi - \int_0^1 u_{mz}^s \xi d\xi \int_0^1 u_{kz}^p \xi d\xi, \tag{28}$$

$$K_m^s = \int_0^1 \left(u(\xi)u_{mz}^s + \tau(\xi)\sigma_{mrz}^s\right) \xi d\xi - 2\int_0^1 u(\xi)\xi d\xi \int_0^1 u_{mz}^s \xi d\xi. \tag{29}$$

The constant C can be expressed via B_k^p as

$$C = \int_0^1 \left(-\sum_{k=1}^{\infty}\sum_{p=1}^{2} B_k^p u_{kz}^p(\xi, 0) + 2u(\xi) \right) \xi d\xi. \tag{30}$$

We used the following notation in the formulas (22)–(30):

$$\begin{array}{llll}
\sigma_{kzz}^1 = \sigma_{kzz}(\xi), & \sigma_{kzz}^2 = \overline{\sigma}_{kzz}(\xi), & \sigma_{krz}^1 = \sigma_{krz}(\xi), & \sigma_{krz}^2 = \overline{\sigma}_{krz}(\xi), \\
\sigma_{mzz}^1 = \sigma_{mzz}(\xi), & \sigma_{mzz}^2 = \overline{\sigma}_{mzz}(\xi), & \sigma_{mrz}^1 = \sigma_{mrz}(\xi), & \sigma_{mrz}^2 = \overline{\sigma}_{mrz}(\xi), \\
u_{kr}^1 = u_{kr}(\xi), & u_{kr}^2 = \overline{u}_{kr}(\xi), & u_{kz}^1 = u_{kz}(\xi), & u_{kz}^2 = \overline{u}_{kz}(\xi), \\
u_{mr}^1 = u_{mr}(\xi), & u_{mr}^2 = \overline{u}_{mr}(\xi), & u_{mz}^1 = u_{mz}(\xi), & u_{mz}^2 = \overline{u}_{mz}(\xi).
\end{array}$$

4. Numerical study the convergence of the reduction method

The system (21) can be solved with the use of the reduction method. To do that we consider retain a finite number of terms in the expansions (11)–(16). This brings to finite system of the algebraic equations for $B_k^1 = B_k$, $B_k^2 = \overline{B}_k$, $k = 1, 2, \ldots, N$

$$\sum_{k=1}^{N}\sum_{p=1}^{2} C_{mk}^{sp} B_k^p = K_m^s. \tag{31}$$

To evaluate numerically the convergence of the reduction method for problems I–IV we consider some examples, taking the functions of the right-hand sides for the boundary conditions (2)–(5) in forms respectively

$$\begin{array}{ll}
\sigma(\xi) = \sigma_0 \arctan(d(\xi - \xi_0)), & \tau(\xi) = 0, \\
u(\xi) = 0, & v(\xi) = v_0 \xi, \\
\sigma(\xi) = \sigma_0 \arctan(d(\xi - \xi_0)), & v(\xi) = 0, \\
u(\xi) = 0, & \tau(\xi) = \tau_0.
\end{array}$$

On the figures 1, 2 some results obtained by solving the problem III at $\xi_0 = 0.5$, $d = 40$ are presented. Fig. 1 display the radial dependencies of the normalized stress σ_{zz}/σ_0 and displacement u_r/u_0 ($u_0 = a\sigma_0/\mu$) on the coordinate ξ for different N. As we can see the dependencies $\sigma_{zz}(\xi, \zeta)\mid_{\zeta=0}$ and $u_r(\xi, \zeta)\mid_{\zeta=0}$ are approaching to the given functions $\sigma(\xi)$ and $v(\xi)$ correspondingly, when number

N is increasing: the curves 3 are practically coincided with the graphs of the given functions $\sigma(\xi)$ and $v(\xi)$ of this problem's boundary conditions.

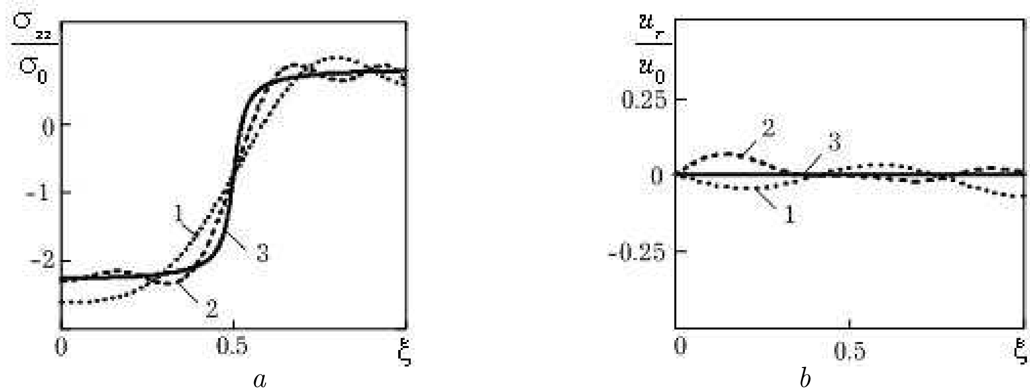
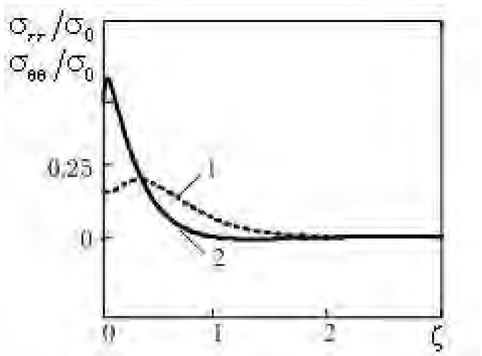


Fig. 1. Radial dependencies of stress component $\sigma_{zz}(\xi, 0)/\sigma_0$ (a) and displacement component $u_r(\xi, 0)/u_0$ (b) for different $N = 2, 4, 20$ (curve 1, 2, 3 correspondingly)

Strain and stress components quickly decay with distance from the end surface. Fig. 2 illustrates the axial dependencies of the stress components $\sigma_{rr}(\xi, \zeta)\mid_{\xi=0}$ and $\sigma_{\theta\theta}(\xi, \zeta)\mid_{\xi=1}$.



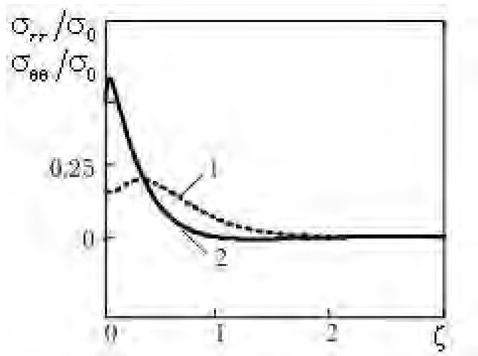


Fig. 2. Axial dependencies of normalized stresses components $\sigma_{rr}(0, \zeta)$ (curve 1) and $\sigma_{\theta\theta}(1, \zeta)$ (curve 2).

Fig. 3. Decaying of the reduction error with growing the number of retained terms in the solution representation for problems I–IV.

So, if an end of finite cylinder is loaded by self-balanced traction and its height is equal or greater of its diameter then we can consider it as a semi-infinite cylinder.

We estimate the error for each solution of the problems I–IV, obtained by solving the system (31), through the value of the corresponding functional as:

$$\varepsilon_I = \frac{1}{\sigma_0}\left(\frac{F_I^N}{2}\right)^{1/2}, \qquad \varepsilon_{II} = \frac{1}{v_0}\left(\frac{F_{II}^N}{2}\right)^{1/2},$$

$$\varepsilon_{III} = \frac{1}{\sigma_0}\left(\frac{F_{III}^N}{2}\right)^{1/2}, \qquad \varepsilon_{IV} = \frac{1}{\tau_0}\left(\frac{F_{IV}^N}{2}\right)^{1/2}.$$

Plots on Fig. 3 demonstrate how the errors ε_I, ε_{II}, ε_{III}, ε_{IV} are decaying with increasing N (curves 1–4 correspondingly). As we can see the convergence of the reduction method depends on the type of boundary conditions and features of the given functions of boundary conditions. On the basis of conducted numerical experiments we can conclude that reduction method's accuracy, sufficient for practical goals, can be achieved at $N \geqslant 10$.

5. Variational method for a finite cylinder

Consider the class of axisymmetric problems of elasticity theory for a finite cylinder $0 \leqslant r \leqslant a$, $0 \leqslant \theta \leqslant 2\pi$, $-b \leqslant z \leqslant b$, with free lateral surface (conditions (1)) and four types of boundary conditions on the ends $\zeta = \pm b$ of cylinder (problems V–VIII):

$$\sigma_{zz}|_{\zeta=\pm b} = \sigma(\xi), \qquad \sigma_{rz}|_{\zeta=\pm b} = \tau(\xi), \tag{32}$$

$$u_z|_{\zeta=\pm b} = u(\xi), \qquad u_r|_{\zeta=\pm b} = v(\xi), \tag{33}$$

$$\sigma_{zz}|_{\zeta=\pm b} = \sigma(\xi), \qquad u_r|_{\zeta=\pm b} = v(\xi), \tag{34}$$

$$u_z|_{\zeta=\pm b} = u(\xi), \qquad \sigma_{rz}|_{\zeta=\pm b} = \tau(\xi). \tag{35}$$

Since the each solution of the problems V–VIII satisfies on the opposite cylinder's ends $\zeta = \pm b$ the conditions of the same types, we can split each problem (32)–(35) on the symmetric and antisymmetric parts with respect to the plane $\zeta = 0$.

To do that we take the solution in the forms:

$$\chi = \frac{1}{2}\sum_{k=1}^{\infty}\left(L_k f_k(\xi)\cosh(\gamma_k\zeta) + \overline{L}_k \overline{f}_k(\xi)\cosh(\overline{\gamma}_k\zeta)\right)$$

for symmetric part and

$$\chi = \frac{1}{2}\sum_{k=1}^{\infty}\left(L_k f_k(\xi)\sinh(\gamma_k\zeta) + \overline{L}_k \overline{f}_k(\xi)\sinh(\overline{\gamma}_k\zeta)\right),$$

for antisymmetric part, where $f_k(\xi) = \xi J_1(\gamma_k\xi)\kappa_k - \frac{2}{\pi\gamma_k}J_0(\gamma_k\xi)$, L_k are undetermined constants.

Using the variational method of homogeneous solutions [3,4] we represent the components $w(\xi,\zeta) \in \{\sigma_{zz}(\xi,\zeta), \sigma_{rr}(\xi,\zeta), \sigma_{\theta\theta}(\xi,\zeta), \sigma_{rz}(\xi,\zeta), u_r(\xi,\zeta), u_z(\xi,\zeta)\}$ for symmetry and antisymmetry problems in the form (36) and (37) correspondingly:

$$w(\xi,\zeta) = \frac{1}{2}\sum_{k=1}^{\infty}\left(L_k w_k(\xi)\cosh(\gamma_k\zeta) + \overline{L}_k \overline{w}_k(\xi)\cosh(\overline{\gamma}_k\zeta)\right), \tag{36}$$

$$w(\xi,\zeta) = \frac{1}{2}\sum_{k=1}^{\infty}\left(L_k w_k(\xi)\cosh(\gamma_k\zeta) + \overline{L}_k \overline{w}_k(\xi)\cosh(\overline{\gamma}_k\zeta)\right), \tag{37}$$

Here $w_k(\xi) \in \{\sigma_{kzz}(\xi), \sigma_{krr}(\xi), \sigma_{k\theta\theta}(\xi), \sigma_{krz}(\xi), u_{kr}(\xi), u_{kz}(\xi)\}$.

To subordinate the solutions for problem V–VIII to the boundary conditions (32)–(35) in the quadratic norm, we introduce corresponding quadratic functional:

$$F_V = \int_0^1 \left[(\sigma_{zz}|_{\zeta=\pm b} - \sigma(\xi))^2 + (\sigma_{rz}|_{\zeta=\pm b} - \tau(\xi))^2\right]\xi d\xi, \tag{38}$$

$$F_{VI} = \int_0^1 \left[(u_z|_{\zeta=\pm b} - u(\xi))^2 + (u_r|_{\zeta=\pm b} - v(\xi))^2\right]\xi d\xi, \tag{39}$$

$$F_{VII} = \int_0^1 \left[(\sigma_{zz}|_{\zeta=\pm b} - \sigma(\xi))^2 + (u_r|_{\zeta=\pm b} - v(\xi))^2\right]\xi d\xi, \tag{40}$$

$$F_{VIII} = \int_0^1 \left[(u_z|_{\zeta=\pm b} - u(\xi))^2 + (\sigma_{rz}|_{\zeta=\pm b} - \tau(\xi))^2\right]\xi d\xi. \tag{41}$$

Using the necessary condition of minimum of the functionals (38)–(41)

$$\frac{\partial F_j}{\partial L_m}=0,\qquad \frac{\partial F_j}{\partial \overline{L}_m}=0,\qquad \frac{\partial F_{VI}}{\partial C}=0,\qquad j=V,VI,VII,VIII\qquad m=1,2,\dots$$

we obtain the infinite system of linear algebraic equations

$$\sum_{k=1}^{\infty}\sum_{p=1}^{2}C_{mk}^{sp}L_k^p=K_m^s. \tag{42}$$

Here, $L_k^1=L_k$, $L_k^2=\overline{L}_k$.

The coefficients of system (42) are defined by the formulas (22)–(29) with following notation

$$w_k^1=w_k(\xi)\cosh(\gamma_k\zeta),\qquad w_k^2=\overline{w}_k(\xi)\cosh(\overline{\gamma_k}\zeta)$$

for symmetric part and

$$w_k^1=w_k(\xi)\sinh(\gamma_k\zeta),\qquad w_k^2=\overline{w}_k(\xi)\sinh(\overline{\gamma_k}\zeta)$$

for antisymmetric part. Here $w_k^1\in\{\sigma_{kzz}^1,\sigma_{krr}^1,\sigma_{k\theta\theta}^1,\sigma_{krz}^1,u_{kr}^1,u_{kz}^1\}$, $w_k^2\in\{\sigma_{kzz}^2,\sigma_{krr}^2,\sigma_{k\theta\theta}^2,\sigma_{krz}^2,u_{kr}^2,u_{kz}^2\}$.

The constant C can be calculated as

$$C=\int_0^1\left(-\sum_{k=1}^{\infty}\sum_{p=1}^{2}L_k^p u_{kz}^p(\xi,b)+2u(\xi)\right)\xi d\xi.$$

6. Axisymmetric bending of a thick disk

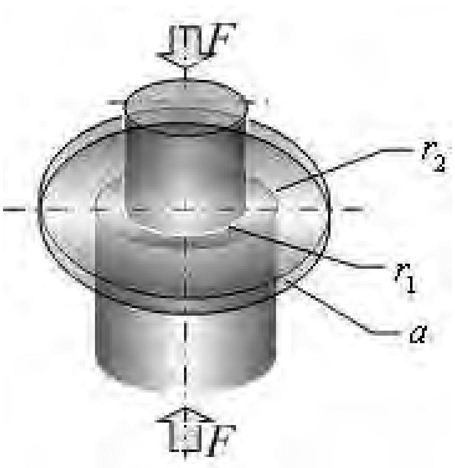


Fig. 4. Axisymmetric bending of the circular disk.

As an example of applying the variational approach we consider the problem of bending of a round disk (Fig. 4). The external axisymmetric loadings $F_1(\xi)$ and $F_2(\xi)$ applying to opposite disc's surfaces, are concentrated in vicinities of the concentric circles of radiuses r_1 and r_2 correspondingly and satisfy the conditions

$$\int_0^1 F_1(\xi)\xi d\xi=\int_0^1 F_2(\xi)\xi d\xi.$$

Under such loading the boundary conditions for symmetric and antisymmetric problems are as follows:

$$\sigma_{zz}|_{\zeta=\pm b}=\frac{1}{2}\left(F_1(\xi)+F_2(\xi)\right),\qquad \sigma_{rz}|_{\zeta=\pm b}=0,$$

$$\sigma_{zz}|_{\zeta=\pm b}=\frac{1}{2}\left(F_1(\xi)-F_2(\xi)\right),\qquad \sigma_{rz}|_{\zeta=\pm b}=0.$$

We solved the problem, taking the functions $F_1(\xi)$ and $F_2(\xi)$ in the form

$$F_1(\xi)=\sigma_0\exp\left(\frac{-\sin(\xi-r_1)^2}{\delta}\right),\qquad F_2(\xi)=\psi\sigma_0\exp\left(\frac{-\sin(\xi-r_2)^2}{\delta}\right),$$

where $\psi=\int_0^1\exp\left(\frac{-\sin(\xi-r_1)^2}{\delta}\right)\xi d\xi\Big/\int_0^1\exp\left(\frac{-\sin(\xi-r_2)^2}{\delta}\right)\xi d\xi$, σ_0 and δ are given parameters.

The solution was obtained at $N = 25$. Some results obtaining for $b = 0.2a$, $r_1 = 0.5a$, $r_2 = 0.8a$, $\delta = 0.04a$ are shown on figures 5–7.

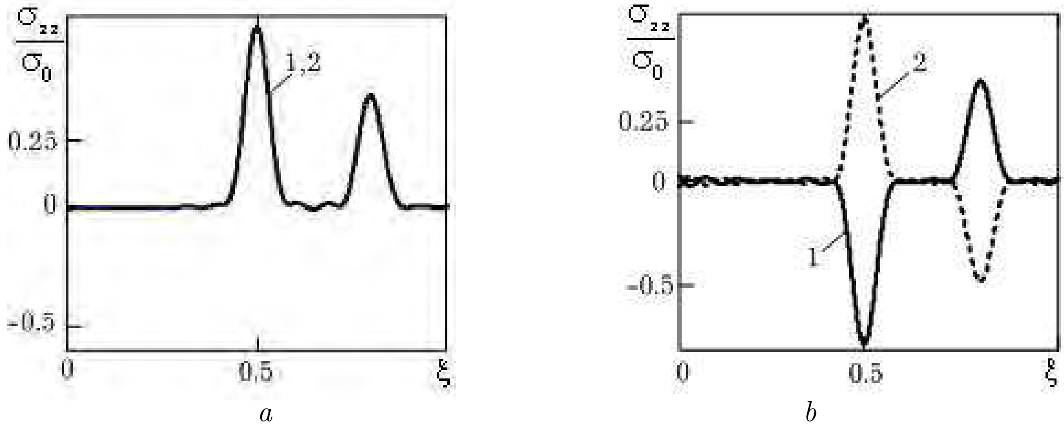


Fig. 5. Radial dependence of the normalized stress component σ_{zz}/σ_0 on the faces $\zeta = b$ (curve 1) and $\zeta = -b$ (curve 2) for symmetric (a) and antisymmetric (b) problems.

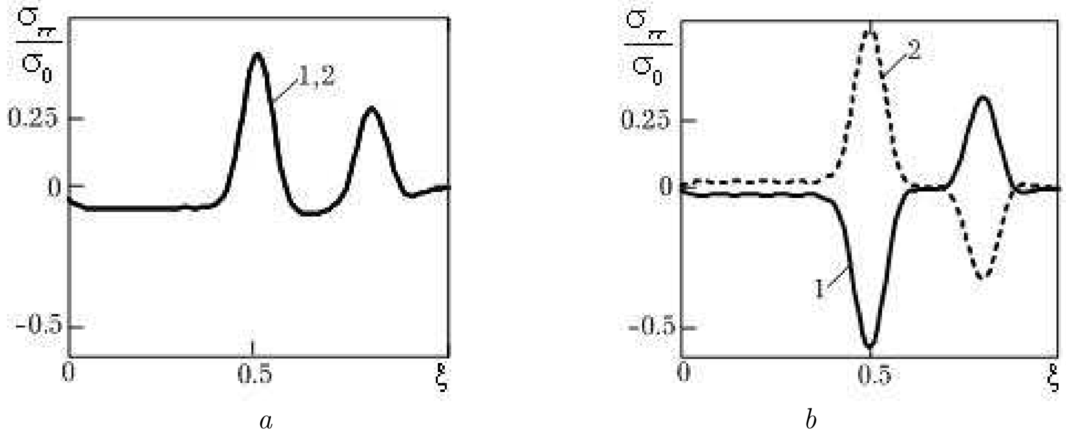


Fig. 6. Radial dependence of the normalized stress component σ_{rr}/σ_0 on the faces $\zeta = b$ (curve 1) and $\zeta = -b$ (curve 2) for symmetric (a) and antisymmetric (b) problems.

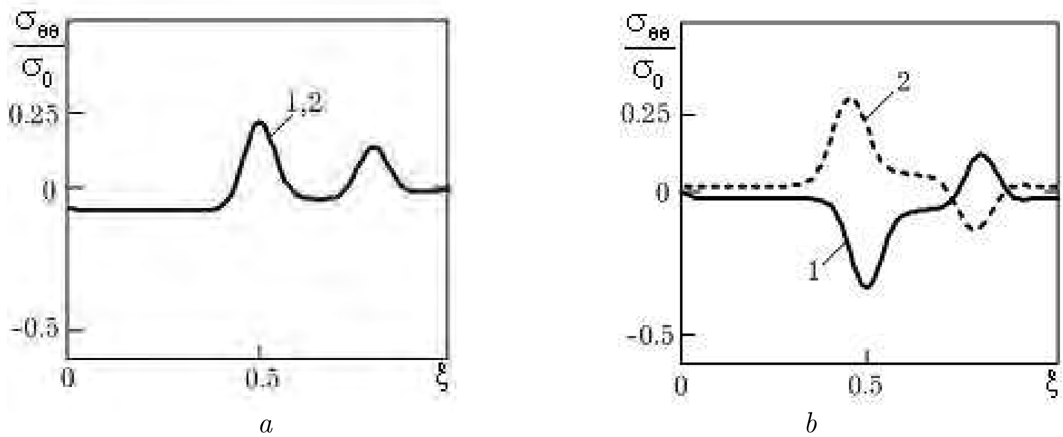


Fig. 7. Radial dependence of the normalized stress component $\sigma_{\theta\theta}/\sigma_0$ on the faces $\zeta = b$ (curve 1) and $\zeta = -b$ (curve 2) for symmetric (a) and antisymmetric (b) problems.

On figure 5 the normalized axial stress component σ_{zz}/σ_0 on surfaces $\xi = \pm b$ for symmetric (a) and antisymmetric (b) problems as the function of radial coordinate are shown. As we can see the calculated values of the stress component are practically coincide with the applied loading.

Graphs of distribution of normalized stress components σ_{rr}/σ_0, $\sigma_{\theta\theta}/\sigma_0$ for symmetric (a) and antisymmetric (b) tasks are shown in Fig. 6, 7.

7. Conclusions

Variational method for solving of axisymmetric elasticity problems for semi-infinite and finite cylinders is developed. According to this method the solution is represented as the series expansion in the eigenfunctions of problem for a cylinder with an unloaded lateral surface. This reduces the problem to an infinite system of algebraic equations for the coefficients of the expansion, which have being solved by the reduction method. Four types of boundary problems have been considered for semi-infinite and finite cylinders with the use of this method. Application of this method to study of the stressed state of a circular disc under its transversal bending by normal forces applying to the disc's faces has been made. That allowed to obtain the analytical solution that reflects the volumetric character of the stressed state. These results can be used in particular for optimizing the geometric parameters of the samples and loading conditions for long-term strength statistical testing of brittle materials [11].

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Варіаційний метод однорідних розв'язків розв'язування осесиметричних задач теорії пружності для циліндра

Чекурін В. Ф., Постолакі Л. І.

Інститут прикладних проблем механіки і математики ім. Я. С. Підстригача НАН України
вул. Наукова, 3-б, 79060, Львів, Україна

Розвинено варіаційний метод однорідних розв'язків для розв'язування осесиметричних задач теорії пружності для півбезмежного та скінченого циліндрів, на торцевих поверхнях яких задані умови навантаження в напруженнях, переміщеннях чи змішані. Розв'язок подано у вигляді розвинення за системами власних функцій відповідної однорідної бігармонічної задачі у циліндричних координатах. Підпорядкування розв'язку умовам, заданим на торцях циліндра, здійснюється за квадратичною нормою. Як приклад застосування цього методу розглянута задача згинання товстого круглого диска зосередженими силами, прикладеними до його лицьових поверхонь.

Ключові слова: *циліндричне тіло, осесиметрична задача теорії пружності, функція Лява, бігармонічне рівняння, варіаційний метод однорідних розв'язків*

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