ECONTECHMOD. AN INTERNATIONAL QUARTERLY JOURNAL – 2016. Vol. 5. No. 1. 97–102

Investigation of mathematical models for vibrations of one dimensional environments with considering nonlinear resistance forces

P. Pukach[1,2], P. Sokhan[1], R. Stolyarchuk[1]

[1] Department of Higher Mathematics; Lviv Polytechnic National University
S. Bandery str., 12, Lviv, 79013, Ukraine; e-mail: ppukach@gmail.com
[2] Hetman Petro Sahaidachny National Army Academy

Received September 29.2015: accepted January 20.2016

Abstract. In this paper we consider important classes of one dimensional environments, bending stiffness of which can be neglected. It is impossible to apply approximate analytical method of solution of mathematical models of dynamic processes. So justification of existence and uniqueness of solutions, carried out a qualitative their evaluation, based on numerical analysis are considering in this paper. Also the features of dynamic processes of some of examined class of systems are analyzed. Methods of qualitative study of oscillations for restricted and unrestricted bodies under the influence of the resistance forces, described in this paper are based on the general principles of the theory of nonlinear boundary value problems – Galerkin method and the method of monotonicity. Scientific novelty consists in generalization these methods of studying for nonlinear problems at new classes of oscillating systems, justification of solution correctness for specified mathematical models that have practical application in real engineering vibration systems.

Key words: mathematical model, nonlinear vibrations, nonlinear boundary value problem, Galerkin method, nonlinear elastic properties.

INTRODUCTION

The development of new technology and the transition to high-speed machinery requires the formulation and solution of new problems, mathematical models of which can not be investigated by the asymptotic methods of nonlinear mechanics: the problem of vibrations of flexible elements belt or chain gears tape systems for recording and reproducing information, conveyor lines, different kind of cable lifts, equipment for rolling paper, metal strip, wire, thread, equipment for drilling oil and gas wells, pipelines and others.

In the case of nonlinear elasticity law, significantly nonlinear dependence of oscillation amplitude from the resistance force and so the problem is related with fundamental mathematical difficulties (even in the case when model fluctuations in limited areas are researching), because there is no general analytical methods for solving this class of problems. This problem is generally solved only for a very narrow class of problems. Therefore, there is no general methods for determining the amplitude– frequency characteristics of oscillatory process.

THE ANALYSIS OF RECENT RESEARCHES AND PUBLICATIONS

On the other hand, qualitative methods of the theory of nonlinear boundary value problems allow for a broad class of oscillatory systems mentioned above to get the results for correct solution of the problem (ie, the existence, uniqueness and continuous dependence on initial data). The above method allows to prove the correctness of the solution of a problem in the model and allows with further research of solution to use various approximate (numerical) methods. Thus, the problem of qualitative research methods for nonlinear systems is relevant.

The work is devoted to the research solutions of problem for nonlinear wave equations and systems, which in the model case [1] have the form:

$$\frac{\partial^2 u}{\partial x^2} - a\,\frac{\partial^2 u}{\partial x^2} + g\left(x,t,\frac{\partial u}{\partial t}\right) = f(x,t), \quad (1)$$

where: a – some function (constant), which characterizes the physical – mechanical parameters of oscillation system, g – nonlinear function, which describes the nonlinear dependence of the oscillation amplitude from the resistance. Problems for hyperbolic equations like (1) and in a form:

$$\frac{\partial^2 u}{\partial x^2} - a\,\frac{\partial^2 u}{\partial x^2} + g_1(x,t,u) = f(x,t), \quad (2)$$

were: g_1 – nonlinear function in restricted and unrestricted areas were considered in [3–13]. Some results of solution correctness interpretation in these works obtained under the assumption of a qualitative behavior of the solution, the initial data and the right-hand side of equation (system) on infinity, another results – without such assumptions. The work [12] is devoted to research of first mixed problem for second-order weakly nonlinear hyperbolic equation of the form (1). The conditions of existence and uniqueness of a generalized solution in spaces of locally integrable functions received in this paper. In [14] have been studied mixed problem for weakly nonlinear hyperbolic systems of equations of the first order with two independent variables. A similar question for a mathematical model that describes the problem for equation (2) was considered in [15].

Note that the issue of substantiation of well-posedness of certain weakly and strongly nonlinear mathematical models of nonlinear oscillating systems has been considered in the works [7–22]. In particular, in those works there have been developed a methodology of investigating the well-posedness (existence and uniqueness of solutions) of mixed problems for quasi-linear and strongly nonlinear evolutional equations of beam vibration type (in the case of presence of dissipative forces in the system) in bounded and unbounded domains.

OBJECTIVES

Lets discuss a method of qualitative research of mathematical model for nonlinear oscillations of semi-stricted inhomogeneous environment under condition Vincler nonlinear force action:

$$\frac{\partial^2 u(x,t)}{\partial t^2} - a\,\frac{\partial^2 u(x,t)}{\partial x^2} + g\left|\frac{\partial u(x,t)}{\partial t}\right|^{p-2}\frac{\partial u(x,t)}{\partial t} = f(x,t),\ p>2, \quad (3)$$

with initial conditions:

$$u(x,0) = u_0(x), \quad (4)$$

$$\frac{\partial u(x,0)}{\partial t} = u_1(x), \quad (5)$$

and boundary condition:

$$u(0,t) = 0. \quad (6)$$

In relations (3)-(6):

– $u(x,t)$ – the longitudinal (transverse) movement of environment with coordinate x at an arbitrary point of time t,

– a – constant, which characterizes cross-sectional area of environment, running mass, elastic properties of the environment , etc.,

– $g>0$ – constant, which takes into account given above characteristics and describes nonlinear resistance forces,

– $f(x,t)$ – the function that describes the distribution of forces along the environment,

– $u_0(x)$ and describing the initial state of the environment (initial rejection – form and initial velocity).

Environment is semi-stricted, therefore $x \in (0,+\infty)$ and the dynamic process we will consider an arbitrarily long time period, so $0 \le t < +\infty$. Everywhere further in this paper use the following notation for arbitrary $R>0$, $t \in (0,T]$:

$Q_{R,t} = (0,R)\times(0,t)$ – rectangle with base $(0,R)$ on the axis Ox and height t ,

$Q_t = (0,+\infty)\times(0,t)$ – halfstrips with base $(0,+\infty)$ on the axis Ox and height t.

For description of qualitative properties of the input data and solution we will use some spaces of generalized functions.

$H_0^1(0,R)$ – space of functions which squares with their derivatives are Lebesgue integrable on the interval $(0,R)$, and at the end of the interval homogeneous boundary conditions fulfilled $u|_{x=0} = u|_{x=R} = 0$. Norm in this space defines as:

$$\|u\|^2_{H_0^1(0,R)} = \int_0^R \left(\frac{\partial u}{\partial x}\right)^2 dx,$$

were: $H^1_{0,loc}(0,+\infty)$ – space of functions belonging to $H_0^1(0,R)$ for arbitrary $R>0$, where $u(0,t)=0$,

$L^r_{loc}(\overline{Q})$ _ space of functions Lebesgue integral with degree p at the interval $(0,R)$ for arbitrary $R>0$, and $r \in (1,+\infty)$.

The aim of this work is, in particular, research of the solution for problem (3) – (6) for second order nonlinear wave equation, namely obtaining of correct solution conditions in mathematical model – sufficient conditions for the existence and uniqueness of the solution in the class of locally integrated functions. The specified equation, in particular, describes the forced oscillation of rod or rope in the environment with resistance [2, p. 234].

THE MAIN RESULTS OF THE RESEARCH

Generalized solution of problem (3) – (6) we will call function u, that satisfies the condition (4) and integral identity:

$$\int_{Q_t}\left[-\frac{\partial u}{\partial t}\frac{\partial v}{\partial t}+a\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\right]dxdt+\int_{Q_t}\left[\left|\frac{\partial u}{\partial t}\right|^{p-2}\frac{\partial u}{\partial t}v-fv\right]dxdt+$$

$$+\int_0^{+\infty}\frac{\partial u}{\partial t}(x,t)v(x,t)dx-\int_0^{+\infty}u_1(x)v(x,0)dx=0\,, \quad (7)$$

where: arbitrary $t\in(0,T]$ and arbitrary function v with limited carrier are such, that the identity (7) makes sense. Concerning coefficients of right side for equation (3) and initial data lets assume fulfillment of following conditions.

1. Function $f\in L^q_{loc}(\overline{Q})$, and q is the conjugate number for p: $\frac{1}{p}+\frac{1}{q}=1.$

2. The initial deviation $u_0(x)$ belong to $H^1_0(0,R)$ for arbitrary $R>0$, and $u_0(0)=0$; primary velocity $u_1(x)$ is a function that belongs to $L^2(0;R)$ for the arbitrary $R>0$.

The main result of qualitative research: if the mathematical model of an oscillatory process described problem (4) – (6) for the equation (3) under conditions that (1)-(2) exist unique generalized solution $u(x,t)$ of problem (3) – (6), moreover: function u – continuous by variable t on the interval $[0,T]$, and by variable x refers to the space $H^1_{0,loc}(0,+\infty)$; derivative $\frac{\partial u}{\partial t}$ – continuous and locally integrable with degree p by variable t at the interval $[0,T]$, and by the variable x – locally integrable with degree p.

Application of methods of the theory of nonlinear boundary problems for proving existence and uniqueness of solution. Let u^1,u^2 – generalized solution of problem (3) – (6) and problem, which differs from (3) – (6) by the fact, that in the right side of (3) compelled force f is replaced by $\overline{f}$ respectively. Then for arbitrary t,R,R_0 such that $0<R_0<R$, $t\in(0,T]$ one can obtain following estimation:

$$\int_0^{R_0}\left(\frac{\partial u^1(x,t)}{\partial t}-\frac{\partial u^2(x,t)}{\partial t}\right)^2dx+$$

$$+C_1\int_0^{R_0}\left(\frac{\partial u^1(x,t)}{\partial x}-\frac{\partial u^2(x,t)}{\partial x}\right)^2dx+$$

$$+C_2\int_{Q_{R_0,t}}\left|\frac{\partial u^1}{\partial t}-\frac{\partial u^2}{\partial t}\right|^p dxdt\le\left(\frac{R}{R-R_0}\right)^b\times$$

$$\times\left(C_3R^{1+(a-1)\frac{2p}{p-2}}+C_4\int_{Q_{R,t}}\left|f-\overline{f}\right|^q dxdt\right), \quad (8)$$

where: $b>\frac{2p}{p-2}$ are arbitrary number; $C_1 – C_4$ are positive numbers, which depends only from p, b.

Lets justify inequality (8). Put $R>R_0>0$, $t\in(0,T]$ are arbitrary numbers. Let function $j(x)$ is defined as follows: $j(x)=\begin{cases}\frac{R^2-x^2}{R}, & x\le R,\\ 0, x>R\end{cases}$.

Directly easy to see that for the function j following estimation is correct $R-x\le j(x)\le 2(R-x)$.

Let $u^1(x,t),u^2(x,t)$ – generalized solution of problem (3) – (6) and problem, which differs from (3)–(6) by the fact, that in the right side of (3) compelled force f is replaced by $\overline{f}\in L^q_{loc}(\overline{Q})$. Suppose further $w=u^1-u^2$ and make the regularization procedure described in [2, pp. 238–239]. Fix $s_0,s_1\in[0,T]$, $s_0<s_1$. Let q_m – неперервна continuous piecewise linear function on $[0,T]$, $q_m=1$, if $s_0+\frac{2}{m}<t<s_1-\frac{2}{m}$, $q_m=0$ for $t>s_1-\frac{1}{m}$ and for $t<s_0+\frac{1}{m}$, $m=1,2,...$ Let r_n $(n=1,2,...)$ – regulative sequence in the space of infinitely differentiable in $\mathbf{R}$ functions with a compact support $r_n(t)=r_n(-t)$, $\int_{-\infty}^{+\infty}r_n(t)dt=1$, supp $r_n(t)\subset\left[-\frac{1}{n},\frac{1}{n}\right]$ for arbitrary $n\in\mathbf{N}$. Suppose now $v=((q_m(\frac{\partial w}{\partial t}))*r_n*r_n)q_mj^b$, $b>0$. Symbol $*$ indicated on the last equality means convolution operation. Subtract from equation (3), generalized solution of which is function u^1 similar equation for u^2 with right-hand side $\overline{f}$. Lets multiply both sides of the resulting difference for v and integrate results from 0 to t, $0<t\le T$. As a result, after calculations and transformations similar to those made in [2, p. 238-239], and in limiting transition $n\to\infty$, $m\to\infty$ we will obtain:

$$\frac{1}{2}\int_0^R\left[\left(\frac{\partial w(x,t)}{\partial t}\right)^2+a\left(\frac{\partial w(x,t)}{\partial x}\right)^2\right]j^b\,dx+$$

$$+\int_{Q_{R,t}}a\frac{\partial w}{\partial x}\frac{\partial w}{\partial t}\frac{\partial j^b}{\partial x}dxdt+$$

$$+\int_{Q_{R,t}} g\left(\left|\frac{\partial u^1(x,t)}{\partial t}\right|^{p-2}\frac{\partial u^1(x,t)}{\partial t}-\right.$$

$$\left.-\int_{Q_{R,\tau}} -\left|\frac{\partial u^2(x,t)}{\partial t}\right|^{p-2}\frac{\partial u^2(x,t)}{\partial t}\right)\times$$

$$\times\left(\frac{\partial u^1(x,t)}{\partial t}-\frac{\partial u^2(x,t)}{\partial t}\right)j^{\,b}\,dxdt=$$

$$=\int_{Q_{R,t}}\left(f-\overline{f}\right)\frac{\partial w}{\partial t}j^{\,b}\,dxdt. \qquad (9)$$

Let's estimate integrals in equality (9) for $\frac{1}{2}+\frac{1}{p}+\frac{1}{p_1}=1$ just as is done in [10]. Considering there evaluations obtained there and using properties of function $j(x)$, we have:

$$(R-R_0)^b\left(\int_0^{R_0}\left(\frac{\partial w(x,t)}{\partial t}\right)^2 j^{\,b}\,dx+\right.$$

$$+C_5\int_0^{R_0}\left(\frac{\partial w(x,t)}{\partial t}\right)^2 j^{\,b}\,dx+$$

$$\left.+C_6\int_{Q_{R_0,t}}\left|\frac{\partial w}{\partial t}\right|^p dxdt\right)\le C_7R^{\,b+1+(a-1)\frac{2p}{p-2}}+$$

$$+C_8R^b\int_{Q_{R,t}}\left|f-\overline{f}\right|^q dxdt,$$

where: C_5-C_8 are positive constants. From the last inequality is easy to obtain the inequality (8).

Consider the sequence of domains $Q^k=(0,k)\times(0,T)$, $k=1,2,\ldots$ and in each domain Q^k respectively problem:

$$\frac{\partial^2u^k(x,t)}{\partial t^2}-a\frac{\partial^2u^k(x,t)}{\partial x^2}+g\left|\frac{\partial u^k(x,t)}{\partial t}\right|^{p-2}\times$$

$$\times\frac{\partial u^k(x,t)}{\partial t}=f^k(x,t), \qquad (10)$$

$$u^k(x,0)=u_0^k(x), \qquad (11)$$

$$\frac{\partial u^k(x,0)}{\partial t}=u_1^k(x), \qquad (12)$$

$$u^k(0,t)=u^k(k,t)=0. \qquad (13)$$

Note that in equation (10) functions $f^k(x,t)=\begin{cases}f(x,t), x\le k,\\ 0, x>k.\end{cases}$ Besides this, instead of function u_0 considered u_0^k, where

$$u_0^k(x)=u_0(x)\cdot x^k(x),\quad x^k\in C^1(\mathrm{R}),$$

$$x^k(x)==\begin{cases}1, x\le k-1,\\ 0, x>k,\end{cases}\ 0\le x^k(x)\le 1.$$

It is clear that functions $u_0^k\in H_0^1(0,k)$ and $\lim\limits_{k\to+\infty}\left\|u_0^k-u_0\right\|_{H_0^1(0,k)}=0$. Instead initial function u_1 were reviewed u_1^k – constriction of function u_1 at $(0,k)$, $u_1^k\in L^2(0,k)$, $\lim\limits_{k\to+\infty}\left\|u_1^k-u_1\right\|_{L^2(0,k)}=0$. Under a generalized solution of problem (10)-(13) we understand the function u^k, which satisfy (10), (11), (13) and the integral identity similar to the identity (7), which is consider in the area Q^k, and function v is chosen integrable with her derivative by time variable, and by spatial variable is locally integrable with degree p.

Note that in the above conditions exist unique generalized solution of the problem (10) – (13) in Q^k [2, p. 234]. Consider now sequence of problems of the form (10) – (13) for $k=1, k=2,\ldots,$ redefinding u^k by zero at $Q\setminus Q^k$. We obtain a sequence of solutions of problem (3) – (6) in Q, which for convenience we again denote as $\{u^k\}$. Similarly as is done in [10] we show that the sequence $\{u^k\}$ and $\left\{\frac{\partial u^k}{\partial t}\right\}$ are fundamental in appropriate functional spaces. It is obviously that for function u, conditions (4)–(6) is true. Consequently, the function $u(x,t)$ is a generalized solution of problem (3)–(6) in sense of the integral identity (7).

The uniqueness of the obtained solution follows from inequality (8) for $R\to+\infty$, if we consider two arbitrary solutions u^1 and u^2 of problem (3)–(6) and consider that

$$u^1(x,0)=u^2(x,0),\ \frac{\partial u^1(x,0)}{\partial t}=\frac{\partial u^2(x,0)}{\partial t}.$$

Note, that for problem (3) – (6) is easy to obtain sufficient conditions for the existence and uniqueness of periodic by linear variable generalized solution.

The results of numerical integration in the model case. Consider the special case of equation (3), namely the case of natural oscillations of a continuous environment, provide by constant along its length physical and mechanical properties, ie

$$\frac{\partial^2u}{\partial t^2}(x,t)=a\frac{\partial^2u}{\partial x^2}(x,t)-g_0\left|\frac{\partial u}{\partial t}(x,t)\right|^{p-2}\frac{\partial u}{\partial t}(x,t).$$

In the latter relation a and g_0 are constants and boundary conditions take the form $u(0,t)=u(l,t)=0.$ Due to original form, it is described as:

$$u_0(x)=\begin{cases}\dfrac{2hx}{l}, & 0\le x\le\dfrac{l}{2}\\ 2h-\dfrac{2hx}{l}, & \dfrac{l}{2}<x\le l\,.\end{cases}$$

We feel that the initial velocity of continuous environment points is zero, i.e., $\frac{\partial u}{\partial t}(x,0)=0$. The problem describes transversal oscillations of threads (rope) that at the initial time loaded by concentrated force at the point with coordinate $x=\frac{l}{2}$. The above problem is of the form (10) – (13). As shown above, there is unique generalized solution of this problem. Therefore, for the numerical integration of the equations of motion is the choice of method is principal only in computer needs. Examined equation we will reduce to a system of two equations like:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t)=v(x,t), \\ \frac{\partial v}{\partial t}(x,t)=a\frac{\partial^2 u}{\partial x^2}(x,t)-g_0|v(x,t)|^{p-2}v(x,t). \end{cases}$$

We divide the interval $[0;l]$ by discretization points $[0;l]$ by discretization points $x_i=i\frac{l}{n}$ at n parts with length $\Delta=\frac{l}{n}$. We approximate the derivative by spatial variable with finite difference:

$$\frac{\partial^2 u}{\partial x^2}(x,t)=\frac{u(x_{i-1},t)-2u(x_i,t)+u(x_{i+1},t)}{\Delta^2}.$$

Numerical solution of system of ordinary differential equations:

$$\begin{cases} u'(t)=v(t), \\ v'(t)=L(t,v), \end{cases}$$

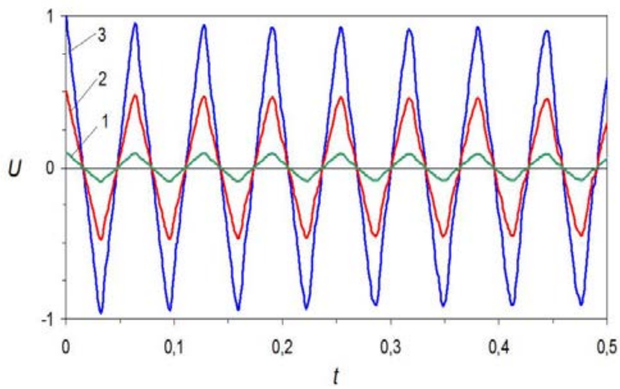
where: $L(t,v)=a\frac{\partial^2 u}{\partial x^2}(x,t)-g_0|v(x,t)|^{p-2}v(x,t)$,

implemented by the Runge-Kutta fourth order method:

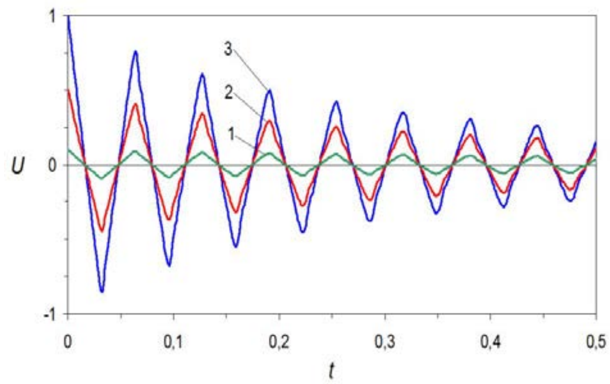
$$\begin{cases} u_{k+1}=u_k+v_k\Delta t+\frac{\Delta t}{6}(k_1+k_2+k_3), \\ v_{k+1}=v_k+\frac{1}{6}(k_1+2k_2+2k_3+k_4), \end{cases}$$

moreover:

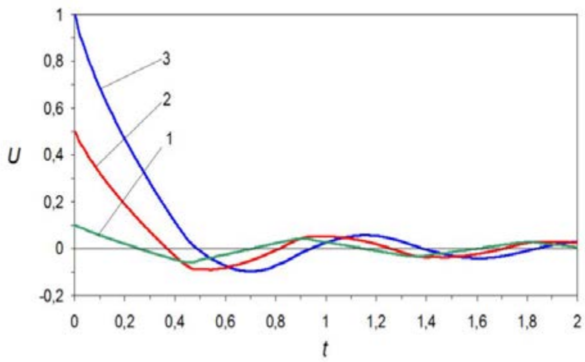
$$t_k=k\Delta t,\ u_k=u(t_k),\ v_k=v(t_k),$$



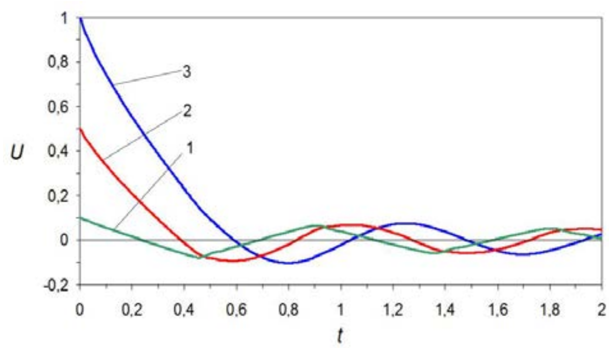
a) $a=1000$, $g_0=1$, $p=2{,}1$.



b) $a=1000$, $g_0=1$, $p=3$.



c) $a=5$, $g_0=6$, $p=3$.



d) $a=5$, $g_0=6$, $p=4$.

Fig. 1. Laws change of environment mid-point deviation with the different parameters a, g_0, p

$$\begin{cases} k_1 = L(t_k, u_k, v_k)\Delta t, \\ k_2 = L\left(t_k + \dfrac{\Delta t}{2}, u_k + v_k\dfrac{\Delta t}{2}, v_k + \dfrac{k_1}{2}\right)\Delta t, \\ k_3 = L\left(t_k + \dfrac{\Delta t}{2}, u_k + v_k\dfrac{\Delta t}{2} + \dfrac{k_1}{4}\Delta t, v_k + \dfrac{k_2}{2}\right)\Delta t, \\ k_4 = L\left(t_k + \Delta t, u_k + v_k\Delta t + \dfrac{k_2}{2}\Delta t, v_k + k_3\right)\Delta t. \end{cases}$$

Fig. 1a – 1d presents the graphic changes over time of deviation in time from the equilibrium position of the environment mid-point under different initial deviation from equilibrium (curve 1- $h = 0{,}1$; curve 2 - $h = 0{,}5$; curve 3- $h = 1$) taking into account the different models of the resistance force and for other values of dimensionless parameters of the system.

CONCLUSIONS

Obtained in proposed work qualitative results and graphical depending show:

1) the presence of the resistance force leads to damping of environment oscillations;

2) the rate of damping depends largely on the degree of nonlinearity of the resistance force;

3) by a considerable nonlinearity resistance force ($p = 3$) dynamic process is aperiodic;

4) the impact of the resistance force at the period of oscillation for small values of parameters g, p and h is insignificant. The latter also confirmed by asymptotic integration of specified differential equations.

REFERENCES

1. **Gajevski H, Greger K., Zakharias K. 1978.** Nichtlineare operator gleichungen und operator differen tialgleichungen. Moscow: Mir, 336. (in Russian).
2. **Lions J.L. 2002.** Some methods for solving nonlinear boundary value problems. Moscow: Editorial URSS, 587 (in Russian).
3. **Pukach P.Ya. 2014.** Qualitative methods of the investigation of nonlinear oscillation systems. Lviv: Publishing House of Lviv National Polytechnic University, 286. (in Ukrainian).
4. **Sleptsova I.P. 2005**. Fragmen – Lindelof principle for some quasi-linear evolution equations of second order. Ukrainian Mathematical Journal, Volume 57, Issue 2, 239–249. (in Ukrainian).
5. **Agre K. and Rammaha M. A. 2001**. Global solutions to boundary value problems for a nonlinear wave equation in high space dimensions. Differential And Integral Equations, Volume 14, 1315−1331.
6. **D'Ancona P. and Manfrin R. 1995**. A class of locally solvable semilinear equations of weakly hyperbolic type. Ann. Math. Pura Appl., Volume 168, 355–372.
7. **Demeio L. and Lenci S. 2007.** Forced nonlinear oscillations of semi-infinite cables and beams resting on a unilateral elastic substrate. Nonlinear Dynamics, Volume 49, 203–215.
8. **Demeio L. and Lenci S. 2008.** Second-order solutions for the dynamics of a semi-infinite cable on a unilateral substrate. Journal of Sound And Vibrations, Volume 315, 414–432.
9. **Ghayesh M.H. 2010.** Parametric vibrations and stability of an axially accelerating string guided by a non-linear elastic foundation. International Journal of Non-Linear Mechanics, Volume 45, 382–394.
10. **Pukach P., Kuzio I. and Sokil M. 2013**. Qualitative methods for research of transversal vibrations of semi-infinite cable under the action of nonlinear resistance forces. Econtechmod: an international quarterly journal on economics in technology, new technologies and modelling processes. – Lublin–Rzeszow, Vol. 2, No. 1., 43–48.
11. **Lavrenyuk S.P. and Pukach P.Ya**. **2007**. Mixed problem for a nonlinear hyperbolic equation in a domain unbounded with respect to space variables. Ukrainian Mathematical Journal, Volume 59, Issue 11, 1708–1718.
12. **Lavrenyuk S.P. and Pukach P.Ya**. **2007.** Variational hyperbolic inequality in the domains unbounded in spatial variables. International Journal of Evolution Equations, Volume 3, Issue 1, 103–122.
13. **Pukach P.Ya. 2014.** Qualitative research methods of mathematical model of nonlinear vibrations of conveyor belt. Journal of Mathematical Sciences, Volume 198, Issue 1, 31–38.
14. **Pukach P.Ya. and Kuzio I.V. 2013.** Nonlinear transverse vibrations of semiinfinite cable with consideration paid to resistance. Scientific Bulletin of National Mining University, Issue 3, 82 – 86. (in Ukrainian).
15. **Chen L.Q. 2005**. Analysis and control of transverse vibrations of axially moving strings. Appl. Mech. Rev, Volume 58, 91–116.
16. **Pukach P., Kuzio I. and Nytrebych Z.M. 2013.** Influence of some speed parameters on the dynamics of nonlinear flexural vibrations of a drill column ECONTECHMOD, Volume 2, Issue 4, 61–66.
17. **Pukach P.Ya. 2012**. On the unboundedness of a solution of the mixed problem for a nonlinear evolution equation at a finite time. Nonlinear Oscillations, Volume 14, Issue 3, 369–378.
18. 18. **Pukach P.Ya. 2007**. Mixed problem for nonlinear equation of beam vibrations type in unbounded domain. Matematychni Studii, Volume 27, no. 2, 139–148. (in Ukrainian).
19. **Pukach P. Ya. 2004**. Mixed problem in unbounded domain for weakly nonlinear hyperbolic equation with growing coefficients. Matematychni metody i fizyko-mekhanichni polya, Vol. 47, no. 4, 149–154. (in Ukrainian).
20. **Pukach P.Ya.**, **Kuzio I.V., Nytrebych Z.M. and Sokhan P.L. 2013.** Nonlinear oscillations of elastic beam including dissipation and the Galerkin method in their investigation. Scientific Bulletin of Lviv Polytechnic National University, Series of Dynamics, Strength and Design of Machines and Devices, Volume 759, 106–111. (in Ukrainian).
21. **Salenger G. and Vakakis A.F. 1998.** Discretness effects in the forced dynamics of a string on a periodic array of non-linear supports. International Journal of Non-Linear Mechanics, Volume 33, 659–673.
22. **Santee D.M. and Goncalves P.B. 2006.** Oscillations of a beam on a non-linear elastic foundation under periodic loads. Shock and Vibrations, Volume 13, 273–284.