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NONLINEAR ANALYSIS IN SYNTHESIS OF THE RESONANT VIBRO-IMPACT SYSTEMS

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Abstract. The paper considers differential equation of the vibro-impact resonance system with an asymmetric piecewise linear elastic characteristic. The time-instant of switching of elastic characteristic is determined on the basis of equality of oscillation period to average value of the corresponding eigenfrequencies. Then, expansion of the asymmetric piecewise linear elastic characteristic into Fourier series was made. The initial differential equation was reduced to a kind of parametric equations of Hill's and Mathieu's type with taking into account the time of elastic characteristic change. Stability analysis of parametric Mathieu equation is shown for the analysis of natural oscillations. For stability analysis of the synthesized by various stiffness coefficients of vibro-impact system, dependencies of Mathieu equations coefficients on the parameter of synthesis are used. The solution of the initial equation with forced oscillations in the form of asymmetric two-frequency vibrations has been obtained by means of Bubnov-Galerkin and Levenberg-Marquardt methods for nonlinear algebraic systems of equations, also amplitude and phase frequency dependence was graphically drawn. The basic equation with an asymmetric elastic response characteristic feature is determined by the fixed natural frequency of oscillations independently of amplitude. Numerical solution of differential equations by means of Runge-Kutta method are presented for comparison. Comparison of the vibro-impact resonance system kinematics characteristics, synthesized by the elastic parameters and solved by the listed methods, is conducted. The feasibility of using nonlinear analysis presented in two harmonics in Fourier series asymmetric elastic characteristic is justified in the article. The suggested approach with Bubnov-Galerkin method for general Hill's equation and correlation analysis of time kinematic characteristics was used. Acceleration frequency spectrum and harmonics are obtained on the basis of Runge-Kutta numerical method simulation of the initial nonlinear differential equation.

Introduction

The subject of nonlinear and vibro-impact systems currently is sufficiently popularized. Vibro-impact systems are belong to the class of strongly nonlinear systems, subject to complex analysis and actually do not has an analytical solutions [1, 2]. Their particularity is determined that the practical realization of systems is performed by various, especially piecewise-linear and piecewise-nonlinear elastic characteristics. That gives them special parametric properties and problems of dynamics stability.

Analysis of modern information sources on the subject of the article

The basic share research of the vibro-impact systems is aimed at improving for the analytical solution methods of basic and subharmonic oscillations of parametrical Mathieu's and Hill's equation, stability research and analysis of bifurcations and chaos phenomena [3–7]. The practical value of the vibro-impact is justified by low technological efficiency in various complex physical materials and mechanical processes [8–11].

Problem statement

Submitted article provides a review of the vibro-impact system's equations based on asymmetric piecewise linear elastic characteristics with following transfer of the equation for the estimation of parametrical oscillation, stability analysis for free vibration. The possibility of solution forced oscillations of reduced

parametric equations as asymmetric two-frequency oscillations is regarded with an followed by a comparative evaluation the solutions of the initial equation of the vibro-impact system with its parametric representation.

Statement of the main material

The known differential equation of the vibratory system with an linear viscous friction is consider [8, 11]:

$$M \cdot \ddot{x}(t) + b \cdot \dot{x}(t) + c(t) \cdot x(t) = F(t), \quad (1)$$

where an elastic asymmetric dependence has the form:

$$c(t) = \begin{cases} c_1, & x(t) \geq 0, \\ c_2, & x(t) < 0. \end{cases} \quad (2)$$

Condition (2) can be written as follows [8]:

$$c(t) = \begin{cases} c_1, & 0 \leq t \leq t_1, \\ c_2, & t_1 < t \leq 2p/w, \end{cases} \quad (3)$$

where t_1 – time when switch of elastic parameter is made (Fig. 1); w – the oscillation frequency of the system; $c_2 > c_1$. Contact time during which the system works with an elastic parameter c_2 defined as $\frac{2p}{w} - t_1$.

It is assumed the coefficient c_1 is defined traditional way for the one-frequency resonant systems as $c_1 = M(w/z)^2$. The coefficient c_2 can take values according to the setting Λ of synthesis $c_2 = M(\Lambda \cdot w/z)^2$, where $z = w/\Omega_0$ is the proportion of resonance adjustment relative eigenfrequency Ω_0 . We will have, in this case a ratio of elasticity coefficients as $c_2/c_1 = \Lambda^2$. So, will evaluated the solutions of the vibro-impact system equation (1) through the synthesis by entering the Λ parameter.

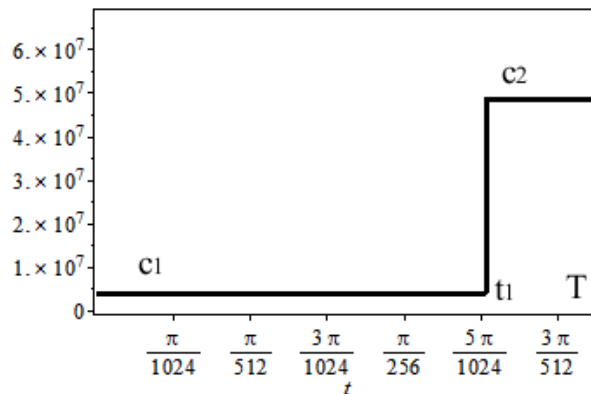


Fig. 1. Stiffness coefficient time change at the period of oscillation

The moment t_1 will be determine on the basis of equality oscillation period average values of the corresponding eigenfrequencies $\omega_{01} = \sqrt{c_1/M}$ and $\omega_{02} = \sqrt{c_2/M}$ with fixed oscillation eigenfrequency $\Omega_0 = 2\omega_{01}\omega_{02}/(\omega_{01} + \omega_{02})$ [6, 7], which is typical for systems with an asymmetric elastic characteristic of type (2):

$$\frac{1}{T} \left(\int_0^{t_1} \omega_{01} dt + \int_{t_1}^{\frac{2\pi}{\omega}} \omega_{02} dt \right) \equiv \Omega_0, \quad (4)$$

is from where, $t_1 = \frac{2\pi}{\omega} \frac{\omega_{02}}{\omega_{01} + \omega_{02}}$.

Expanding dependence (2) by Fourier series is received in a form:

$$c(t) = \frac{a_0}{2} + \sum_{k=1}^m c_k \sin(k\omega t + \gamma_k),$$

and the coefficients of the series are: $a_0 = 2c_2 + \frac{c_1 - c_2}{\pi} \omega t_1$, $c_k = \sqrt{a_k^2 + b_k^2}$,
 $a_k = (c_1 - c_2) \sin(k\omega t_1) / \pi k$, $b_k = 2(c_1 - c_2) \sin(k\omega t_1 / 2)^2 / \pi k$, $\gamma_k = \pi + a \tan(a_k / b_k)$,
 $c_k = -2(c_1 - c_2) \sqrt{\sin(k\omega t_1 / 2)^2} / \pi k$.

Stiffness coefficient change in expanded form:

$$c(t) = c_2 + \frac{c_1 - c_2}{T} t_1 + \frac{2(c_2 - c_1)}{\pi} \sum_{k=1}^m \frac{\sqrt{\sin\left(\frac{k\omega t_1}{2}\right)^2}}{k} \sin(k\omega t + \gamma_k).$$

Equation (1) can be written as:

$$\ddot{x}(t) + 2n \cdot \dot{x}(t) + \left[\begin{array}{l} \omega_{02}^2 + \frac{t_1}{T} (\omega_{01}^2 - \omega_{02}^2) + \\ + 2 \frac{(\omega_{02}^2 - \omega_{01}^2)}{\pi} \sum_{k=1}^m \frac{\sqrt{\sin\left(\frac{k\omega t_1}{2}\right)^2}}{k} \sin(k\omega t + \gamma_k) \end{array} \right] \cdot x(t) = f(t), \quad (5)$$

where appropriate notation adopted $2n = b/M$, $T = 2\pi/\omega$, $f(t) = F(t)/M$.

Equation (5) is the Hill's equation [2, 5, 12, 13] in the next form:

$$\ddot{x}(t) + 2n \cdot \dot{x}(t) + [\delta + \varepsilon\psi(t)] \cdot x(t) = f(t), \quad (6)$$

where δ , ε are constant system parameters; $\psi(t)$ is disturbance function: $\delta = \omega_{02}^2 + \frac{t_1}{T} (\omega_{01}^2 - \omega_{02}^2) = \omega_{01}\omega_{02}$,

$$e = 2(\omega_{02}^2 - \omega_{01}^2) / \pi, \quad y(t) = \sum_{k=1}^m \Psi_k \sin(k\omega t + g_k), \quad \Psi_k = \sqrt{\sin(k\omega t_1 / 2)^2} / k = \sin(k\omega t_1 / 2) \cdot c \operatorname{sgn}(\sin(k\omega t_1 / 2)) / k,$$

$$\psi(t) = \psi(t + T), \quad \int_0^T \psi(t) dt = 0, \quad c \operatorname{sgn}(z) = \begin{cases} 0, & \text{if } z = 0, \\ 1, & \text{if } \operatorname{Re}(z) > 0 \text{ or } (\operatorname{Re}(z) = 0 \text{ and } \operatorname{Im}(z) > 0), \\ -1 & \text{otherwise.} \end{cases}$$

Then will coming to the equation in dimensionless form, taking $\tau = \omega t$ and $f(t) = f_0 \cdot \sin(\omega t)$:

$$\frac{d^2}{d\tau^2} \tilde{x}(\tau) + 2h \cdot \frac{d}{d\tau} \tilde{x}(\tau) + [\tilde{\delta} + \tilde{\varepsilon}\psi(\tau)] \cdot \tilde{x}(\tau) = \tilde{f} \sin \tau, \quad (7)$$

where $h = n/\omega$, $\tilde{\delta} = \omega_{01}\omega_{02} / \omega^2$, $\tilde{\varepsilon} = 2(\omega_{02}^2 - \omega_{01}^2) / \pi\omega^2$, $\tilde{f} = f_0 / \omega^2$.

After replacing $\tilde{x}(\tau) = e^{-h\tau} \cdot \tilde{z}(\tau)$ [3] the equation (7) is reduced to dimensionless form with the right side:

$$\frac{d^2}{d\tau^2} \tilde{z}(\tau) + [\delta_1 + \tilde{\varepsilon}\psi(\tau)] \cdot \tilde{z}(\tau) = \tilde{f} e^{n\tau} \sin \tau, \quad (8)$$

where $\delta_1 = \tilde{\delta} - h^2$.

Let's consider the possibility of bringing equation of the form (6) to the Mathieu equation [2, 3, 11, 14]. Number of harmonics for this is $m = 1$. Now, the disturbance function will have the form:

$$\psi(t) = \Psi_1 \sin(\omega t + \gamma_1) = \sin(\omega t_1 / 2) \sin(\omega t + \gamma_1),$$

where $\gamma_1 = \pi + a \tan\left[\sin(\omega t_1) / 2 \sin(\omega t_1 / 2)^2\right]$, and equation (6) will be:

$$\ddot{x}(t) + 2n \cdot \dot{x}(t) + [\delta + v \cdot \sin(\omega t + \gamma_1)] \cdot x(t) = f(t), \quad (9)$$

where $v = \varepsilon \cdot \sin(\omega t_1 / 2) = \frac{2(\omega_{02}^2 - \omega_{01}^2)}{\pi} \cdot \sin(\omega t_1 / 2)$.

Natural oscillations and stability analysis. Equation (9) is written without regard of the right side:

$$\ddot{x}(t) + 2n \cdot \dot{x}(t) + \left[\delta + v \cdot \cos\left(\frac{\pi}{2} - \omega t - \gamma_1\right) \right] \cdot x(t) = 0. \quad (10)$$

Denoting for $\frac{\pi}{2} - \omega t - \gamma = 2\tau$, we obtain $t = \frac{\frac{\pi}{2} - 2\tau - \gamma}{\omega}$, $dt = -\frac{2}{\omega} d\tau$ and $dt^2 = \frac{4}{\omega^2} d\tau^2$, then

equation (10) takes the form:

$$\frac{d^2}{d\tau^2} \tilde{x}(\tau) + \left(-\frac{4n}{\omega}\right) \cdot \frac{d}{d\tau} \tilde{x}(\tau) + \left[\frac{4\delta}{\omega^2} + \frac{4\varepsilon}{\omega^2} \sin\left(\frac{\omega t_1}{2}\right) \cos(2\tau)\right] \cdot \tilde{x}(\tau) = 0. \quad (11)$$

Denoting for $2h = -4n / \omega$ and completing a procedure similar to (8) will have the Mathieu equation in dimensionless variables:

$$\frac{d^2}{d\tau^2} z(\tau) + \left[\left(\frac{4\delta}{\omega^2} - h^2\right) - 2\left(-\frac{2\varepsilon}{\omega^2} \sin\left(\frac{\omega t_1}{2}\right)\right) \cos(2\tau)\right] \cdot z(\tau) = 0,$$

also the classical form:

$$\frac{d^2}{d\tau^2} z(\tau) + [a - 2q \cos(2\tau)] \cdot z(\tau) = 0, \quad (12)$$

where $a = \frac{4\delta}{\omega^2} - h^2 = \frac{4\delta}{\omega^2} - 4\left(\frac{n^2}{\omega^2}\right) = \frac{4}{\omega^2}(\omega_{01}\omega_{02} - n^2)$,

$q = -\frac{2\varepsilon}{\omega^2} \sin(\omega t_1 / 2) = \frac{4(\omega_{01}^2 - \omega_{02}^2)}{\pi\omega^2} \sin(\omega t_1 / 2)$.

Equation (12) is analyzed for parametric stability according to the diagram (Fig. 2), built by Mathieu functions and Maple software. The diagram was built we counted the symmetry a -axis. For the vibration system analysis was selected following parameters: $M = 41,435[kg]$; $\omega = 314,15[rad / s]$; $z = 0,94$; $b = 2M\omega\zeta$; $\zeta = 0,2$; $f_0 = 43,442[N / kg]$. For stability analysis of the synthesized by Λ vibro-impact system used dependency of Mathieu equations coefficients a and q with Λ . Parametric graph $a = f(q)$ that has the form approximate to line is constructed. The point $A(q; a)$ can be located on the line, with the appropriate values Λ . The stability of the solution is determined by the point provision in relevant chart area. When the value of the synthesis parameter is $\Lambda = 2$ the point coordinates are $A(3,74; 8,89)$ and is located in a stable zone of stability diagrams. Stability and instability of the system proves a view at the Fig. 3 of special functions MathieuC, MathieuS – even and odd Mathieu functions. The graphics are periodic in stable zones (Fig. 3, a for $\Lambda = 2$) or those that grow to infinity for unstable (Fig. 3, b for $\Lambda = 3$).

Note, that the use of the Mathieu equation may be restricted due to precision of the result. Otherwise, to apply over number of harmonics and directly stability analysis spending by the Hill's equation (7) [2, 5, 12, 13].

Forced oscillations. Solution of equation (9) is looking in the form:

$$x(t) = X_0 + X_1 \sin(\omega t + \varphi_1) + X_2 \sin(2\omega t + \varphi_2), \quad (13)$$

here is accounted dissymmetry X_0 , amplitude values X_1 and X_2 , harmonics ω and 2ω , the initial phases φ_1 and φ_2 .

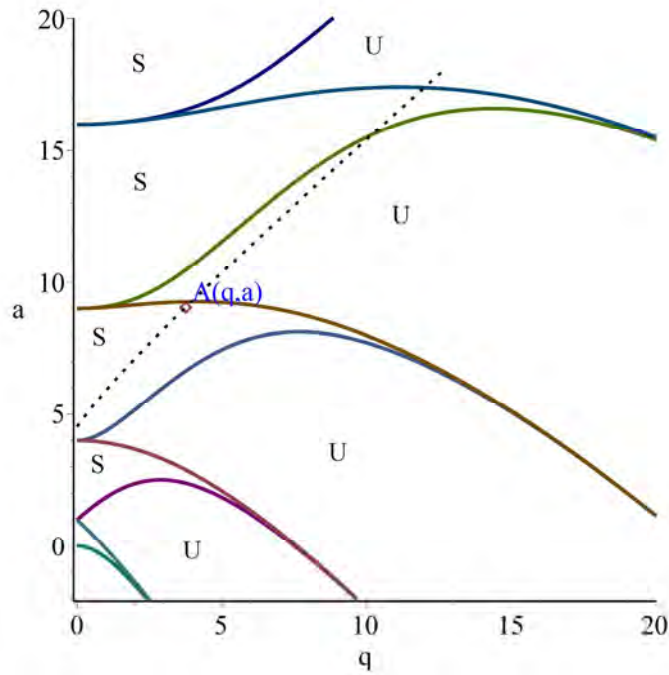


Fig. 2. Stability chart for the linear Mathieu equation: S – stable zone, U – unstable zone

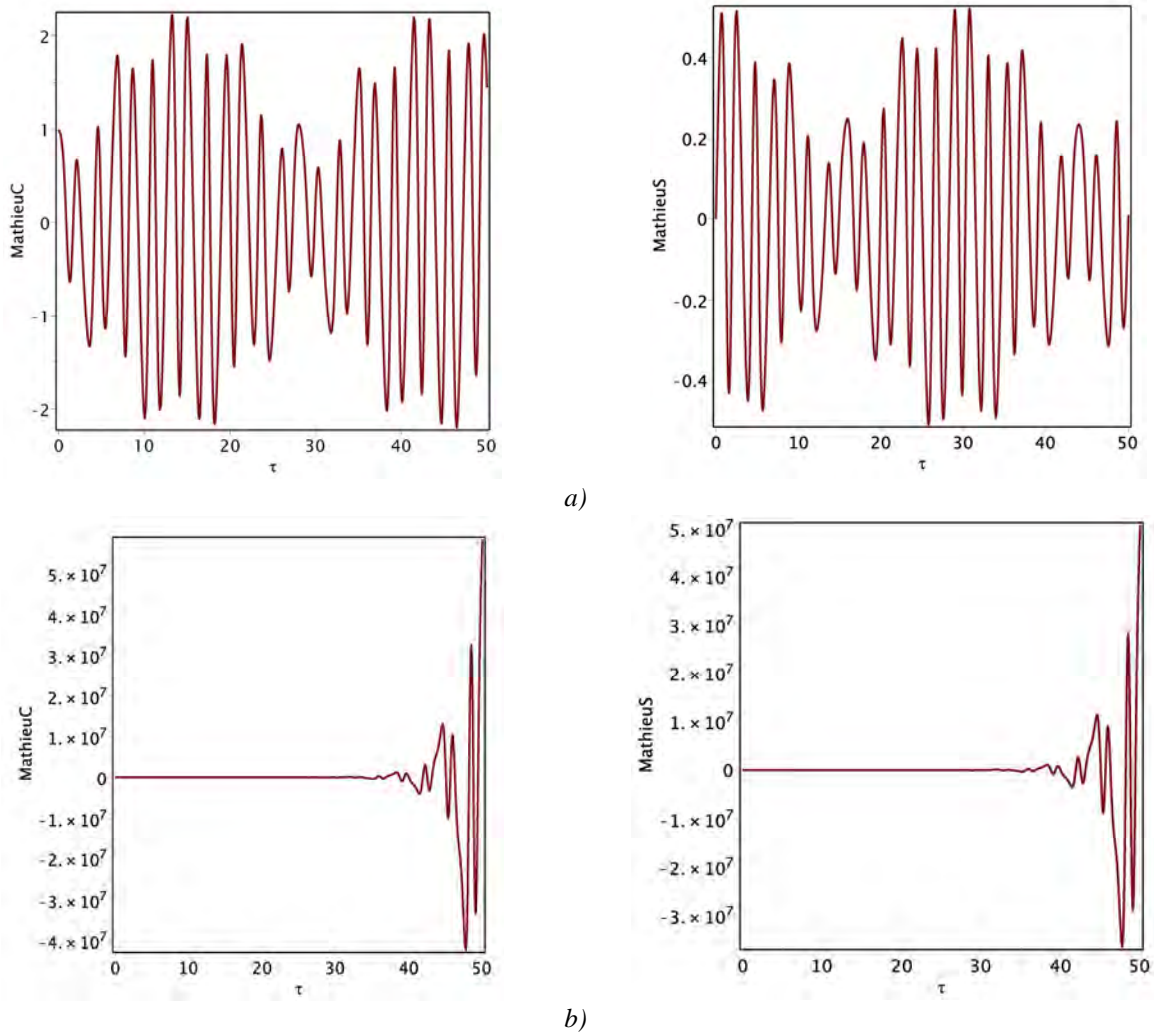


Fig. 3. Stable (a) and unstable (b) solutions of Mathieu functions with the different settings Λ : $a - \Lambda = 2$, $b - \Lambda = 3$

Substituting (13) in to (9), we get the following result with an parameters of the solution and oscillatory system as a function:

$$L = \left[\begin{array}{l} -f_0 \sin(\omega t) + X_0 \cdot (\delta + v \cdot \sin(\omega t + \gamma_1)) + \\ + X_1 \cdot \left[(\delta - \omega^2) \sin(\omega t + \varphi_1) + 2\omega n \cdot \cos(\omega t + \varphi_1) + \frac{v \cdot \cos(\varphi_1 - \gamma_1)}{2} - \frac{v \cdot \cos(2\omega t + \varphi_1 + \gamma_1)}{2} \right] + \\ + X_2 \cdot \left[(\delta - 4\omega^2) \sin(2\omega t + \varphi_2) + 4\omega n \cdot \cos(2\omega t + \varphi_2) + \frac{v \cdot \cos(\varphi_1 - \gamma_1)}{2} - \frac{v \cdot \cos(2\omega t + \varphi_1 + \gamma_1)}{2} \right] \end{array} \right].$$

The Bubnov-Galerkin method is suited to the analysis of nonlinear systems with soft elastic characteristic and analysis of subharmonic oscillations [16, 17]. Used the following procedure and orthogonalization of the result of substitutions L relative to the desired solution (13):

$$\int_0^{2\pi/\omega} L dt = 0, \quad \int_0^{2\pi/\omega} L \sin(\omega t + \varphi_1) dt = 0, \quad \int_0^{2\pi/\omega} L \cos(\omega t + \varphi_1) dt = 0, \\ \int_0^{2\pi/\omega} L \sin(2\omega t + \varphi_2) dt = 0, \quad \int_0^{2\pi/\omega} L \cos(2\omega t + \varphi_2) dt = 0,$$

that allowed to receive the following system of nonlinear equations:

$$\left. \begin{array}{l} X_0 \delta + \frac{X_1 v \cdot \cos(\varphi_1 - \gamma_1)}{2} = 0, \\ X_1 (\delta - \omega^2) - f_0 \cos(\varphi_1) + \frac{X_2 v \cdot \sin(\varphi_1 - \varphi_2 + \gamma_1)}{2} + X_0 \cdot v \cdot \cos(\varphi_1 - \gamma_1) = 0, \\ f_0 \sin(\varphi_1) + \frac{X_2 v \cdot \cos(\varphi_1 - \varphi_2 + \gamma_1)}{2} + X_0 \cdot v \cdot \sin(\varphi_1 - \gamma_1) = 0, \\ X_2 (\delta - 4\omega^2) + \frac{X_1 v \cdot \sin(\varphi_1 - \varphi_2 + \gamma_1)}{2} = 0, \\ -4X_2 \omega n + \frac{X_1 v \cdot \cos(\varphi_1 - \varphi_2 + \gamma_1)}{2} = 0. \end{array} \right\} \quad (14)$$

The system's result is solved numerically by known methods for nonlinear equations (Levenberg-Marquardt Method). The equations (14) solutions are included in (13) and allow to construct of the kinematics parameters time dependence. Vector of the calculated parameters at the fixed disturbance frequency $\omega = 314,15 [rad / s]$ synthesized by the value parameter $\Lambda = 2$ is next:

$$\begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} 1,703 \cdot 10^{-4} [m] \\ 4,12 \cdot 10^{-4} [m] \\ 2,02 \cdot 10^{-4} [m] \\ -0,558 [rad] \\ 0,921 [rad] \end{bmatrix}.$$

The proposed approach is somewhat simplified, as seen equation (9), which was received by consideration one harmonic in disorder of elastic characteristics in Fourier series, and imposes find the result as a two-frequency function (13). Accordingly the tolerance of the solutions can be limited.

Another way to obtain the basic time characteristics is possible by numerical solution of the initial equation (1) by known methods for differential equations (such as Rkadapt).

Pearson correlation coefficient was used for the convergence comparison of the received dependences by methods Levenberg-Marquardt and Rkadapt. The table lists the value of the Pearson's criterion for synthesized vibro-impact systems.

The proposed approach with Bubnov-Galerkin method for general Hill's equation (6) and correlation analysis of time characteristics for more harmonics ($m = 2, m = 3$) was used. This greatly improves the convergence of kinematic characteristics for the harmonic's number of $m = 2$.

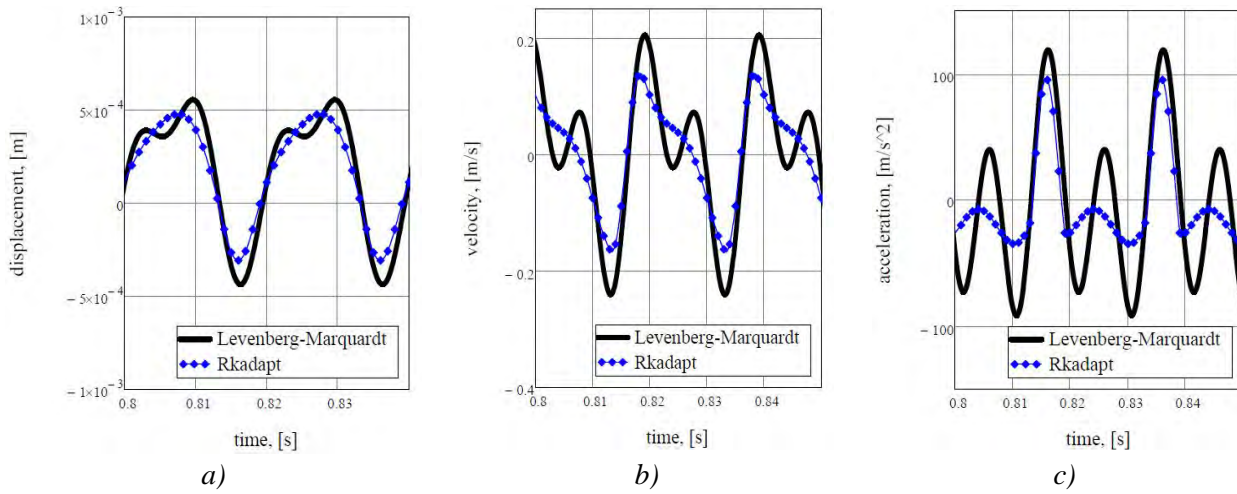


Fig. 4. Time dependencies of kinematics parameters of the vibro-impact system with ratio $c_2 / c_1 = 4$

The proposed approach with Bubnov-Galerkin method for general Hill's equation (6) and correlation analysis of time characteristics for more harmonics ($m = 2, m = 3$) was used. This greatly improves the convergence of kinematic characteristics for the harmonic's number of $m = 2$.

Note, that such procedures can be conducted to analyze the subharmonic $\omega/2$ oscillations and increasing the number of harmonics in the solution (13).

Table

Pearson correlation coefficients basic kinematic parameters of the vibro-impact systems

The parameter of synthesis Λ	The m number of harmonics in the equation (5)	Kinematics characteristics		
		$x(t)$	$v(t)$	$a(t)$
$\Lambda = 2$	$m = 1$	0,964	0,921	0,856
	$m = 2$	0,994	0,981	0,916
$\Lambda = 3$	$m = 1$	0,66	0,615	0,531
	$m = 2$	0,964	0,898	0,728
$\Lambda = 4$	$m = 1$	0,436	0,469	0,381
	$m = 2$	0,953	0,855	0,618
	$m = 3$	0,785	0,752	0,572

The system (14) was used to construct the amplitude-frequency and phase-frequency characteristics of the solution (13). Fig. 5 shows the dependence of these parameters for the solution of two-frequency vibrations (13). The initial phases φ_1 and φ_2 leap is observed at the resonant frequencies. The equation (1) with an elastic response (2) characteristic feature is fixed natural frequency of oscillations with independent of amplitude. Linear frequency response for the amplitudes (Fig. 5, a) this shows.

Vibro-impact system, although linear frequency response is endowed with a lot of frequency spectrum, which is particularly manifests itself on the acceleration characteristic. Acceleration frequency spectrum and harmonics (Fig. 6) obtained by Runge–Kutta numerical method simulation of the differential equation (1).

Equation (13) considers only the two multiples harmonics $\omega_1 = 314,15 [rad / s]$ and $2\omega_1 = 628,3 [rad / s]$ unlike to the direct solution of equation (1) by integrated numerical method.

Therefore, for a complete range of acceleration's amplitude values and for more adequate analysis of the vibro-impact processes worth in solution (13) to increase the number of unknown parameters with an $3\omega_1$ and $4\omega_1$ harmonics.

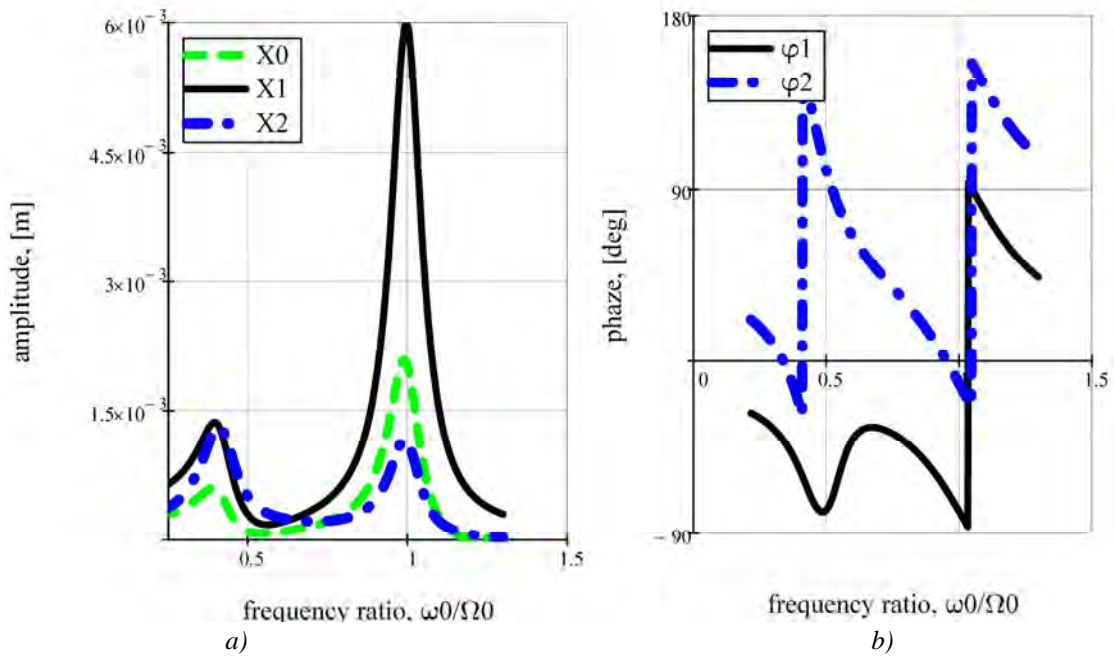


Fig. 5. Amplitude-frequency (a) and phase-frequency (b) descriptions of the vibro-impact system with stiffness coefficients ratio $c_2 / c_1 = 4$

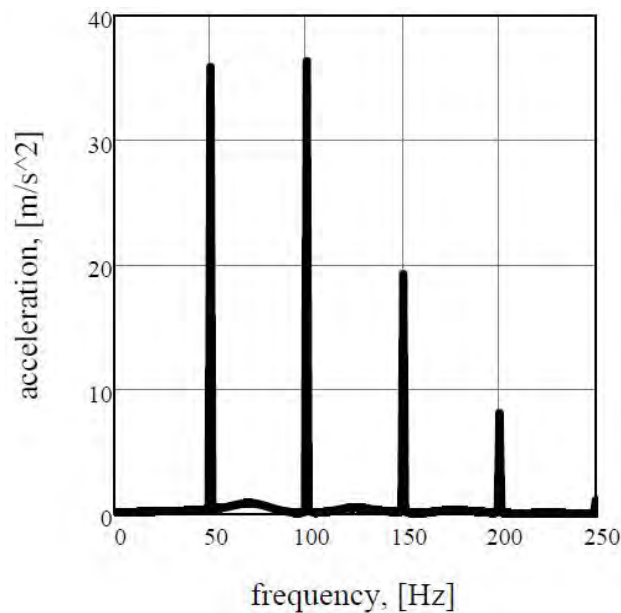


Fig. 6. Spectral analysis of the vibro-impact system's acceleration

Conclusions

We can affirm the following as a result of nonlinear analysis:

- the time-instant of contact in vibro-impact system with an asymmetric elastic characteristic is defined, and then the initial piecewise linear equation is presented as a general parametric Mathieu-Hill equations;

– the Bubnov-Galerkin's method was used for the solving parametric equations as a system of nonlinear algebraic equations, asymmetric two-frequency oscillation, amplitude and phase-frequency characteristics of synthesized presented vibro-impact system were obtained by means of numerical solution;

– the correlation convergence is evaluate for the solutions of initial piecewise linear differential equations and reduced Hill-Mathieu equations conducted by the Pearson correlation coefficient.

Established, that the incorporation of two multiple harmonic in the disturbance function of the Hill's equation provides a highly accurate results.

Nonlinear analysis was conducted during the synthesis of vibro-impact systems creates conditions for its use for other asymmetric piecewise linear and nonlinear resonant vibratory systems.

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