Engineering Simulation, 1999, Vol.16, pp. 311-318.

Synthesis of DeterminedSystems ReconstractingtheGiven Random Sequence

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(Received February 6, 1997)

A synthesis method for the ordinary differential equations is proposed.Their solutions reconstructthe statistical properties of the given pseudo-random sequence. The method is illustrated by an example of simulating singular Lorenz attractor and the sequence of pseudo-random numbers whish are uniformly distributed within [0,1] interval.

Key words: mathematical simulation, identification, strange attractor, random processes.

Lorentz inhis published article [1]showedthe possibilityof obtaining one-dimensional discrete sequences corresponding to the chaotic motion of the certain continuous system of the third order. The comparison of continuous and discrete systemsis evidently useful for many applications. Her the solution of inverse problem is proposed: the synthesisof a continuous system which is in some sense equivalent to one-dimentional random sequence.

Statement of the problem. There is an object – black box – at whish output the sequence of random numbers appear. An autonomous continuous dissipate system with lumped parameters should be found which can reconstruct the statistical properties of the given random sequence. This system is known to have a strange attractor and its order chouldnot be less than three [2].

The work consist of three sections: in the first, the structure of the mathematical model and a general method of its identification is justified; in the second section the model synthesis is made for Lorenz attractor in a special and general cases; the sources of possible incorrectnessesand methods of their elimination are shown; in the third section the model of pseudo-random process is constructed.

Structure of mathematical model and its identification.Consider a nonlinear system with a rather general description

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u},t), \qquad \boldsymbol{y} = \boldsymbol{\varphi}(\boldsymbol{x},\boldsymbol{u},t), \tag{1}$$

where $\boldsymbol{u}$, $\boldsymbol{y}$ are the vectors of input and output sygnals;$\boldsymbol{x}$is the vector of the state variables of the k dimensionality;$\boldsymbol{f}$(.)is the differentiable vector-function;$\boldsymbol{\varphi}$(.)isthe vector-function differentiatedwhich respect to t.

In [3] the theorem is proved by which the system involving linear stationary dynamic and nonlinear inertialess subsystems, is equivalent to (1) as to its input-output:

$$a(\lambda)\boldsymbol{y} = \boldsymbol{D}(\lambda)\boldsymbol{\psi}(\boldsymbol{y}, \boldsymbol{y}^{(1)}, ..., \boldsymbol{y}^{(\boldsymbol{k}-1)}, \boldsymbol{v}, \boldsymbol{v}^{(1)}, ..., \boldsymbol{v}^{(\boldsymbol{k}-2)}, \boldsymbol{u}, \dot{\boldsymbol{u}}, t). \qquad (2)$$

Here $\boldsymbol{u}, \boldsymbol{y}$ correspond to (1) description; $\boldsymbol{v}$ is its inner vector (but not the state vector $\boldsymbol{x}$); $\boldsymbol{\psi}(.)$ is the nonlinear vector-function.

The matrix of transfer function $\boldsymbol{W}(\lambda)=\boldsymbol{D}(\lambda)/a(\lambda)$ describes the linear subsystem with the output vector $\boldsymbol{y},\ldots, \boldsymbol{y}^{(k-1)}, \boldsymbol{v}^{(1)},\ldots, \boldsymbol{v}^{(k-2)}$ and input vector $\boldsymbol{v}$. The nonlinear vector-function $\boldsymbol{\psi}(.)$ corresponds to the nonlinear subsystem with vector inputs $\boldsymbol{y},\ldots, \boldsymbol{y}^{(k-1)}, \boldsymbol{v},\ldots, \boldsymbol{v}^{(k-2)}, \boldsymbol{u}, \dot{\boldsymbol{u}}$ and output vector $\boldsymbol{v}$.

The modelling structure (2)may be used for mathematical model synthesis of the systems which assumethe description(1). If there isan identification algorithm for definition of model parameters on the basis of information about the input vector $\boldsymbol{u}$ and object reactions $\boldsymbol{y}$, a macromodel can be built which recon-structs which a certain accuracy the reaction $\boldsymbol{y}$ to perturbations $\boldsymbol{u}$.

Sometimes the equations (1) may be transformed into (2) in an analytical way. The Lorentz equations

$$\begin{cases} \dot{x}_1 = 10(x_2 - x_1)\ ; \\ \dot{x}_2 = 40x_1 - x_2 - x_1x_3\ ; \\ \dot{x}_3 = -\dfrac{8}{3}x_3 + x_1x_2\ ; \\ x_1(0) = 1;\ x_2(0) = 0;\ x_3(0) = 0; \end{cases} \qquad (3)$$

is written in the form corresponding to (2) with $\boldsymbol{y}=x_1=y_1$, $\boldsymbol{v}=v$, $\boldsymbol{u}=0$, k=3, $\boldsymbol{W}(\lambda)= -1/(1040\text{-}88/3*\lambda-41/3*\lambda^2-\lambda^3)$:

$$\begin{cases} \dot{y}_1 = y_2\ ; \\ \dot{y}_2 = y_3\ ; \\ \dot{y}_3 = 1040y_1 - \dfrac{88}{3}y_2 - \dfrac{41}{3}y_3 + v; \\ v = 11\dfrac{y_2^2}{y_1} + \dfrac{y_2y_3}{y_1} - y_1^2y_2 - 10y_1^3\ ; \\ y_1(0) = 1;\ y_2(0) = -10;\ y_3(0) = 500. \end{cases} \qquad (4)$$

The structure (4) makes it possible to built a simple reconstruction algorithm for the equation set coefficients only by the output signal y_1. To make it, the signal variable $y_1$shuld be calculated and a metric shuld be selected where the distance $\left\| \ddot{y}_1 - (a_1 y_1 + a_2 \dot{y}_1 + a_3 \ddot{y}_1 + c_1 \dot{y}_1^2 / y_1 + c_2 \dot{y}_1 \ddot{y}_1 / y_1 + c_3 y_1^2 \dot{y}_1 + c_4 y_1^3) \right\|$ is to be minimized in a certain given domain. If the metric is quadratic and the signal $y_1(t_m) \equiv y_m$ is discrete at M points, the problem becomes

$$\min_{\bar{a},\bar{c}} \sum_{m=1}^{M} (\dddot{y}_m - (a_1 y_m + a_2 \dot{y}_m + a_3 \ddot{y}_m + c_1 \dot{y}_m^2 / y_m + c_2 \dot{y}_m \ddot{y}_m / y_m + c_3 y_m^2 \dot{y}_m + c_4 y_m^3))^2 \qquad (5)$$

and is reduced to solving the system of the Mlinear algebraic equation with the seven unknown components of the vectors $\boldsymbol{a}$ and $\boldsymbol{c}$.

Model identification for Lorentz system. The problem (5) has been solved in [4], but it is not explained why such basis of approximation. The variables were calculated by the method [5] in terms of the eigen vectors of covariance matrix built with discrete values of the signal y_1. The coefficient values were reconstructed with rather small errors of initial data. Some other problems of inverse nonlinear dynamics were solved in [5], including those with additive noises available in initial data.

The numerical differentiation of the digital function is known to be an incorrect pocedure, in particular, unstable with respect to perturbations of initial data. This fact necessitates the use of regularizing algoritms for achieving steady results [6]. The numerical procedures of smoothing initial sygnal were used in [4] which made it possible to eliminate incorrectnesses associated with noises of initial signal. But incorrectness becomes apparent by solving the problems of the type (5) even if initial data are “noiseless”. Therefore regularization should involve all stage of the above problem statement and solution. In [5] ther are no indications as to regularizing procedures and in [4] the regularization was not used for solving (5) and similar problems. The correctness of the obtained results is explaned by the small dimension of the problems to be solved.

Numerical differentiation is performad by varios regularized methods [7]. We show here the use of splines. The interpolating spline can be always built with the set of y_m, m=1,...,Mvalues. This spline is of the $n<M$ power with the coefficients number $p=M$ and variable continuity of the $(n–1)$ order inclusive [8]. For the problem (5) the derivatives up to the third order should be calculated. If the derivative continuity at least up to the fourth order is required, the interpolating spline should be of the fifth power. The regularization of the spline construction procedure and consequently, of the analytical spline derivative calculation is achieved by means of approximating spline [8] with $p<M$coefficient number

which can filter noises of initial signal. The texts of FORTRAN codes are also given in the approximating spline which realise all necessary algorithms.

When the derivatives in (5) are calculated by spline-interpolation of the fifth power the reconstruction accuracy of the coefficients in (4) is not lower than in [4].

The problem (5) is simple because the exact approximate basis of nonlinear function is known. But if the basis is not known, the approximation is performed as a rule by many-dimentional power polynomial [9]. In our case it is the polynomial of the third order:

$$\dot{y}_1 = y_2; \quad \dot{y}_2 = y_3; \quad \dot{y}_3 = \sum_{i,j,k=0}^{r} a_{ijk} y_1^i y_2^j y_3^k; \tag{6}$$

We restrict ourself by conditions r=3, i+j+k<=3. Then the number of unknown coefficients is 20 and the problem similar to (5) becomes

$$\min_{\boldsymbol{a}} \sum_{m=1}^{M} \left(\dddot{y}_m - \sum_{i,j,k=0}^{3} a_{ijk} y_m^i \dot{y}_m^j \ddot{y}_m^k \right)^2, \quad i+j+k \le 3. \tag{7}$$

But it is impossible to obtain the stable results by direct solution (7) because of its incorrectness. By the initial regularization idea [6], instead of the problem (7), the minimum is sought for:

$$\min_{\boldsymbol{a}} \left(\sum_{m=1}^{M} \left(\dddot{y}_m - \sum_{i,j,k=0}^{3} a_{ijk} y_m^i \dot{y}_m^j \ddot{y}_m^k \right)^2 + \alpha \sum_{i,j,k=0}^{3} a_{ijk}^2 \right), \quad i+j+k \le 3. \tag{8}$$

The regularization parameter α is selected by an empirical way based on the incorrectness of the problem (8).

The approximation coefficients are given below which are the solution of (8) with $M = 500$ and $\alpha = 1.0$:

$$\begin{array}{lll}
a_{000} = \ 3.4120_{10}1; & a_{100} = \ 5.8604_{10}2; & a_{010} = \ 1.0572_{10}2; \\
a_{001} = -5.6501_{10}0; & a_{200} = -1.7036_{10}-1; & a_{020} = -5.8876_{10}-3; \\
a_{002} = \ 1.4002_{10}-4; & a_{110} = \ 2.1736_{10}-2; & a_{101} = \ 1.5655_{10}-2; \\
a_{011} = \ 7.8429_{10}-4; & a_{300} = -5.3826_{10}0; & a_{030} = \ 2.4692_{10}-3; \\
a_{003} = \ 5.6768_{10}-7; & a_{210} = -2.2706_{10}0; & a_{201} = \ 6.6169_{10}-3; \\
a_{120} = \ 7.7593_{10}-2; & a_{021} = \ 3.4417_{10}-4; & a_{102} = \ 1.1896_{10}-4; \\
a_{012} = -8.3498_{10}-6; & a_{111} = -3.4414_{10}-3. &
\end{array} \tag{9}$$

Fig. 1a shows the phase tragectory of the system (3), projected on the plane $(x_1, \dot{x}_1)$, and Fig.1b – the phase tragectory of the system (6) with the coefficients (9), projected on the plane (y_1, y_2) with the initial conditions $y_1(0)=1$; $y_2(0)= -10$; $y_3(0)=500$. Fractal dimention [2] for the the system (6) is 2.33, with correspond to value 2.1 for the system (3). The projections of phase portraits in Fig. 1 show topological similarity.

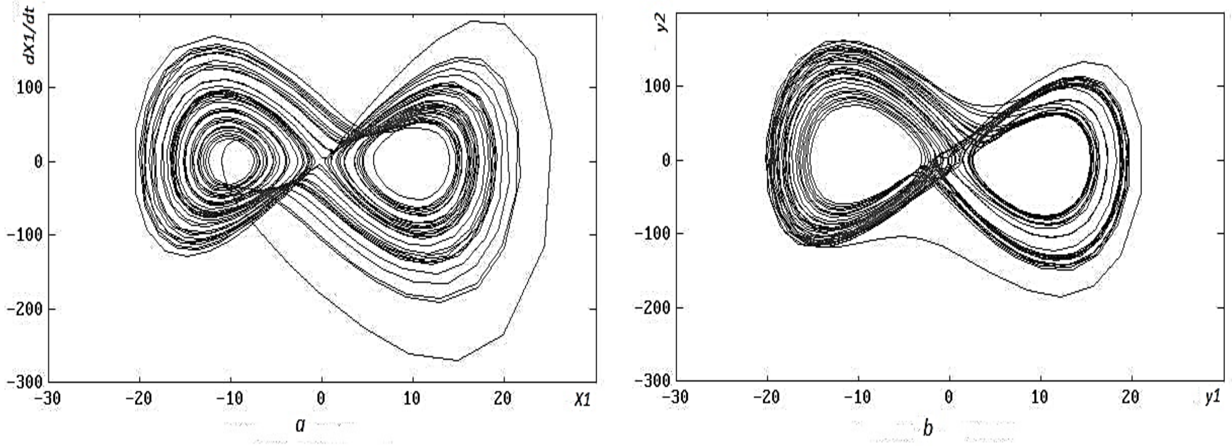

Figure 1. Phase portrait of the Lorenz attractor (*a*) and its reconstraction (*b*)

Simulation of pseudo-random sequence. A model reconstruction method [10] by one output variable is used for the sequence of pseudo-random numbers distributed uniformly within the interval [0,1].

We suppose that a pseudo random numbers come at intervals of 1 second. An interpolation fifth power spline is constructed in a sequence section [8]. This spline and the three derivatives are found an interpolation system for the random sequence. Fig. 2a presents the phase portrait for two first variables of this system, which correspond the pseudo-random sequence in the sense of the above spline interpolation.

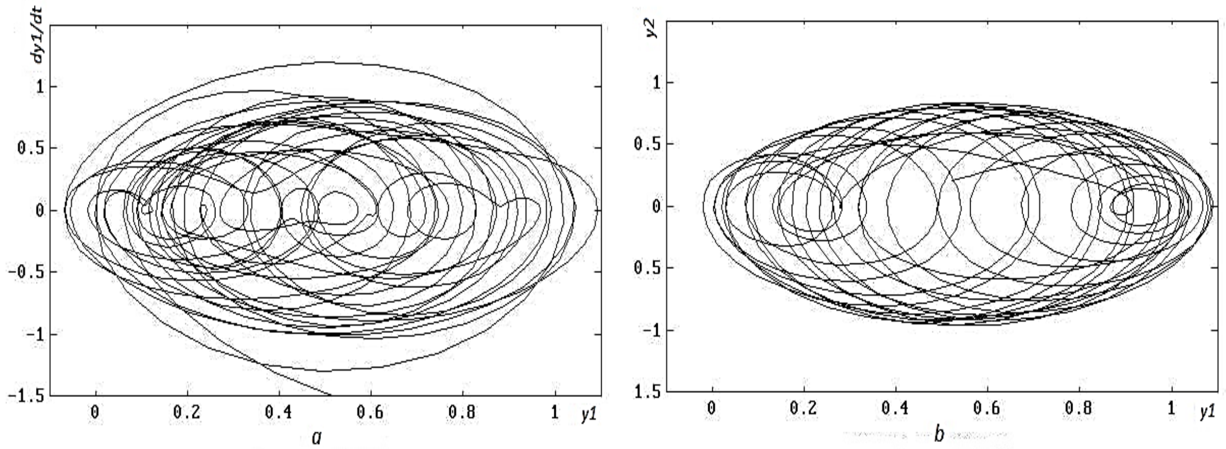

Figure 2. Interpolation of pseudo-random sequence (a) and its macromodel (b).

The problem (8) is solved by a set values of spline and its derivatives, i. e. the coefficients of simulating system (6) are found. Numerical stable solutions which correspond to differential equations having chaotic motions, were obtained for various pseudo-random sequences containing 25 – 100 numbers and with the regularization parameter value $\alpha = 0.001 – 0.42$.

More than 10 systems with strange attractors are obtained for different lengths of initial sections and different values α. One of the systems (6) obtained from (8) for the 29 values of pseudo-random sequence with $\alpha = 0.001$ has the following coefficients:

$$\begin{aligned} &a_{000} =-0.11817;\ a_{100} = 6.55425;\ a_{010} =-11.0487;\ a_{001} =-0.38934;\\ &a_{200} =-9.41420;\ a_{020} =-4.23326;\ a_{002} =-0.03089;\ a_{110} = 38.6623;\\ &a_{101} = 2.12705;\ a_{011} = 9.24564;\ a_{300} = 2.82030;\ a_{030} =-6.76449;\\ &a_{003} =-0.08485;\ a_{210} =-34.6874;\ a_{201} =-2.19388;\ a_{120} = 3.47065;\\ &a_{021} = 0.46197;\ a_{102} =-0.47460;\ a_{012} =-3.26361;\ a_{111} =-16.9036 \end{aligned} \tag{10}$$

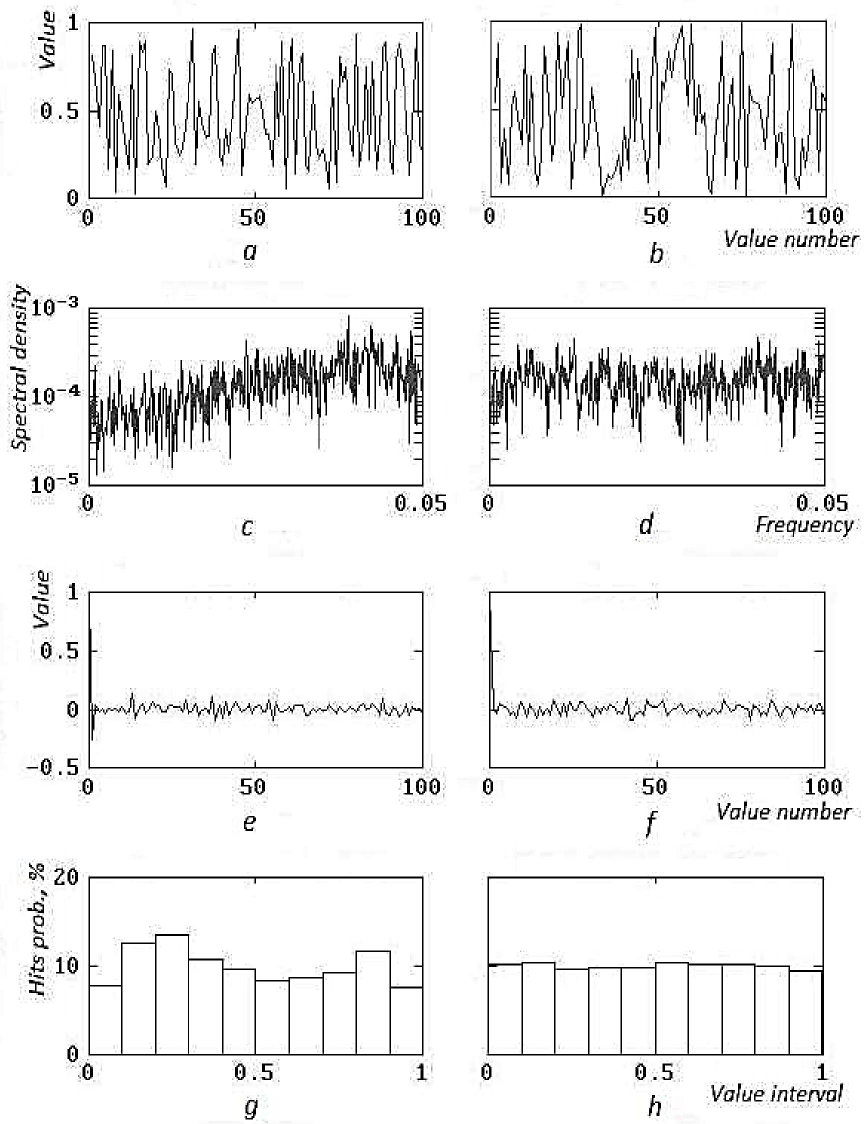

Figure 3. Comparison of macromodel and pseudo-random sequence: “a” is the macromodel; “b” is the random process; “c,d” are the spectra of model and random process; “e,f” are autocorrelation functions of model and random process; “g,h” are the histograms of model and random process.

The system (6) with the coefficients (10) was integrated by an implicit method of the second order with the integration step not more the 0.1 and maximal local error 0.00001 under the initial conditions: $y_1(0)=0.5$; $y_2(0)=0.2$; $y_3(0)=0$. The projection of phase portrait on the plane y_1 - y_2 is presented in Fig. 2*b*. This phase portrait correspond to the strange attractor for which the major Lyapunov number is 0.23 [2].

An initial pseudo-random sequence is simulated by the sequence of y_1 values with time interval 1 sec and mapped within the interval values [0,1]. Certain results of their comparison are given in Fig. 3. Spectral densities were calculated for 4096 points of the appropriate sequence. Autocorrelation functions were calculated for 500 points, 50 initial points are given in Fig. 3. By and large the statistical properties of pseudo-random sequence and simulating sequences are similar.

It should by noted that all the above strange attractors proved to be rather brittle. They were destroyed when the explicit integration methods were used.

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