Vibrations of axially compressed sandwich beam due to a moving force

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Abstract – The dynamic response of a finite, simply supported axially compressed sandwich beam subject to a force moving with a constant velocity is investigated. The classical solution has a form of an infinite series. The main goal of this paper is to present that the aperiodic part of the solution can be presented in a closed form instead of an infinite series. The shown method of finding the solution when the form is closed is based on the observation that the solution of the system of partial differential equations in the form of an infinite series is also a solution of an appropriate system of ordinary differential equations. The closed solutions take different forms depending on the velocity v of the moving force is smaller or bigger than the shear-wave velocity of the beam. The dynamic response of the sandwich beam under moving force is very important solution. It is because that it can be used also in order to find the solution for other types of moving loads.

Key words – Vibration, sandwich beam, moving force, closed solutions.

I. Introduction

Over the past decades the sandwich beams had widespread applications in the fields of aerospace, automotive, marine, civil and mechanical engineering. Sandwich construction offers high strength to weight ratios, as well as good buckling resistance, formability to complex shapes and easy reparability. Due to their many advantages over traditional materials, dynamics response of the sandwich beam has been studied by many authors in the recent decades [1]. The problem of a dynamic response of a structure subjected to moving loads is very important and interesting. Many authors have considered the problem of vibrations in structural engineering, resulting from the moving load, because it has a significant importance for practice, for example in bridge designing and also is interesting from theoretical point of view. Different types of structures and girders like beams, plates, shells, frames have been considered. Also different models of moving loads have been assumed [2]. Deterministic and stochastic approaches have been considered [3]. The problem of vibration of the laminate plates, sandwich and graded beams has been presented in the papers [4-9].

In this paper is concerned the problem of dynamic response of a finite, simply supported, axially compressed sandwich beam subject to a force moving with a constant velocity. The classical solution receive a form of an infinite series. The main goal of this paper is to present that the aperiodic part of the solution can be introduced in a closed form instead of an infinite series. By making use of the method of superimposed deflections Z. Kączkowski [10] has shown for a simply supported Euler-Bernoulli beam that it is possible to present the aperiodic part of the solution in a closed form. Next Z. Reipert obtained a closed solution for a beam with arbitrary boundary conditions [11] and for a frame [12]. In this paper is used a different method to get the solutions in the closed form [13], [14]. The shown method of finding the solution when the form is closed is based on the observation that the solution of the system of partial differential equations in the form of an infinite series is also a solution of an appropriate system of ordinary differential equations. The dynamic response of the sandwich beam under moving force is very important solution. It is because that it can be used also in order to find the solution for other types of moving loads.

II. Statement of the problem. Governing equations

In this paper is considered finite, simple supported, axially compressed sandwich beam with a rectangular cross-section consisting of two thin, stiff, elastic sheets and a thick core layer. A further key assumptions are made at a time when developing the differential equations of motions of a sandwich beam under load excitation:

- the theory of linear elasticity applies,
- transverse direct strains in the face sheets and core are negligible and hence, transverse displacements are the same for all points in a normal section,
- the face sheets carry only axial forces,
- the core carries only shear,
- there is no slippage or delamination between the core and the face sheets,
- only transverse inertia is taken into account,
- the undamped vibrations are considered.

The displacement and forces are shown on Figs.1, 2

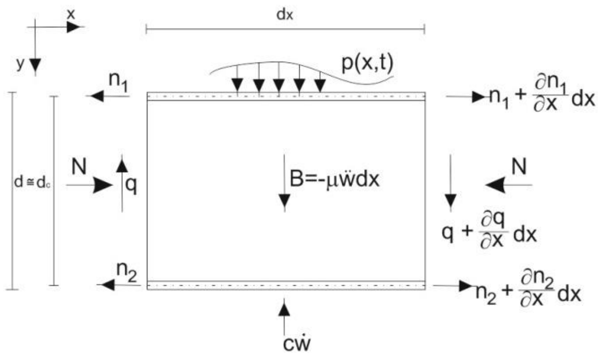

Fig. 1. Geometry of a sandwich beam section and internal forces

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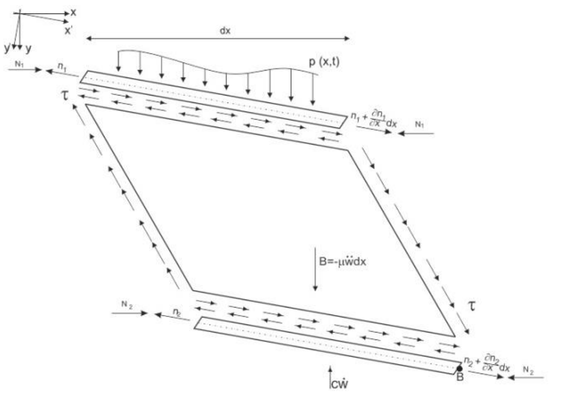

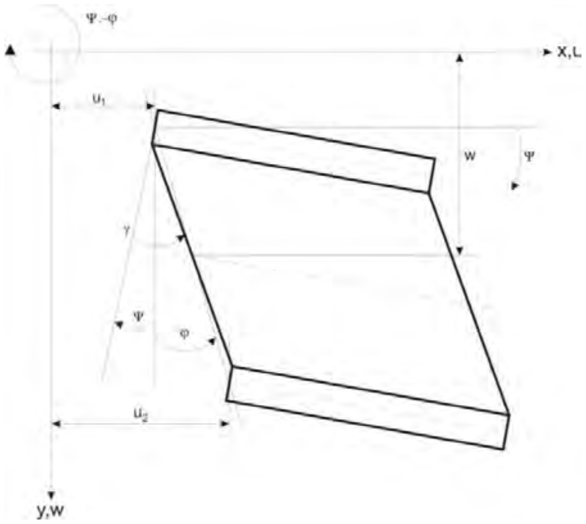

Fig. 2. Deformations and displacements in a sandwich beam section

Taking into account equilibrium equations (equations of horizontal, vertical and moment equilibrium):

$$\frac{\partial n_1(x,t)}{\partial x}+\frac{\partial n_2(x,t)}{\partial x}=0, \tag{1}$$

$$\frac{\partial q(x,t)}{\partial x}=-p\left(x,t\right)+\mu\frac{\partial^2 w(x,t)}{\partial t^2}, \tag{2}$$

$$q(x,t)=\frac{\partial n(x,t)}{\partial x}d-N\frac{\partial w_c\left(x,t\right)}{\partial x}, \tag{3}$$

$n\left(x,t\right)=n_2\left(x,t\right)=-n_1\left(x,t\right)$ oraz $N=N_1+N_2$.

Further analysis of the sandwich beam assumed equal face sheets made of the same material, with the same thickness and $N_1=N_2=\dfrac{N}{2}$.

Dependencies between axial forces and shear stresses

$$\frac{\partial n_i\left(x,t\right)}{\partial x}=(-1)^i\frac{\partial n\left(x,t\right)}{\partial x}=(-1)^i\tau b, \tag{4}$$

where $n=\left|n_i\right|$, $(i=1,2)$,

and the constitutive relations for axial forces of the face sheets and the core shear

$$n_i(x,t)=E_iA_i\frac{\partial u_i(x,t)}{\partial x}, \tag{5}$$

$$\gamma=\frac{\tau}{G}=\left(\frac{\partial w(x,t)}{\partial x}+\Phi(x,t)\right) \tag{6}$$

and after some mathematical transformation one obtains a set of differential equations which describe vibrations of the sandwich beam in the form

$$\begin{gathered}-Gbd\left[\frac{\partial^2 w_c\left(x,t\right)}{\partial x^2}+\frac{\partial\Phi_c\left(x,t\right)}{\partial x}\right]+N\frac{\partial^2 w\left(x,t\right)}{\partial x^2}+\\+c_c\dot{w}\left(x,t\right)+\mu_c\ddot{w}\left(x,t\right)=p\left(x,t\right)\end{gathered} \tag{7}$$

$$\frac{\partial^2\Phi_c\left(x,t\right)}{\partial x^2}-2\frac{Gb}{\varsigma d}\Phi_c\left(x,t\right)-2\frac{Gb}{\varsigma d}\frac{\partial w_c\left(x,t\right)}{\partial x}=0 \tag{8}$$

where:

$$\varsigma=\frac{2E_1A_1E_2A_2}{E_1A_1+E_2A_2},\ \Phi(x,t)=\frac{u_2(x,t)-u_1(x,t)}{d},$$

E_1, E_2 are the Young modulus of the upper and the lower sheets, G is the shear modulus of the beam core, A_1, A_2 are the areas of the sheets cross-sections, b is the beam width and μ is the mass of the beam, μ_1,μ_2 are the masses of the sheets, μ_c is the mass of the beam core. The quantity ς denotes the harmonic average of the axial stiffnesses E_1A_1 and E_2A_2 of the sheets. Taking into account Eq. (5) and that $\Phi(x,t)=\dfrac{u_2(x,t)-u_1(x,t)}{d}$ the axial force in the sheets is equalto:

$$\begin{aligned}n(x,t)&=\frac{1}{2}\left(\frac{2E_1A_1E_2A_2}{E_1A_1+E_2A_2}\right)d\frac{\partial\Phi(x,t)}{\partial x}-\hat{N}=\\&=\frac{1}{2}\varsigma d\frac{\partial\Phi(x,t)}{\partial x}-\hat{N}.\end{aligned} \tag{9}$$

Where $\widehat{N}=\left(-1\right)^{i+1}N_i$.

The boundary conditions for a simply supported beam have forms

$$w(0,t)=w(L,t)=0, \tag{10}$$

$$\left.\frac{\partial\Phi(x,t)}{\partial x}\right|_{x=0}=\left.\frac{\partial\Phi(x,t)}{\partial x}\right|_{x=L}=0, \tag{11}$$

where L is the beam length.

III. Vibration of the beam under moving force

The vibrations of a sandwich beam excited by a point force P moving with a constant velocity v as it is show on Fig.3 are considered.

In this case the load function in Eq. (7) receive the form

$$p(x,t)=P\delta(x-vt), \tag{12}$$

where $\delta(.)$ is the Dirac delta.

After introducing the dimensionless variables

$$\xi=\frac{x}{L},\ T=\frac{vt}{L},\ \xi\in[0,1],\ \ T\in[0,1] \tag{13}$$

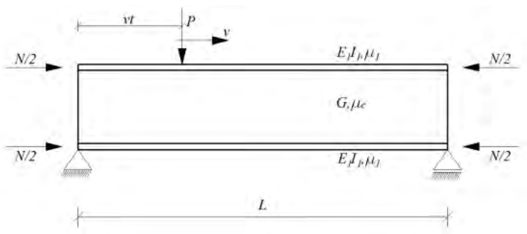

Fig. 3. Sandwich beam system under moving force

the Eqs. (7), (8) take the forms

$$-\frac{\partial^2 w(\xi,T)}{\partial \xi^2} - b_1^2 \frac{\partial \Phi(\xi,T)}{\partial \xi} + \sigma_s^2 \ddot{w}(\xi,t) = P_0 \delta(\xi - T), \qquad (14)$$

$$\frac{\partial^2 \Phi(\xi,T)}{\partial \xi^2} - b_2^2 L \Phi(\xi,T) - b_2^2 \frac{\partial w(\xi,T)}{\partial \xi} = 0, \qquad (15)$$

where: $\sigma_s^2 = \frac{\mu v^2}{Gbd - N} = \frac{v^2}{v_s^2}$, $b_1^2 = \frac{GbdL}{Gbd - N}$,

$b_2^2 = 2\frac{GbL}{\varsigma d}$, $P_0 = \frac{PL}{Gbd - N}$ and where $v_s = \sqrt{\frac{Gbd - N}{\mu}}$

is the shear wave velocity of the beam.

Dots denote differentiation with respect to the time T.

The boundary conditions (10), (11) get the form

$$w(0,T) = w(1,T) = 0, \qquad (16)$$

$$\left.\frac{\partial \Phi(\xi,T)}{\partial \xi}\right|_{\xi=0} = \left.\frac{\partial \Phi(\xi,T)}{\partial \xi}\right|_{\xi=1} = 0. \qquad (17)$$

And the initial conditions have the form

$$w(\xi,0) = 0, \quad \dot{w}(\xi,0) = 0, \qquad \Phi(\xi,0) = 0, \quad \dot{\Phi}(\xi,0) = 0. \qquad (18)$$

The solutions of Eqs. (14,15) for boundary conditions (16,17) are assumed to be in the form of the sine and cosine series

$$w(\xi,T) = \sum_{n=1}^{\infty} y_n(T) \sin n\pi\xi, \qquad (19)$$

$$\Phi(\xi,T) = \sum_{n=1}^{\infty} \varphi_n(T) \cos n\pi\xi. \qquad (20)$$

After exchanging the expressions (19), (20) for equations (14), (15) and making use of the orthogonality method received the set of ordinary differential equations

$$\ddot{y}_n(T) + \varpi_n^2 y_n(T) + \frac{b_1^2}{\sigma_s^2}(n\pi)\varphi_n(T) = \frac{2}{\sigma_s^2} P_0 \sin n\pi T, \qquad (21)$$

$$b_2^2(n\pi) y_n(T) + \left[(n\pi)^2 + b_2^2 L\right]\varphi_n(T) = 0, \qquad (22)$$

where $\varpi_n = \frac{n\pi}{\sigma_s}$.

These functions fulfill the initial conditions

$$y_n(0) = 0,\ \dot{y}_n(0) = 0,\ \phi_n(0) = 0,\ \dot{\phi}_n(0) = 0. \qquad (23)$$

Finally, the solutions of the system of equations (14), (15) are sums of the particular integrals $w_A = (\xi,T)$, $\Phi_A = (\xi,T)$ and general integrals $w_S = (\xi,T)$, and $\Phi_S = (\xi,T)$, have form

$$w(\xi,T) = w_A(\xi,T) + w_S(\xi,T), \qquad (24)$$

$$\Phi(\xi,T) = \Phi_A(\xi,T) + \Phi_S(\xi,T), \qquad (25)$$

where

$$w_A(\xi,T) = 2P_0 \sum_{n=1}^{\infty} \frac{\left[(n\pi)^2 + b_2^2 L\right] \sin n\pi T \sin n\pi\xi}{\left\{\begin{array}{l}(1-\sigma_s^2)(n\pi)^2\left[(n\pi)^2 + b_2^2 L\right] \\ -b_1^2 b_2^2 (n\pi)^2\end{array}\right\}}, \qquad (26)$$

$$w_S(\xi,T) = -2P_0 \sum_{n=1}^{\infty} \frac{(n\pi)\left[(n\pi)^2 + b_2^2 L\right] \sin \omega_{ns} T \sin n\pi\xi}{\omega_{ns}\left\{\begin{array}{l}(1-\sigma_s^2)(n\pi)^2\left[(n\pi)^2 + b_2^2 L\right] \\ -b_1^2 b_2^2 (n\pi)^2\end{array}\right\}}, \qquad (27)$$

$$\Phi_A(\xi,T) = -2P_0 b_s^2 \sum_{n=1}^{\infty} \frac{(n\pi) \sin n\pi T \cos n\pi\xi}{\left\{\begin{array}{l}(1-\sigma_s^2)(n\pi)^2\left[(n\pi)^2 + b_2^2 L\right] \\ -b_1^2 b_2^2 (n\pi)^2\end{array}\right\}}, \qquad (28)$$

$$\Phi_S(\xi,T) = 2P_0 b_s^2 \sum_{n=1}^{\infty} \frac{(n\pi)^2 \sin \omega_{ns} T \cos n\pi\xi}{\omega_{ns}\left\{\begin{array}{l}(1-\sigma_s^2)(n\pi)^2\left[(n\pi)^2 + b_2^2 L\right] \\ -b_1^2 b_2^2 (n\pi)^2\end{array}\right\}}, \qquad (29)$$

where $\omega_{ns}^2 = \varpi_{ns}^2 - \frac{b_1^2 b_2^2 (n\pi)^2}{\sigma_s^2\left[(n\pi)^2 + b_2^2 L\right]} = \frac{(n\pi)^4 + (n\pi)^2[b_2^2 L - b_1^2 b_2^2]}{\sigma_s^2\left[(n\pi)^2 + b_s^2 L\right]}.$

When the following condition

$$(1-\sigma_s^2)(n\pi)^2\left[(n\pi)^2 + b_2^2 L\right] - b_1^2 b_2^2 (n\pi)^2 = 0 \qquad (30)$$

is fulfilled then the solutions (24)-(29) tend to infinity. Thus the resonance velocity v_{cr} is equal to

$$v_{cr} = \sqrt{\frac{(Gbd - N)\left[(n\pi)^2 + b_2^2\left(L - b_1^2\right)\right]}{\mu_c\left[(n\pi)^2 + b_2^2 L\right]}} \qquad (31)$$

It should be noticed that the resonance velocity is less than the shear wave velocity of the beam $(v_{cr} < v_s)$.

The functions $w_A(\xi,T)$ and $\Phi_A(\xi,T)$ are aperiodic vibrations and satisfy the nonhomogeneous differential Eqs. (14)-(15) but do not satisfy the initial conditions of motion (18). These functions are free vibrations of the sandwich beam which satisfy the homogeneous differential Eqs. (14)-(15) $(P_0 = 0)$ and together with the aperiodic, satisfy also functions of the initial conditions of motion (18). Next will be presented the aperiodic solutions $w_A(\xi,T)$ and $\Phi_A(\xi,T)$ given by the expressions (26) and (28) in closed forms.

It should be noticed an important fact that these functions are solutions not only the system of partial differential system equations (14)-(15) but also the system of ordinary system equations (see [13], [14])

$$-(1-\sigma_s^2)\frac{d^2 w_A(\xi,T)}{d\xi^2}-b_1^2\frac{d\Phi_A(\xi,T)}{d\xi}=P_0\delta(\xi-T),$$
$$\frac{d^2\Phi_A(\xi,T)}{d\xi^2}-b_2^2L\Phi_A(\xi,T)-b_2^2\frac{dw_A(\xi,T)}{d\xi}=0, \qquad (32)$$

for the boundary conditions (16)-(17).

The variable T in the equations (32) is the only parameter which describes the location of the moving force on the beam. The system of equations (32) has been created from the system of the partial differential equations (14)-(15) by changing differentiation with respect to the time T to differentiation with respect to the geometrical coordinate ξ, namely

$$\ddot{w}_A(\xi,T)\rightarrow\frac{\partial^2 w_A(\xi,T)}{\partial\xi^2} \text{ and } \ddot{\Phi}_A(\xi,T)\rightarrow\frac{\partial^2\Phi_A(\xi,T)}{\partial\xi^2}.$$

After solving Eqs. (32) by using, for example, the Laplace transform received the functions $w_A(\xi,T)$ and $\Phi_A(\xi,T)$ in the closed form instead of series. The closed form of the solutions depends on the velocity of moving force.

In the case if $\sigma_s<1$ $(v<v_s)$ the solutions have forms for $\xi\le T$

$$w_A(\xi,T)=\frac{P_0}{(1-\sigma_s^2)}\left[\begin{array}{l}\left(1+\frac{b_2^2L}{a^2}\right)\frac{\sin a\xi\sin a(1-T)}{a\sin a}\\ -\frac{Lb_2^2}{a^2}(1-T)\xi\end{array}\right], \qquad (33)$$

for $\xi\ge T$

$$w_A(\xi,T)=\frac{P_0}{1-\sigma_s^2}\left[\begin{array}{l}\left(1+\frac{b_2^2L}{a^2}\right)\frac{\sin aT\sin a(1-\xi)}{a\sin a}\\ -\frac{Lb_2^2}{a^2}T(1-\xi)\end{array}\right], \qquad (34)$$

and for $\xi<T$

$$\Phi_A(\xi,T)=\frac{P_0b_2^2}{a^2(1-\sigma_s^2)}\left[\begin{array}{l}(1-T)\\ -\frac{\sin a(1-T)}{\sin a}\cos a\xi\end{array}\right], \qquad (35)$$

for $\xi>T$

$$\Phi_A(\xi,T)=\frac{P_0b_{2c}^2}{a^2(1-\sigma_s^2)}\left[-T+\frac{\sin aT\cos a(1-\xi)}{\sin a}\right] \qquad (36)$$

where: $a^2=b_2^2\left[\frac{b_1^2}{1-\sigma_s^2}-L\right]$ i $\sigma_c<1$.

In the case if $\sigma_s>1$ $(v>v_s)$ the solutions have forms for $\xi\le T$

$$w_A(\xi,T)=-\frac{P_0}{\sigma_s^2-1}\left(1-\frac{b_2^2L}{\bar{a}^2}\right)\frac{\sinh\bar{a}\xi\sinh\bar{a}(1-T)}{\bar{a}\sinh\bar{a}} -\frac{P_0}{\sigma_s^2-1}\frac{b_2^2L}{\bar{a}^2}(1-T)\xi \qquad (37)$$

for $\xi\ge T$

$$w_A(\xi,T)=\frac{P_0}{\sigma_s^2-1}\left(1-\frac{b_2^2L}{\bar{a}^2}\right)\frac{\sinh\bar{a}T\sinh\bar{a}(\xi-1)}{\bar{a}\sinh\bar{a}} +\frac{P_0}{\sigma_s^2-1}\frac{b_2^2L}{\bar{a}^2}T(\xi-1), \qquad (38)$$

and for $\xi<T$

$$\Phi_A(\xi,T)=\frac{P_0}{\sigma_s^2-1}\frac{b_2^2}{\bar{a}^2}\left[(1-T)-\frac{\sinh\bar{a}(1-T)\cosh\bar{a}\xi}{\sinh\bar{a}}\right] \qquad (39)$$

for $\xi>T$

$$\Phi_A(\xi,T)=\frac{P_0}{\sigma_s^2-1}\frac{b_2^2}{\bar{a}^2}\left[-T+\frac{\sinh\bar{a}T\cosh\bar{a}(\xi-1)}{\sinh\bar{a}}\right] \qquad (40)$$

where $\bar{a}^2=b_{2c}^2\left(\frac{b_{1c}^2}{\sigma_{2c}^2-1}+L\right)$.

The closed solution is particularly important if the axial forces in the sheets and the shear force in the core. If the velocity v of the moving force is less than the shear wave velocity of the beam $v_s=\sqrt{\frac{Gbd-N}{\mu}}$ $(\sigma_s<1)$, the axial force in the sheetscan be obtained from Eq. (9) and has the form

$$n(\xi,T)=\frac{1}{2}\frac{\varsigma d}{L}\frac{\partial\Phi(\xi)}{d\xi}-\hat{N}=n_A(\xi,T)+n_s(\xi,T)-\hat{N}, \qquad (41)$$

where

$$n_s(\xi,T)=-P_0b_2^2\frac{\varsigma d}{L}\sum_{n=1}^{\infty}\frac{(n\pi)\sin\omega_{ns}T\sin n\pi\xi}{\omega_{ns}(1-\sigma_s^2)[(n\pi)^2-a^2]} \qquad (42)$$

for $\xi<T$

$$n_A(\xi,T)=\frac{P_0\varsigma db_2^2}{2L(1-\sigma_s^2)}\frac{\sin a\xi\sin a(1-T)}{a\sin a}, \qquad (43)$$

for $\xi>T$

$$n_A(\xi,T)=\frac{P_0\varsigma db_2^2}{2L(1-\sigma_s^2)}\frac{\sin a(1-\xi)\sin aT}{a\sin a}, \qquad (44)$$

Shear stress in the beam core can be received from Eq. (6) in the form

$$\tau(\xi,T)=G[\frac{1}{L}\frac{\partial w(\xi,T)}{d\xi}+\Phi(\xi,T)] =\tau_A(\xi,T)+\tau_s(\xi,T), \qquad (45)$$

Where

$$\tau_s(\xi,T) = -2\frac{GP_o}{L}\sum_{n=1}^{\infty}\frac{(n\pi)^2 \sin\omega_{ns}T \sin n\pi\xi}{\omega_{ns}(1-\sigma_s^2)[(n\pi)^2 - a^2]} \qquad (46)$$

for $\xi < T$ $\quad \tau_A(\xi,T) = \frac{P}{bd}\frac{\cos a\xi \sin a(1-T)}{(1-\sigma_s^2)\sin a}, \qquad (47)$

for $\xi > T$ $\quad \tau_A(\xi,T) = \frac{-P}{bd}\frac{\cos a(1-\xi)\sin aT}{(1-\sigma_s^2)\sin a}, \qquad (48)$

If the velocity v of the moving force is bigger than the shear wave velocity of the beam then $\sigma_s > 1$ and the aperiodic solutions for the axial force in the sheets and the shear stress in the beam core have forms

$$n_s(\xi,T) = -P_0 b_2^2 \frac{\varsigma d}{L}\sum_{n=1}^{\infty}\frac{(n\pi)\sin\omega_{ns}T \sin n\pi\xi}{\omega_{ns}(1-\sigma_s^2)[(n\pi)^2 + \bar{a}^2]} \qquad (49)$$

for $\xi < T$

$$n_A(\xi,T) = -\frac{P_0\varsigma d b_2^2}{2L\left(\sigma_s^2 - 1\right)}\frac{\sinh\bar{a}\xi \sinh\bar{a}(1-T)}{\bar{a}\sinh\bar{a}}, \qquad (50)$$

for $\xi > T$

$$n_A(\xi,T) = \frac{P_0\varsigma d b_2^2}{2L\left(\sigma_s^2 - 1\right)}\frac{\sinh\bar{a}(\xi-1)\sinh\bar{a}T}{\bar{a}\sinh\bar{a}}, \qquad (51)$$

$$\tau_s(\xi,T) = -2\frac{GP_o}{L}\sum_{n=1}^{\infty}\frac{(n\pi)^2 \sin\omega_{ns}T \sin n\pi\xi}{\omega_{ns}(1-\sigma_s^2)[(n\pi)^2 + \bar{a}^2]}. \qquad (52)$$

for $\xi < T$ $\quad \tau_A(\xi,T) = -\frac{GP_o}{L}\frac{\cosh\bar{a}\xi \sinh\bar{a}(1-T)}{(\sigma_s^2 - 1)\sinh\bar{a}}, \qquad (53)$

for $\xi > T$ $\quad \tau_A(\xi,T) = \frac{GP_0}{L}\frac{\cosh a(1-\xi)\sinh aT}{(\sigma_s^2 - 1)\sinh a}, \qquad (54)$

Conclusion

The dynamics response of a finite, simply supported, axially compressed sandwich beam loaded by a force moving with a constant velocity has been investigated. The classical solution for transverse displacement and the rotation of the cross section has a form of a sum of two infinite series. It has been shown that one of the series (the one which represents aperiodic vibrations of the beam) can be presented in a closed form. The closed solutions take different forms depending on the velocity v of the moving force is smaller or bigger than the shear wave velocityof the beam. This follows from the fact that for a sandwich beam with thin sheets wave phenomena can occur. The shear wave of the beam is less than the shear wave in the core of the beam. The presented closed solutions have very important meaning in the case when we consider the axial forces in the sheets and shear stresses in the core of the beam.

The closed solutions improve the accuracy of the conventional sinus and cosines series expansion of the sandwich beam response by considering the aperiodic part as the solution not only partial of the differential equation but also appropriate ordinary differential equation. The closed solution allow to obtain the discontinuities in the axial and shear forces ("jump") under moving force.

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