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# MODELING THE ELECTROMAGNETIC WAVE SCATTERING BY SMALL IMPEDANCE PARTICLE

# МОДЕЛЮВАННЯ РОЗСІЮВАННЯ ЕЛЕКТРОМАГНІТНИХ ХВИЛЬ МАЛИМ ІМПЕДАНСНИМ ВКЛЮЧЕННЯМ

The problem of electromagnetic waves scattering on the small particle is reduced to solving the Fredholm integral equation of the second kind. Integral representation of solutions to the diffraction problem implies in determination of some auxiliary function which contains in integrand of this equation. The respective linear algebraic system for the components of this auxiliary function is derived and solved by the successive approximation method. The region of convergence of the proposed method is substantiated numerically. The numerical results show rapid convergence in the wide region of the physical and geometrical parameters of problem. The numerical results of scattering on the particles of various forms and sizes are presented.

Key words: EM wave scattering, small impedance particle, Fredholm integral equation, method of successive approximations, computational results.

Задача електромагнітного розсіювання малим включенням зведена до розв'язування інтегрального рівняння Фредгольма ІІ-го роду. Інтегральне подання розв'язку задачі дифракції приводить до необхідності знаходження певної допоміжної функції, яка міститься у підінтегральному виразі даного рівняння. Відповідна система лінійних алгебраїчних рівнянь для компонент цієї допоміжної функції розв'язується методом послідовних наближень. Область збіжності запропонованого методу визначається на основі отриманих числових результатів. Числові результати демонструють швидку збіжність запропонованого методу в широкому діапазоні фізичних та геометричних параметрів задачі. Наведено числові результати розсіювання на включеннях різної форми і величини.

Ключові слова: розсіювання електромагнітних хвиль, імпедансне включення малого розміру, інтегральне рівняння Фредгольма, метод послідовних наближень, числові результати.

## Introduction

In this paper wave scattering by one small impedance particle is studied. Because of high intrinsic interest for this problem, the developed theory allows one to generalize it to the case of many particles and to obtain some physically interesting conclusions about the changes of the material properties of the medium in which many small particles are embedded [1]-[7]. These results were presented firstly in paper [8] and were used for developing a method for creating materials with a desired refraction coefficient by embedding many small impedance particles into a given medium for scalar wave scattering [9], [10], as well as for the electromagnetic wave scattering [11], [12]. In contrast to above papers, where the explicit expression of solution to the diffraction problem was obtained, we deal here with another method of solving the diffraction and scattering problem. The paper discusses the possibility to solve the formulated diffraction problem using the solution of the obtained Fredholm integral equation of the second kind. The problems, related to investigation of the method of the successive approximation for the respective to this equation linear algebraic system, are discussed.

The impedance boundary conditions widely applicable in physics and do not require that the body be small or large. One can pass to the limit in the equation for the effective field [3] in the medium, obtained by embedding many small impedance particles into a given medium. Such theory is applied for the scalar wave scattering, where the equation for the limiting field in the medium is derived. The similar theory was developed for the EM wave scattering by a small impedance particle embedded in a given material [11] This provides to carry out the numerical calculations showing the possibility to use the developed theory in the wide range of the parameters, such as the radius a of the particle, its boundary impedance $\zeta$, the distance $d$ between neighboring particles, and the wavelength. The numerical results obtained on the basis of the solution to Fredholm integral equation will allow to testify that the theory developed in the mentioned above papers is rigorous and its application to creating media with desired refraction coefficient or permeability is physically defensible.

## Statement of Problem

The electromagnetic field scattered by the limited particle with smooth boundary $S$ satisfies outside the Maxwell equations

$$\nabla \times E = i\omega\mu_0 H, \nabla \times H = -i\omega\varepsilon_0 E \quad \text{in} \quad D' = \mathrm{R}^3 \setminus D, \tag{1}$$

where $D$ is the particle region. The outside medium is described by constant permittivity $\varepsilon_0 > 0$ and constant permeability $\mu_0 > 0$, $\omega$ is the frequency. The impedance boundary conditions have form

$$[N,[E,N]] = \zeta[H,N] \quad \text{on} \quad S, \tag{2}$$

where $\zeta$ is the surface impedance of $S$, $N$ is outside normal on $S$; and radiation conditions:

$$E = E_0 + E_s, \quad H = H_0 + H_s, \tag{3}$$

where $E_0, H_0$ are the components of incident field, a $E_s, H_s$ are the components of scattered field.

In formula (2) $[E,N] = E \times N$ is cross product of two vectors.

Eq. (1) and boundary conditions (2) can be written as:

$$\nabla \times \nabla \times E = k^2 E \quad \text{in} \quad D', \quad H = \frac{\nabla \times E}{i\omega\mu_0}, \tag{4}$$

$$[N,[E,N]] = \frac{\zeta}{i\omega\mu_0}[\nabla \times E, N] \quad \text{on} \quad S, \tag{5}$$

where $k$ is wavenumber, $k = \omega(\varepsilon_0\mu_0)^{1/2}$, smallness of particle means that $ka < 1$, where $a$ is effective radius of scattering.

Hence, the problem can be reduced to determination of one vector $E$, after its determination, $H$ components are determined by the second formula of Eq. (4).

## Method of Solution

The vector $E$ is sought in the form [4]

$$E = E_0 + \nabla \times \int_S g(x,t)J(t)dt, \ g(x,t) = \frac{e^{ik|x-t|}}{4\pi|x-t|}, \tag{6}$$

where $J(t)$ is unknown function which will be determined below, $g(x,t)$ is Green function of free space.

In order to obtain the integral equation for function $J(t)$, we substitute first expression of (6) into boundary conditions (5). Using the know formula [5]

$$[N, \nabla \times \int_S g(x,t)J(t)dt \mid_{\mathbf{m}} = \int_S [N,[\nabla_x g(x,t)J(t)]]dt \pm \frac{J(t)}{2}, \tag{7}$$

where by $\mathbf{m}$ are marked limit values at passing variable $x$ to boundary $S$ of the domain $D$ outside and inside, and $\nabla_x g(x,t)$ is derivative of Green function with respect to variable $x$, we obtain in the general operator form

$$J = AJ + f. \tag{8}$$

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Eq. (8) is Fredholm integral equation of the second kind.

Action of operator $A$ and function $f$ are defined as

$$AJ = -2[N, BJ], \quad f = 2[f_e, N],$$ (9)

Operator $B$ acts in such a way

$$B[J(S)] = [\int_S [N, [\nabla_x g(x,t), J(t)]] dt, N] + \zeta i \omega \varepsilon_0 [\int_S g(x,t) J(t) dt, N],$$ (10)

and function $f_e$ is presented in form

$$f_e(S) = [N_S, [E_0(S), N]] - \frac{\xi}{i\omega\mu_0} [\nabla \times E_0(S), N].$$ (11)

The cross product in Eqs. (10) and (11) mismatches the components of function $J$; this yields in use of the method of successive approximations while passing from Eq. (8) to the respective linear algebraic system (see Eq. (12) below).

### Determination of the matrix coefficients of integral equation for vector $J$

In the matrix form Eq. (8) looks like as follows

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \begin{pmatrix} A_{xx} & A_{xy} & A_{xz} \\ A_{yx} & A_{yy} & A_{yz} \\ A_{zx} & A_{zy} & A_{zz} \end{pmatrix} + \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}.$$ (12)

The rights parts of (12) are expressed by the next formulae

$$f_x = [E_{0y} \cos\theta - E_{0z} \sin\theta\sin\varphi] - \frac{\zeta}{i\omega\mu_0} [(\frac{\partial E_{0z}}{\partial y} - \frac{\partial E_{oy}}{\partial z})(-\cos^2\theta - \sin^2\theta\sin^2\varphi) +$$

$$+ (\frac{\partial E_{0y}}{\partial x} - \frac{\partial E_{ox}}{\partial y})\sin\theta\cos\theta\cos\varphi + (\frac{\partial E_{0x}}{\partial z} - \frac{\partial E_{oz}}{\partial x})\sin^2\theta\sin\varphi\cos\varphi],$$ (13)

$$f_y = [E_{0z} \sin\theta\cos\varphi - E_{0x} \cos\theta] - \frac{\zeta}{i\omega\mu_0} [(\frac{\partial E_{0x}}{\partial z} - \frac{\partial E_{oz}}{\partial x})(-\sin^2\theta\cos^2\varphi - \cos^2\theta) +$$

$$+ (\frac{\partial E_{0z}}{\partial y} - \frac{\partial E_{oy}}{\partial z})\sin^2\theta\sin\varphi\cos\varphi + (\frac{\partial E_{0y}}{\partial x} - \frac{\partial E_{ox}}{\partial y})\sin\theta\cos\theta\sin\varphi],$$ (14)

$$f_z = [E_{0x} \sin\theta\sin\varphi - E_{0y} \sin\theta\cos\varphi] - \frac{\zeta}{i\omega\mu_0} [(\frac{\partial E_{0y}}{\partial x} - \frac{\partial E_{ox}}{\partial y})(\sin^2\theta\sin^2\varphi - \sin^2\theta\cos^2\varphi) +$$

$$+ (\frac{\partial E_{0x}}{\partial z} - \frac{\partial E_{oz}}{\partial x})\sin\theta\cos\theta\sin\varphi + (\frac{\partial E_{0z}}{\partial y} - \frac{\partial E_{oy}}{\partial z})\cos\theta\sin\theta\cos\varphi].$$ (15)

The coefficients of matrix $A$ are determined by

$$A_{xx} = 2\int_S (\frac{\partial g(x,y,z,t')}{\partial y} \sin\theta' \sin\varphi' + \frac{\partial g(x,y,z,t')}{\partial z} \cos\theta') dt' +$$

$$+ 2i\zeta\omega\mu_0 \int_S g(x,y,z,t') dt' (-\sin^2\theta\sin^2\varphi - \cos^2\theta),$$ (16)

$$A_{xy} = -2\int_S (\frac{\partial g(x,y,z,t')}{\partial x} \sin\theta' \sin\varphi' dt' + 2i\zeta\omega\mu_0 \int_S g(x,y,z,t') dt' \cdot \sin^2\theta\sin\varphi\cos\varphi,$$ (17)

$$A_{xz} = -2\int_S (\frac{\partial g(x,y,z,t')}{\partial x} \cos\theta' dt' + 2i\zeta\omega\mu_0 \int_S g(x,y,z,t') dt' \cdot \sin\theta\cos\theta\cos\varphi,$$ (18)

$$A_{yx} = -2\int_S (\frac{\partial g(x,y,z,t')}{\partial x} \sin\theta' \cos\varphi' dt' + 2i\zeta\omega\mu_0 \int_S g(x,y,z,t') dt' \cdot \sin^2\theta\sin\varphi\cos\varphi,$$ (19)

$$A_{yy} = 2\int_S (\frac{\partial g(x,y,z,t')}{\partial z}\cos\theta' + \frac{\partial g(x,y,z,t')}{\partial x}\sin\theta'\cos\varphi')dt' +$$
$$2i\zeta\omega\mu_0\int_S g(x,y,z,t')dt'(-\cos^2\theta - \sin^2\theta\cos^2\varphi), \tag{20}$$

$$A_{yz} = -2\int_S (\frac{\partial g(x,y,z,t')}{\partial y}\cos\theta' dt' + 2i\zeta\omega\mu_0\int_S g(x,y,z,t')dt' \cdot \sin\theta\cos\theta\sin\varphi, \tag{21}$$

$$A_{zx} = -2\int_S (\frac{\partial g(x,y,z,t')}{\partial z}\sin\theta'\cos\varphi' dt' + 2i\zeta\omega\mu_0\int_S g(x,y,z,t')dt' \cdot \sin\theta\cos\theta\cos\varphi, \tag{22}$$

$$A_{zy} = -2\int_S (\frac{\partial g(x,y,z,t')}{\partial z}\sin\theta'\sin\varphi' dt' + 2i\zeta\omega\mu_0\int_S g(x,y,z,t')dt' \cdot \sin\theta\cos\theta\sin\varphi, \tag{23}$$

$$A_{zz} = 2\int_S (\frac{\partial g(x,y,z,t')}{\partial x}\sin\theta'\cos\varphi' + \frac{\partial g(x,y,z,t')}{\partial y}\sin\theta'\sin\varphi')dt' -$$
$$-2i\zeta\omega\mu_0\int_S g(x,y,z,t')dt' \cdot \sin^2\theta. \tag{24}$$

Let us write first line of (8) for determination of $J_x$ component

$$J_x = A_{xx}J_x + A_{xy}J_y + A_{xz}J_z + f_x, \tag{25}$$

or

$$(E - A_{xx})J_x = A_{xy}J_y + A_{xz}J_z + f_x. \tag{26}$$

It is follows from (26) that for determination of component $J_x$ it is necessary to have the values of the rest components $J_y$ and $J_z$. In this connection, we will use for solving the system (12) the method of successive approximations. In the first stage, we prescribe $J_{y0}$ and $J_{z0}$; having these both values, we determine $J_{x1}$. In the next stage, we determine $J_{y1}$ by $J_{x1}$ and $J_{z0}$. In the third stage, we determine $J_{z1}$, using found $J_{x1}$ and $J_{y1}$.

In accordance with contraction mapping method, the proposed iterative method converges if $\|A\| < 1$. As long as, determination of $\|A\|$ is complicate procedure, we will change the physical parameters $a$, $\zeta$, and $\omega$ of problem in order to determine the convergence domain of the approach proposed.

### Numerical results

The numerical algorithms for solving the diffraction problem (1)-(3) need the accurate taking into account of requirements related to convergence of the proposed iterative procedure for determination of the $J(t)$ components by the proposed iterative procedure (see [12]). The numerical results show that domain of convergence for this procedure is enough wide. In Figs. 1-3, the characteristic of convergence are shown for $J_x, J_y, J_z$ component for various values of radius $a$ of particle. The rest parameters of problem are the following: $\varepsilon_0 = 8.85 \times 10^{-12} \text{F/m}$, $\mu_0 = 4\pi \times 10^{-7} \text{H/m}$, $\omega = 229.86 \text{GHz}$ ($k = 0.1 \text{m}^{-1}$), $\zeta = 500$.

One can see that iterative process converges very rapidly. The relative error of solution is determined as degree of accuracy. This error is determined as

$$RE = \frac{|V_{n+1} - V_n|}{|V_{n+1}|}, \tag{27}$$

where $V_n, V_{n+1}$ are the values of the respective components in $n$-th and $n+1$-iteration, $V = \{J_x, J_y, J_z\}$. At $a = 1.0$, the relative error for $J_x$ component (see Fig. 1) decreases from 4.28% in the first iteration to 0.015% in the seventh iteration. This error diminishes if $a$ decreases. So, at $a = 0.1$, the values of relative

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error are equal to $0.52\%$ and $0.004\%$, respectively. The respective values of the relative error for $J_y$ component are equal to $8.32\%$ and $0.013\%$ for $a = 1.0$, and $1.07\%$ and $0.004\%$ for $a = 0.1$ (Fig. 2). These values for $J_z$ are equal to $2.03\%$ and $0.006\%$ for $a = 1.0$, and $0.78\%$ and $0.002\%$ for $a = 0.1$ (Fig. 3). Summarizing these results, we can conclude that exactness of the obtained solutions does not exceed several thousandths of percent; therefore the values of respective $E$ components are calculated with guaranteed high accuracy.
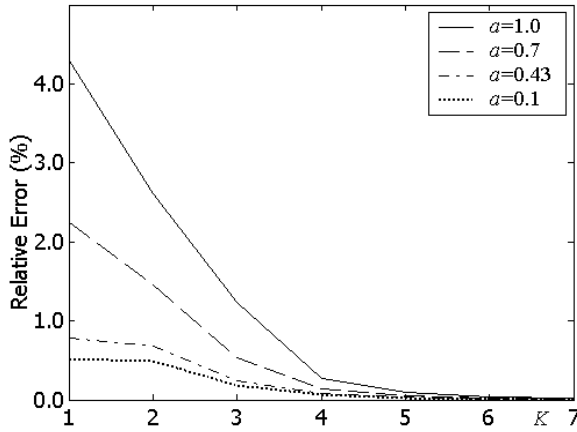


Fig. 1. The relative error of $J_x$ component versus number $K$ of iteration
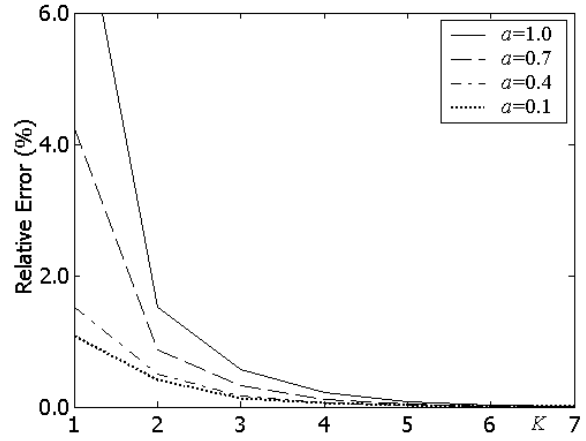
Fig. 2. The relative error of $J_y$ component versus number $K$ of iteration

In order to determine the region of the applicability of the proposed method of successive approximation, we carry out the calculations in wide region of parameter $a$. Be found that at the growth of $a$ the iterative procedure becomes instability; and it becomes divergent at $a > 10.0$. In Fig. 4, the behavior of convergence is shown at $a = 5.0$. One can see that the convergence for $J_y$ and $J_z$ component is non-monotone. Moreover, the values of relative error are greater than for small values of $a$ (in the first stage, they are equal to $9.39\%$, $18.51\%$, and $7.49\%$ for $J_x$, $J_y$ and $J_z$ components, respectively). The number of necessary iterations to achieve the same degree of accuracy as for small $a$ grows twice. So the relative error equal to the thousandths of percent is achieved on 15 iteration; and it is equal to $0.008\%$, $0.007\%$, and $0.005\%$ for $J_x$, $J_y$ and $J_z$ components, respectively.
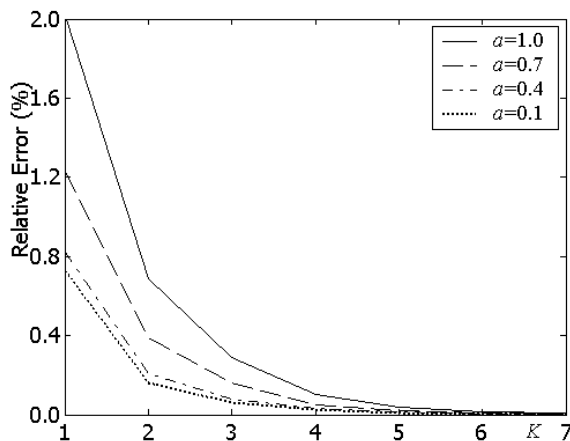


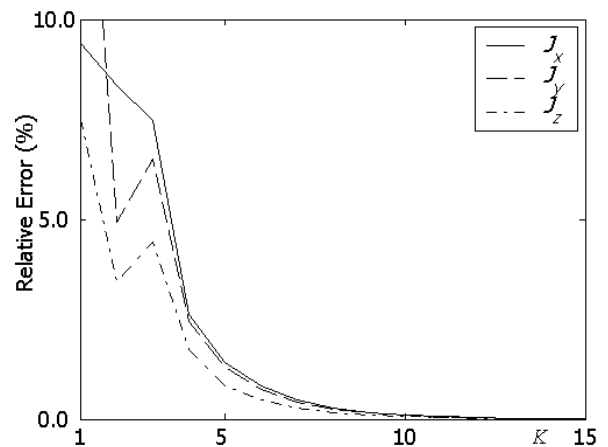Fig. 3. The relative error of $J_z$ component versus number $K$ of iteration

Fig. 4. The relative error of $J$ component versus number $K$ of iteration, $a = 5.0$

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After determination of the area of convergence of the problem parameters, one can deal with solving the diffraction problem. First step consists of solving Eq. (8) by means of passing to linear equation system (12). Let us consider the case of plane wave $E_0 = \beta e^{ik\alpha \cdot x}$, $\beta$ is constant vector, $\alpha$ is unit vector, and $\alpha \cdot \beta = 0$. If the components of vector-function $J$ are determined, the values of vector $E$ are determined explicitly by formula (6). For this case the vector $E$ has only $x$–component, and $E_y = E_z = 0$. In practice, the spherical components of electromagnetic field are much of interest. In this connection, we pass to the spherical components of field using the know formulas (see, for example, [13]). In Fig. 5, the amplitude of $E_r$ component is shown for $a = 0.1$, the rest of parameters are the same as in the previous example.

The electrical size of particle at such $a$ is small: in our case $k = 0.1$, therefore total $ka = 0.01$ that corresponds to very small diameter of scattering, and amplitude of scattered field is fifth order lower than the amplitude of incident field that is equal to 1.0 in our case of plane incident wave. The maximal value of $E_r$ component is equal to $3.87 \times 10^{-5}$: there is value on the distance $d = 2a$ from the surface of particle, and the amplitude diminishes if $d$ grows. In the case if $a$ grows, the form of scattered field changes slightly, but the amplitude increases on several orders. For example, this amplitude is equal to 0.0913 at $a = 5.0$.
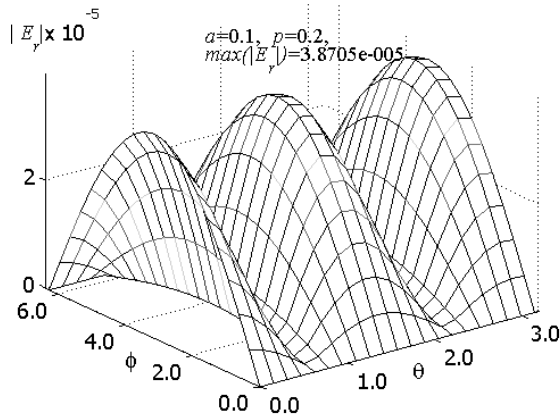


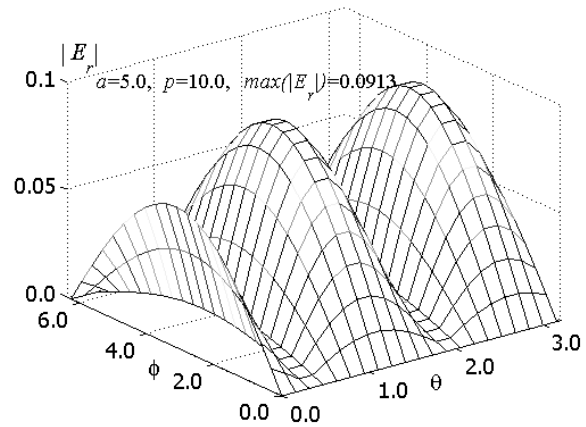Fig. 5. Amplitude of $E_r$ component of the scattered field, $a = 0.1$

Fig. 6. Amplitude of $E_r$ component of the scattered field, $a = 5.0$

The above results concern to the case of spherical particle. If we wish to solve the problem for the particle with another geometry, we should take into account the change of the particle form. For example, in the process of numerical solving Fredholm Eq. (8) and respective to it the linear algebraic system, we should use the actual representation of the element of surface area. So if we pass from investigation of spherical particle to particle in the ellipsoid form, we should change the element of spherical area $dS = a^2 \sin\theta \, d\varphi \, d\theta$ by

$$dS = \sqrt{b^2 c^2 \cos^4\theta \cos^2\varphi + ac \cos^4\theta \sin^2\varphi + a^2 b^2 \sin^2\theta \cos^2\theta} \, d\varphi \, d\theta, \qquad (28)$$

and to take into account that derivative respectively to $r$ coordinate is not equal to zero on the surface of ellipsoid, in contrast of a spherical particle. The numerical results show that the properties of convergence of the method of successive approximations are similar to the case of spherical particle; this method converges very rapidly for case if the semiaxes of ellipsoid do not differ in the limits of 20 %. If this difference grows, the convergence becomes slowly; and we need twice number of iteration to achieve the same accuracy as for spherical particle if $b \geq 3a$, where $a$ is value of semiaxis along $x$ coordinate, and $b$ is value of semiaxis along $y$ coordinate. The amplitude form of the scattered field does not differ considerably if the semiaxes $a, b, c$ of ellipsoid vary in the limits of 10 %.

**Conclusions**

The analytical-numerical method for solving the electromagnetic wave diffraction on the impedance particle of arbitrary shape is elaborated and tested. The integral equation with respect to unknown function in the representation of solution is derived. The respective to this equation linear algebraic system is solved effectively by the method of successive approximations. It is shown numerically that the region of convergence of the proposed method is enough wide. The physical characteristics of scattering are investigated for the particles with different form and size. The approach can be generalized for the case of many particles and it can be used for solving the problems of forming the media with various electromagnetic properties by embedding in the initial medium the big number of small particles.

*1. Andriychuk M. I., Indratno S. W., Ramm A. G. Electromagnetic wave scattering by a small impedance particle: theory and modeling // Optics Communications. – 2012. – Vol. 285. – P. 1684–1691. 2. Ramm A. G. Many-body wave scattering by small bodies and applications // J. Math. Phys. – 2007. – Vol. 48, no. 10, 103511. 3. Ramm A. G. Wave scattering by many small particles embedded in a medium // Phys. Lett. A. – 2008. – Vol. 372/17. – P. 3064–3070. 4. Ramm A. G. Wave scattering by many small bodies and creating materials with a desired refraction coefficient // Afrika Matematika. – 2011. – Vol. 22, no. 1. – P. 33–55. 5. Ramm A. G. Scattering by many small inhomogeneities and applications // “Topics in Chaotic Systems: Selected Papers from Chaos 2010 International Conference”, Editors C. Skiadas, I. Dimotikalis, Char. Skiadas, World Sci.Publishing. – 2011. – P. 41–52. 6. Andriychuk M. I., Ramm A. G. Numerical Solution of Many-Body Wave Scattering Problem for Small Particles and Creating Materials with Desired Refraction Coefficient // in book “Numerical Simulations of Physical and Engineering Processes”, ed. by Jan Awrejcewich, Rieka: InTech. – 2011. – P. 3–28. 7. Андрійчук М., Ткачук В. Моделювання середовища із заданою магнітною проникністю на основі асимптотичного розв’язку задачі електромагнітного розсіяння // Комп’ютерні системи проектування. Теорія і практика. Вісник Нац. ун-ту “Львівська політехніка”. – 2011. – № 711. – С. 58–63. 8. Andriychuk M. I., Ramm A. G. Numerical Modeling in Wave Scattering Problem for Small Particles // Proc. of 18ᵗʰ International Conference on Microwave, Radar and Wireless Communications, June 14-16, 2010, Vilnius, Lithuania. Geozondas. – Vol. 1. – P. 224–227. 9. M. I. Andriychuk M. I., Ramm A. G. Scattering by many small particles and creating materials with a desired refraction coefficient // Intern. Journ. of Computing Science and Mathematics, 2010. – Vol. 3, no. 1/2. – P. 102–121. 10. Андрійчук М. І. Моделювання середовищ із заданим коефіцієнтом заломлення // Відбір і обробка інформації. – 2010. – Вип. 32 (108). – С. 9–15. 11. Ramm A. G. Electromagnetic wave scattering by many small bodies and creating materials with a desired refraction coefficient // Progress in Electromagnetic Research M (PIER M), 2010. – Vol. 13. – P. 203–215. 12. Ramm A. G.,·Andriychuk M. I. Application of the asymptotic solution to EM field scattering problem for creation of media with prescribed permeability // J. Appl. Math. Comput., 2014. – Vol. 45, no 1. – P. 461–485. 13. Корн Г., Корн Т. Справочник по математике для научных работников и инженеров. – М. : ГРФМЛ, 1984.*