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# APPROXIMATION OF FRACTIONAL ORDER DIFFERENTIAL-INTEGRAL CONTROLLERS BY INTEGER ORDER TRANSFER FUNCTIONS

**Yaroslav Marushchak[1], Bohdan Kopchak[2]**

[1]Rzeszow Univesity of Technology, Poland, [2]Lviv Polytechnic National University, Lviv, Ukraine

*jamaru@prz.edu.pl, kopchak@mail.ru*



**Abstract:** The article is devoted to the approximation of fractional order differential-integral controllers by integer order transfer functions using the Oustaloup transformation. The dependence of the accuracy of practical approximation of fractional differential and integral units by the ratio of integer order polynomials on the Oustaloup transformation order has been examined. A modification of the Oustaloup method in which the order of the numerator polynomial is decremented by one has been proposed, and the recommendations for practical implementation of analog fractional order PI controllers have been made.

**Key words:** Oustaloup transformation, approximation, transfer function, fractional order.

## 1. Introduction

Analysis and synthesis of linear and linearized automatic control systems (ACS) are usually carried out by an integer order transfer functions (TF). In [1] it is shown that such a system is a particular case of a more general representation using the fractional order TF. As a result of the synthesis of different systems described by fractional order TFs, there are obtained appropriate fractional controllers, i.e. controllers described by the fractional order TF. Using fractional controllers provides better quality regulation [2] and technological process control and their application will increase with the development of methods and ways of the technical implementation of relevant controllers. Application of fractional order controllers for this purpose provides better flexibility in setting the control circuits than the use of an integer order controller [2]. This flexibility makes the fractional order control systems a powerful tool in the design of robust control systems with fewer parameter adjustments. The main advantage is that by using a few buttons for adjusting, the fractional order controller approaches the same level of robustness as the controller of integer but very high order. At the same time, the implementation of the integer order controllers is well developed in both analog and digital variants. Therefore, the problem of technical implementation of fractional controllers can be solved by means of equivalent replacement (approximation) of their TF by the integer order TF to ensure the same transition functions and the same Bode diagrams in the relevant frequency range for both TF representations.

## 2. Analysis of previous research and publications

Solving the problem of approximation of the fractional order TF by the integer order TF is examined in [3] using the so-called Oustaloup transformations. Such transformations provide the equivalence of Bode diagrams in the frequency range $[\omega_l, \omega_h]$. In this case $\omega_l, \omega_h$ represent lower and upper levels of frequency bands, respectively. Thus, it is presumed as possible to represent the degree of fractional operator $s^{\pm\alpha}$ as follows:

$$s^{\alpha} = \left(\frac{\omega_u}{\omega_h}\right)^{\alpha} \prod_{k=-N}^{k=N} \frac{1 + s/\omega'_k}{1 + s/\omega_k}, \qquad (1)$$

where: $\omega_u = \sqrt{\omega_l \omega_h}$, $N$ is order approximation, which should be set; $\omega'_k$, $\omega_k$ are zeros and poles of equivalent TF of integer order, respectively, $(\omega_u / \omega_h)^{\alpha} = k_n$ [3].

The order of approximation is selected at the level (2N +1). The originality of the idea is that a coefficient, zeros and poles of the expected TF are initially calculated to replace the TF of fractional order by integer order. Next, being based on the calculated poles, the TF is established as follows:

$$W(s) = k_n \frac{(s-\omega'_1)(s-\omega'_2)\ldots(s-\omega'_{2N+1})}{(s-\omega_1)(s-\omega_2)\ldots(s-\omega_{2N+1})}, \qquad (2)$$

where $\omega'_1, \omega'_2 \ldots \omega_{2N+1}$ are calculated zeros of the integer order TF; $\omega_1, \omega_2 \ldots \omega_{2N+1}$ are the calculated poles of the integer order TF of [3].

The expression for the TF (2) can be represented as a ratio of polynomials:

$$W(s) = k_n \frac{s^{2N+1} + \ldots + b_1 s + b_0}{s^{2N+1} + \ldots + a_1 s + a_0} = \frac{P(s)}{Q(s)}. \qquad (3)$$

The disadvantage of this approach is the equal number of poles and zeros in the approximating integer order TF.

### 3. Purpose of the research

Based on simulation studies in MATLAB Simulink environment, this paper aims at solving the following objectives:

– to analyze the dependence of the approximation accuracy of differential-integral units on the order of the approximating polynomial;

– to modify Oustaloup method for the approximating fractional order TF using the integer order TF, if the order of the numerator is decremented by one.

### 4. Basic material and research results

The expression $s^{\pm\alpha}$ can be interpreted as an expression of fractional differentiator (+ α) or integrator (-α), by means of which the total TF of fractional controllers used in the following analysis is formed. To conduct the analysis in MATLAB environment there has been developed a software program that implements the Oustaloup method [3] according to (1) to approximate $s^{\pm\alpha}$ using the integer order TF. The application of the developed program enables transforming the differential-integral units of the fractional order TF with different power α in the frequency range (0,01 ÷ 100) $s^{-1}$ on condition that the order of approximation is N = 1 ÷ 5. To serve as an example, representation of fractional order differential and integral units with power, respectively, α = 0,25; 0,5 and α = - 0,25; - 0.5 by units with the integer order TF using Oustaloup transformation is carried out in the paper.

Below the relevant expressions for the integer order TF with N = 1 ÷ 3 are shown with regard to the fractional order differential and integral units. Their Bode diagrams for different N are constructed, as well as the transition functions to assess the degree of adequacy of such replacement:

1) approximation of the differential fractional unit $W(s) = s^{0,5}$

$$N=1- \quad \frac{31,62s^3 + 1010s^2 + 319,4s + 1}{s^3 + 319,4s^2 + 1010s + 31,62}; \qquad (4)$$

N=2

$$\frac{31,62s^5 + 4249s^4 + 33890s^3 + 16980s^2 + 534,9s + 1}{s^5 + 534,9s^4 + 16980s^3 + 33890s^2 + 4249s + 31,62}; \qquad (5)$$

N=3

$$\frac{10s^7 + 509.4s^6 + 5487s^5 + 14990s^4 + 10790s^3 + ...}{s^7 + 98.34s^6 + 2045s^5 + 10790s^4 + 14990s^3 + ...}$$

$$\frac{... + 2045s^2 + 98.34s + 1}{... + 5487s^2 + 509.4s + 10}. \qquad (6)$$

2) approximation of the fractional integral unit $W(s) = s^{-0,5}$:

$$N=1 - \frac{0,03162s^3 + 10,1s^2 + 31,94s + 1}{s^3 + 31,94s^2 + 10,1s + 0,03162}; \qquad (7)$$

N=2

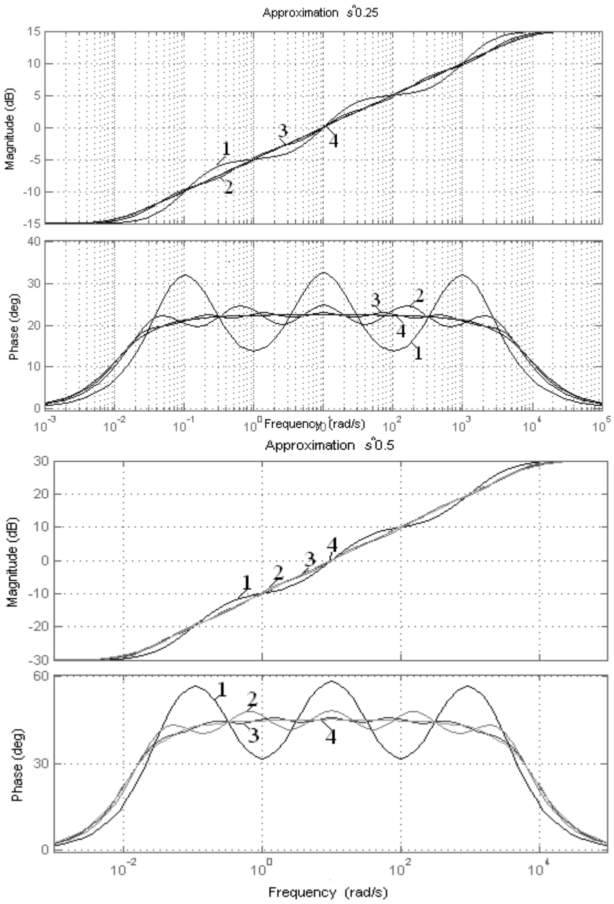
$$\frac{0,03162s^5 + 16,92s^4 + 537,1s^3 + 1072s^2 + 134,4s + 1}{s^5 + 134,4s^4 + 1072s^3 + 537,1s^2 + 16,92s + 0,03162}; \qquad (8)$$

N=3

$$\frac{0.1s^7 + 9.834s^6 + 204.5s^5 + 1079s^4 + 1499s^3 + ...}{s^7 + 50.94s^6 + 548.7s^5 + 1499s^4 + 1079s^3 + ...}$$

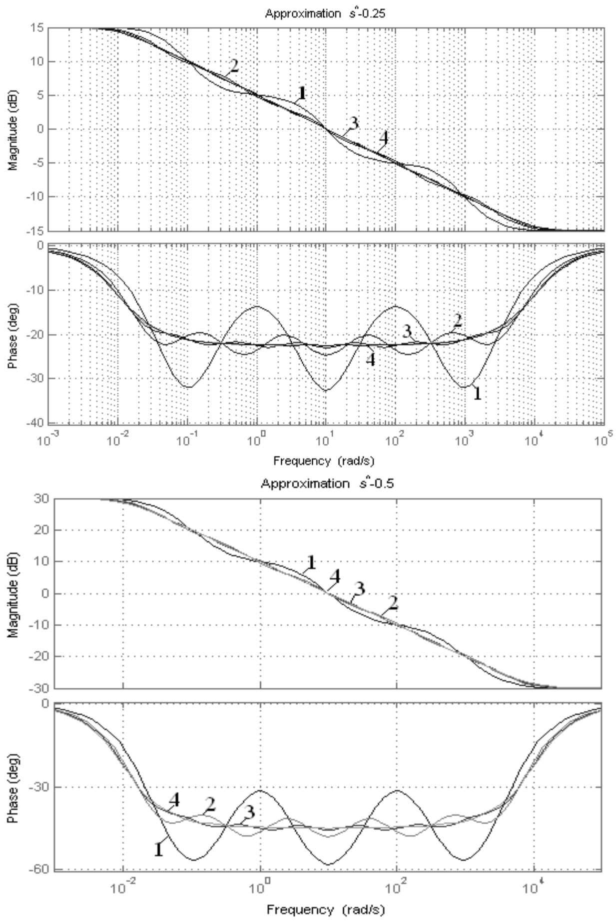
$$\frac{... + 548.7s^2 + 50.94s + 1}{... + 204.5s^2 + 9.834s + 0.1}. \qquad (9)$$

Fig. 1. shows Bode diagrams of differential "a" and integral "b" fractional order units, respectively, for α = 0,25; 0.5 and α = -0,25; -0.5. The calculation was based on TF expressions (4) - (9). Curve 1 corresponds to N = 1, curve 2 to N = 2, curve 3 to N = 3. Curve 4 corresponds to N = 4 and it virtually coincides with the real (theoretical) characteristic of the fractional order unit.



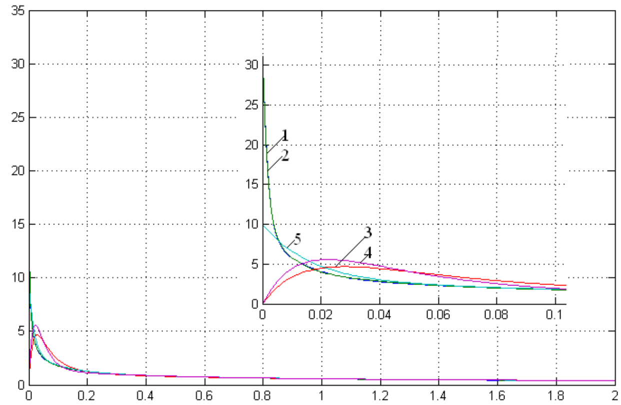

*a)*

*Fig. 1. Bode diagrams of approximated TF (1–3) and real (4) differential and integral fractional order units of.*

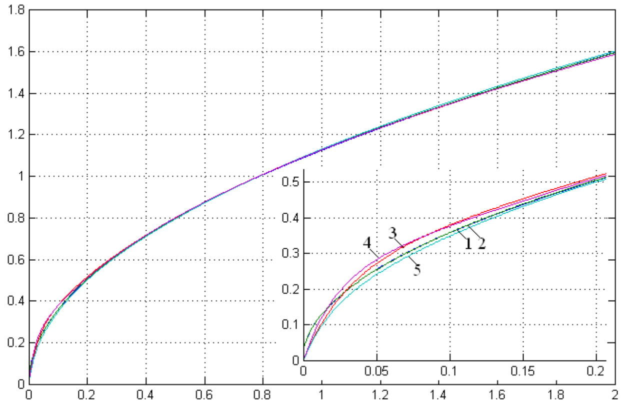



*b)*

*Fig. 1. (continued) Bode diagrams of approximated TF (1–3) and real (4) differential and integral fractional order units of.*



*a)*



*b)*

*Fig. 2. Transition functions of a TF unit W(s)=s$^{0,5}$ (a) and W(s)=s$^{-0,5}$ (b) with fragments in an enlarged scale.*

Fig.2. shows the transition function of fractional order units which, in accordance with the PF W(s)=s$^{0,5}$ (a) and W(s)=s$^{-0,5}$ (b), are obtained by using the Oustaloup transformation**:** for N = 3 (curves 1), for N = 4 (curves 2) and the transition functions obtained by using the optional NINTEGER V.2.3 package, specifically designed for the study of fractional order control systems (curves 5). Curves 3 and 4 will be explained below.

To compare the adequacy of the obtained transition functions, we can calculate the approximation error as the absolute standard deviation $\sigma$ :

$$\sigma = \sqrt{\frac{1}{M}\sum_{i=1}^{M}(y_i - y_{ie})^2} \ ,$$

where: $y_i$ is the value of approximating transition function in the i-th point; $y_{ie}$ is the actual value of the transition function obtained by using the additional NINTEGER V.2.3 package in the i-th point; M is a number of points of transition processing. Also, we can calculate the relative error $\delta = \sigma / y_y \cdot 100\%$, (where $y_y$ is the fixed vaue of transition function of approximating unit**).** The results of evaluating approximation errors for N = 1, 2, 3, 4 are given in Table 1.

*Table 1*

| TF of fract. order unit | Approximation order | Points on the interval [0,t] | t, s. | σ | δ, % |
|---|---|---|---|---|---|
| s$^{-0,5}$ | N=1 | 201 | 2 | 0,01822 | 1,822 |
| | N=2 | 201 | 2 | 0,01368 | 1,368 |
| | N=3 | 201 | 2 | 0,01355 | 1,355 |
| | N=4 | 201 | 2 | 0,01339 | 1,339 |
| s$^{-0,25}$ | N=1 | 201 | 2 | 0,02905 | 2,905 |
| | N=2 | 201 | 2 | 0,02765 | 2,765 |
| | N=3 | 201 | 2 | 0,02762 | 2,762 |
| | N=4 | 201 | 2 | 0,02766 | 2,766 |
| s$^{0,25}$ | N=1 | 201 | 2 | 0,03184 | 3,184 |
| | N=2 | 201 | 2 | 0 | 0 |
| | N=3 | 201 | 2 | 0,003655 | 0,366 |
| | N=4 | 201 | 2 | 0,004026 | 0,403 |
| s$^{0,5}$ | N=1 | 201 | 2 | 0,0765 | 7,65 |
| | N=2 | 201 | 2 | 0 | 0 |
| | N=3 | 201 | 2 | 0,01165 | 1,165 |
| | N=4 | 201 | 2 | 0,01441 | 1,441 |

The analysis of graphs and the calculation of errors (see *Table 1*) has revealed that the frequency characteristics of equivalent units coincide with the characteristics of fractional units for N ≥ 4, and the transition functions show relative mismatch δ<3% even for N ≥ 2.

As it can be seen from formulas (4) – (9), the order of the TF numerator is equal to the order of denominator.

This in turn may cause some difficulties with their practical application.

That is why we have examined the modification of the Oustaloup method of approximation of the fractional order TF by the integer order TF in which the order of the numerator polynomial is decremented by one. To this effect, there have been conducted studies on the possibility of neglecting one zero in the resulting integer order TF or of lowering of polynomial order of the numerator unit by removing part of it at the highest degree of $s$. In this paper the results of such studies for N = 3, 4 are produced. In the case of N = 3 TF (6) for the differential unit looks as follows:

$$W(s)=\frac{509.4s^6+5487s^5+14990s^4+10790s^3+...}{s^7+98.34s^6+2045s^5+10790s^4+14990s^3+...}$$

$$\frac{...+2045s^2+98.34s+1}{...+5487s^2+509.4s+10}, \qquad (10)$$

while TF for the integral unit (9) looks like that:

$$W(s)=\frac{9.834s^6+204.5s^5+1079s^4+1499s^3+...}{s^7+50.94s^6+548.7s^5+1499s^4+1079s^3+...}$$

$$\frac{...+548.7s^2+50.94s+1}{...+204.5s^2+9.834s+0.1}. \qquad (11)$$

For N = 4 TF expressions are not given due to their being cumbersome.

The results of studies of such units using MATLAB are presented in Fig. 2. Fig. 2 (a) shows the transition functions of the fractional order differential unit with $W(s)=s^{-0,5}$ for N = 3 (curve 3) according to (10) and for N = 4 (curve 4). With respect to the error value, satisfactory results can be considered as those obtained by using the Oustaloup transformation with the order of approximation $N \geq 2$. Hence, the implementation of fractional differential analog controllers by reducing the order of the numerator polynomial unit is not feasible.

Fig.2 (b) shows the transition functions of integral unit of fractional order $W(s)=s^{-0,5}$ for N = 3 (curve 3) according to (14) and for N = 4 (curve 4). The resulting transition functions almost coincide with transition processes in accordance with Oustaloup transformation for $N \geq 2$. Approximation error values are within $\sigma$ <0,01 and $\delta$ <0,5 %. So the problem of the practical implementation of both analog and digital integral fractional order controllers is solved, in particular, by decrementing the order of the numerator polynomial by one.

**5. Conclusion**

1) Obtained Bode diagrams and transition functions imply that the Oustoloup transformation above first-order ($N \geq 2$) provides sufficient accuracy for the practical approximation of fractional differential and integral units $s^{\alpha}$ by the ratio of integer order polynomials.

2) Suggested modification of the Oustaloup method in which the order of the numerator polynomial is decremented by one can be recommended for practical realization of only analog fractional order PI controllers, while fractional PID controllers must be applied as digital by using modern microcontrollers.

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## АПРОКСИМАЦІЯ ДИФЕРЕНЦІАЛЬНО-ІНТЕГРУВАЛЬНИХ РЕГУЛЯТОРІВ ДРОБОВОГО ПОРЯДКУ ПЕРЕДАВАЛЬНИМИ ФУНКЦІЯМИ ЦІЛОГО ПОРЯДКУ

Ярослав Марущак, Богдан Копчак

Розглянуто апроксимацію диференціально-інтегрувальних регуляторів дробового порядку передавальними функціями цілого порядку за використання перетворення Оусталоупа. Досліджено залежність точності практичної апроксимації дробових диференціальної та інтегральної ланок відношенням поліномів цілого порядку від порядку перетворення Оустолоупа. Запропоновано модифікацію методу Оусталоупа, в якій порядок поліному чисельника зменшений на одиницю і розроблено рекомендації щодо практичної реалізації аналогових ПІ-регуляторів дробового порядку.

**Yaroslav Marushchak** – D. Sc., professor of Rzeszow Univesity of Technology, Poland. Graduated from Lviv Polytechnic National University, Ukraine. Research interests: synthesis of electromechanichal systems and auto-matization of arc steel-smelting furnace.

**Bohdan Kopchak** – Ph. D., Associate Professor. He works at the Department of Electric Machines at the Institute of Energetics and Control Systems, Lviv Polytechnic National University, Ukraine. Research interests: fractional order control systems.