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LINEAR DYNAMICAL SYSTEMS OF THE N-TH ORDER IN RANDOM CONDITIONS

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Abstract: In the article, the linear dynamic random model of n-th order described by the random state equation is considered. The method for determining the probabilistic characteristics of a stochastic process which is the solution of that equation, is shown. These characteristics, such as mean values and correlation functions, are determined by auxiliary solutions – the deterministic systems of ordinary differential equations.

Key words: moments of stochastic processes, n-th order random system.

1. Introduction

Random dynamical systems are the models of many systems occurring in electrical engineering and electronics [10]. They are usually described by the stochastic differential equations [13]. In the article the method of determining the mean value and the correlation function of the stochastic processes is described, these characteristics being the solutions of linear n-th order random differential equations. The presented results are the generalization of previous works [4, 5, 6, 7, 8, 9, 10] concerning this subject.

2. The formalization of the problem.

A random dynamical system described by the state equation is given:

$$\begin{cases} \dfrac{dX(t)}{dt} = \mathbf{A}X(t) + \mathbf{B}F(t) \\ X(0) = X_0 \end{cases} \tag{1}$$

where: $\mathbf{A}, \mathbf{B}$ – are deterministic matrices, $\mathbf{X}_0$ – is the random or deterministic vector of initial conditions, $\mathbf{F}(t)$ – is the random vector of stochastic processes- excitations satisfying the mean-square Lipschitz condition, $\mathbf{X}(t)$ – is the random vector which is the solution of (in mean-square sense) the equation (1),
and the output equation:

$$\mathbf{Y}(t) = \mathbf{CX}(t) + \mathbf{DF}(t) \tag{2}$$

where: $\mathbf{C}, \mathbf{D}$ – are deterministic matrices, $\mathbf{Y}(t)$ – is the random vector of output stochastic processes.

For solving the equation (1) various methods can be used: the direct method or the method of transformation equations to the moment equation. To solve the equation (1), the method of moments is used [11].

3. Method of moments.

Using the expected value of the operator E [ ] in the both sides of the equation (1) the deterministic system of differential equations in respect to the vector $\boldsymbol{\mu}_X(t)$ is obtained, which is the expected value of the process $\mathbf{X}(t)$:

$$\begin{cases} \dfrac{d\mu_X(t)}{dt} = \mathbf{A}\mu_X(t) + \mathbf{B}\mu_F(t) \\ \mu_X(0) = \mu_{X0} \end{cases} \tag{3}$$

The system (3) is solved by classical methods of mathematical analysis.

Cross correlation of the processes $\mathbf{F}(t)$ and $\mathbf{X}(t)$ is the function:

$$\mathbf{R}_{FX}(t_1, t_2) = E\left[\mathbf{F}(t_1)\mathbf{X}^T(t_2)\right] \tag{4}$$

Then the next operations are to be performed:

– bilateral transposition of the equation (1) and substitution $t=t_2$:

$$\begin{cases} \dfrac{d\mathbf{X}^T(t_2)}{dt_2} = \mathbf{X}^T(t_2)\mathbf{A}^T + \mathbf{F}^T(t_2)\mathbf{B}^T \\ \mathbf{X}^T(0) = \mathbf{X}_0^T \end{cases} \tag{5}$$

– multiplying both sides of the equation by the process $\mathbf{F}(t_1)$:

$$\begin{cases} \mathbf{F}(t_1)\dfrac{d\mathbf{X}^T(t_2)}{dt_2} = \mathbf{F}(t_1)\mathbf{X}^T(t_2)\mathbf{A}^T + \\ \qquad + \mathbf{F}(t_1)\mathbf{F}^T(t_2)\mathbf{B}^T \\ \mathbf{F}(t_1)\mathbf{X}^T(0) = \mathbf{F}(t_1)\mathbf{X}_0^T \end{cases} \tag{6}$$

– applying the expected value of the operator to the result of the previous step. So, as a result the deterministic equation system is obtained:

$$\begin{cases} \dfrac{\partial \mathbf{R}_{FX}(t_1, t_2)}{\partial t_2} = \mathbf{R}_{FX}(t_1, t_2)\mathbf{A}^T + \\ \qquad + \mathbf{R}_F(t_1, t_2)\mathbf{B}^T \\ \mathbf{R}_{FX}(t_1, 0) = \mu_F(t_1)\mu_{X_0}^T \end{cases} . \tag{7}$$

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In system (7), the variable t_1 is treated as a parameter. For a fixed value of the variable t_1, this system is solved by classical methods.

Autocorrelation of the process $\mathbf{X}(t)$ is the function:

$$\mathbf{R}_X(t_1,t_2) = E\left[\mathbf{X}(t_1)\mathbf{X}^T(t_2)\right]. \quad (8)$$

Using the described procedure, in a similar way one can obtain a deterministic system of ordinary differential equations in respect to the vector function $\mathbf{R}_X(t_1, t_2)$ which is the autocorrelation function of the process $\mathbf{X}(t)$:

$$\begin{cases} \dfrac{\partial \mathbf{R}_X(t_1,t_2)}{\partial t_1} = \mathbf{A}\mathbf{R}_X(t_1,t_2) + \\ \qquad + \mathbf{B}\mathbf{R}_{FX}(t_1,t_2) \\ \mathbf{R}_X(0,t_2) = E\left[\mathbf{X}_0\mathbf{X}^T(t_2)\right] \end{cases}. \quad (9)$$

In system (9), the variable t_1 is treated as a parameter. System (9) is treated as a deterministic ordinary differential equation.

If the initial condition of the system (9) is a vector of real numbers, the system (9) can be simplified to the form:

$$\begin{cases} \dfrac{\partial \mathbf{R}_X(t_1,t_2)}{\partial t_1} = \mathbf{A}\mathbf{R}_X(t_1,t_2) + \\ \qquad + \mathbf{B}\mathbf{R}_{FX}(t_1,t_2) \\ \mathbf{R}_X(0,t_2) = \mathbf{X}_0\mu^T(t_2) \end{cases}. \quad (10)$$

If the initial condition of the system (9) is a vector of random variables, it is necessary to find the initial condition of the system (9). It can be obtained by multiplying the equation (1) by $\mathbf{X}_0$ and substituting t to t_2 (assuming that the initial condition is independent of forcing):

$$\begin{cases} \dfrac{\partial \mathbf{R}_X(0,t_2)}{\partial t_2} = \mathbf{R}_X(0,t_2)\mathbf{A}^T + \\ \qquad + \mu_{X_0}\mu_F^T(t_2)\mathbf{B}^T \\ \mathbf{R}_X(0,0) = E\left[\mathbf{X}_0\mathbf{X}_0^T\right] \end{cases}. \quad (11)$$

Covariance and variance of the process $\mathbf{X}(t)$ can be obtained from the relationship:

$$\mathbf{C}_X(t_1,t_2) = \mathbf{R}_X(t_1,t_2) - \mu_X(t_1)\mu_X^T(t_2), \quad (12)$$

$$\sigma_X^2(t) = \mathbf{C}_X(t,t). \quad (13)$$

Applying the expectation operator to equation (2) one can obtain the equation for the expected values of the process $\mathbf{Y}(t)$:

$$\mu_Y(t) = \mathbf{C}\mu_X(t) + \mathbf{D}\mu_F(t). \quad (14)$$

Cross correlation of the processes $\mathbf{F}(t)$ and $\mathbf{Y}(t)$ is the function:

$$E\left[\mathbf{F}(t_1)\mathbf{Y}^T(t_2)\right] = E\left[\mathbf{F}(t_1)\mathbf{X}^T(t_2)\right]\mathbf{C}^T + \\ + E\left[\mathbf{F}^T(t_1)\mathbf{F}(t_2)\right]\mathbf{D}^T, \quad (15)$$

which means:

$$\mathbf{R}_{FY}(t_1,t_2) = \mathbf{R}_{FX}(t_1,t_2)\mathbf{C}^T + \mathbf{R}_F(t_1,t_2). \quad (16)$$

Autocorrelation of the process $\mathbf{Y}(t)$ is the function:

$$\mathbf{R}_Y(t_1,t_2) = E\left[\mathbf{Y}(t_1)\mathbf{Y}^T(t_2)\right]. \quad (17)$$

Taking into account the identity:

$$\mathbf{X}(t_1)\mathbf{F}^T(t_2) = \left(\mathbf{F}(t_2)\mathbf{X}^T(t_1)\right)^T. \quad (18)$$

The equation (17) can be expressed as:

$$\mathbf{R}_Y(t_1,t_2) = \mathbf{C}\mathbf{R}_X(t_1,t_2)\mathbf{C}^T + \\ + \mathbf{C}\mathbf{R}_{FX}(t_2,t_1)\mathbf{D}^T + \mathbf{D}\mathbf{R}_{FX}(t_1,t_2)\mathbf{C}^T + . \quad (19) \\ + \mathbf{C}\mathbf{R}_F(t_1,t_2)\mathbf{D}^T.$$

Covariance and variance of the process $\mathbf{Y}(t)$ can be obtained from the relationship:

$$\mathbf{C}_Y(t_1,t_2) = \mathbf{R}_Y(t_1,t_2) - \mu_Y(t_1)\mu_Y^T(t_2), \quad (20)$$

$$\sigma_Y^2(t) = \mathbf{C}_Y(t,t). \quad (21)$$

4. Example

The series *RLC* circuit is supplied by the ideal voltage source $U(t)$:

$$U(t) = \sin(t+\phi) \quad (22)$$

where ϕ is a random variable with known probability density function:

$$f_\phi(x) = \frac{1}{\pi}\left(H(x) - H(x-\pi)\right) \quad (23)$$

where $H(x)$ is the Heaviside steep function. The probability density function of random variable ϕ is shown in Fig 1.

The mean value of the force process $U(t)$ is given:

$$\mu_U(t) = \mathbf{E}\left[U(t)\right] = \\ = \int_{-\infty}^{\infty} \sin(t+x) f_\phi(x)\,dx = \frac{2}{\pi}\cos(t). \quad (24)$$

The mean value of the force process $U(t)$ is shown in Fig. 2. The variance of the force process $U(t)$ is given:

$$\sigma_U^2(t) = \mathbf{E}\left[U^2(t)\right] - \mu_U^2(t) = \\ = \int_{-\infty}^{\infty} \sin^2(t+x) f_\phi(x)\,dx - \mu_U^2(t) = . \quad (25) \\ = \frac{1}{2} - \frac{4}{\pi^2}\cos^2(t).$$

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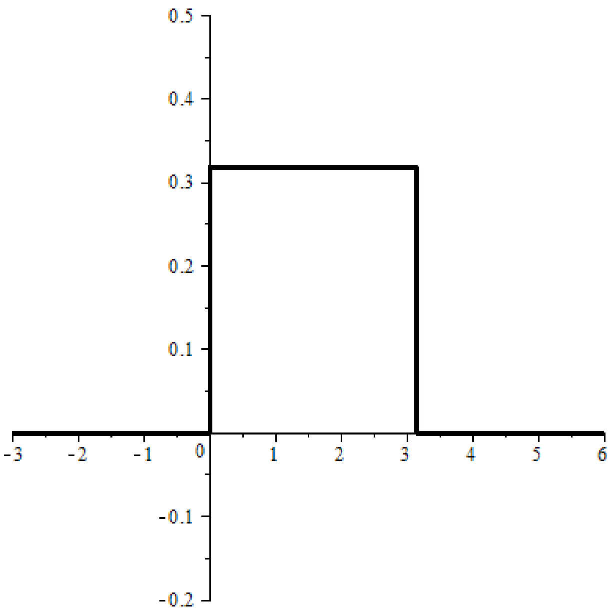

Fig. 1. Probability density function of random variable ϕ *.*

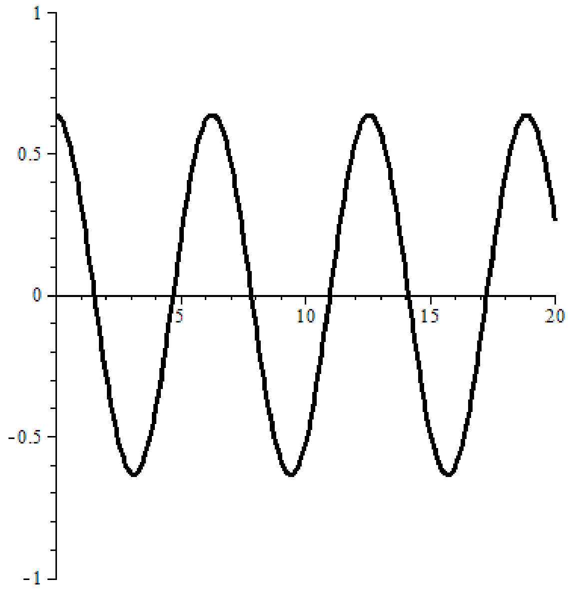

Fig. 2. Mean value of the force process U(t).

Autocorrelation $\mathbf{R}_F(t_1,t_2)=\mathbf{R}_U(t_1,t_2)$ of the force processes
is given by the equation:

$$\begin{aligned}\mathbf{R}_U(t_1,t_2)&=\mathbf{E}\left[U(t_1)U(t_2)\right]=\\&=\sin(t_1+t_2)/4+\pi\cos(t_1)\cos(t_2)/2+\\&+\pi\sin(t_1)\sin(t_2)/2-\sin(t_1)\cos(t_2)/4-\\&-\cos(t_1)\sin(t_2)/4.\end{aligned} \quad (26)$$

Autocorrelation $\mathbf{R}_U(t_1,t_2)$ is used to determine the cross correlation of the force and response (33).

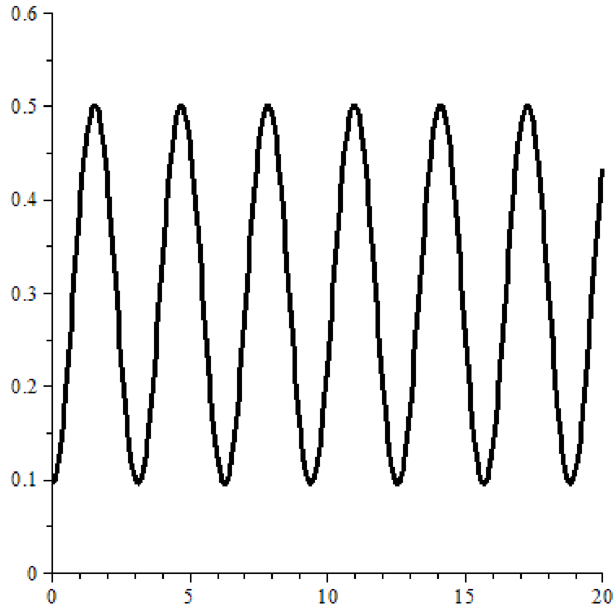

Fig. 3. Variance of the force process U(t).

For simplicity it is assumed that $R=L=C=1$ and $I(0)=U_C(0)=0$. The state equation of the *RLC* system is given:

$$\begin{bmatrix}\frac{d}{dt}I(t)\\ \frac{d}{dt}U_C(t)\end{bmatrix}=\begin{bmatrix}-R/L & -1/L\\ 1/C & 0\end{bmatrix}\begin{bmatrix}I(t)\\ U_C(t)\end{bmatrix}+ , \quad (27)$$

$$+\begin{bmatrix}1/L\\ 0\end{bmatrix}\left[U(t)\right],$$

where:

$$\mathbf{X}(t)=\begin{bmatrix}I(t)\\ U_C(t)\end{bmatrix}, \quad (28)$$

Applying the formula (3) to the equation (27), one can obtain the expected value of the output process:

$$\begin{bmatrix}\frac{d}{dt}\mu_I(t)\\ \frac{d}{dt}\mu_{U_C}(t)\end{bmatrix}=\begin{bmatrix}-1 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\mu_I(t)\\ \mu_{U_C}(t)\end{bmatrix}+ \quad (29)$$

$$+\begin{bmatrix}1\\ 0\end{bmatrix}\left[\frac{2}{\pi}\cos(t)\right].$$

The solution of the equation (29) is given by the formulas:

$$\begin{aligned}\mu_I(t)&=\frac{2\sqrt{3}}{3\pi}\sin\left(\sqrt{3}t/2\right)e^{-t/2}-\\&-\frac{2}{\pi}\cos\left(\sqrt{3}t/2\right)e^{-t/2}+\frac{2}{\pi}\cos(t),\end{aligned} \quad (30)$$

$$\mu_{U_C}(t)=-\frac{4\sqrt{3}}{3\pi}\sin\left(\sqrt{3}t/2\right)e^{-t/2}+\frac{2}{\pi}\sin(t). \quad (31)$$

The cross correlation of the force and response is the function:

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$$\mathbf{R}_{FX}(t_1,t_2)=\left[\mathbf{R}_{UI}(t_1,t_2)\ \mathbf{R}_{UU_C}(t_1,t_2)\right]. \quad (32)$$

Applying formula (7) to the equation (27), one can obtain the cross correlation of the force and response:

$$\begin{cases} \dfrac{\partial R_{UI}(t_1,t_2)}{\partial t_2}=-R_{UI}(t_1,t_2)-R_{UU_C}(t_1,t_2)+ \\ \qquad +R_U(t_1,t_2) \\ \dfrac{\partial R_{UU_C}(t_1,t_2)}{\partial t_2}=R_{UI}(t_1,t_2) \\ R_{UI}(t_1,0)=R_{UU_C}(t_1,0)=0 \end{cases}. \quad (33)$$

The solution of the equations (33) is given by the formulas:

$$\begin{aligned}\mathbf{R}_{UI}(t_1,t_2)=\frac{\pi}{6}\Big(&3\cos(t_1-t_2)+ \\ &+\sqrt{3}\cos(t_1)\sin\left(\sqrt{3}t_2/2\right)e^{-t_2/2}- \\ &-2\sqrt{3}\sin(t_1)\sin\left(\sqrt{3}t_2/2\right)e^{-t_2/2}- \\ &-3\cos(t_1)\cos\left(\sqrt{3}t_2/2\right)e^{-t_2/2}\Big)\end{aligned}, \quad (34)$$

$$\begin{aligned}\mathbf{R}_{UU_C}(t_1,t_2)=\frac{-\pi}{6}\Big(&3\sin(t_1-t_2)+ \\ &+2\sqrt{3}\cos(t_1)\sin\left(\sqrt{3}t_2/2\right)e^{-t_2/2}- \\ &-\sqrt{3}\sin(t_1)\sin\left(\sqrt{3}t_2/2\right)e^{-t_2/2}- \\ &-3\sin(t_1)\cos\left(\sqrt{3}t_2/2\right)e^{-t_2/2}\Big).\end{aligned} \quad (35)$$

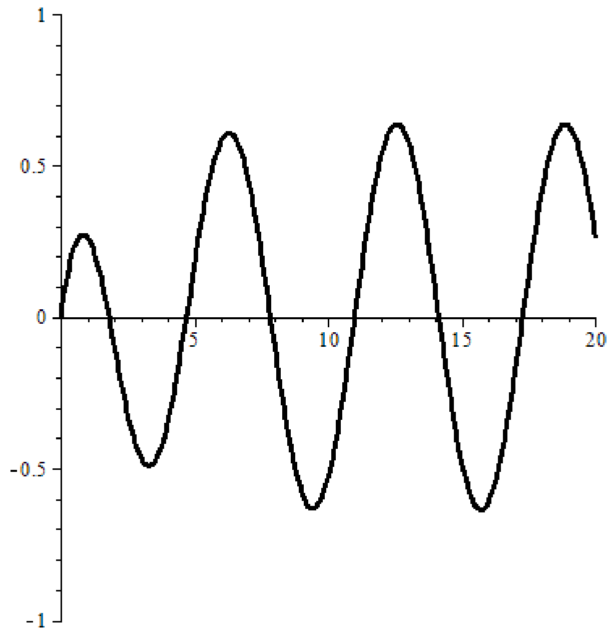

Fig. 4. Mean value of the response process I(t).

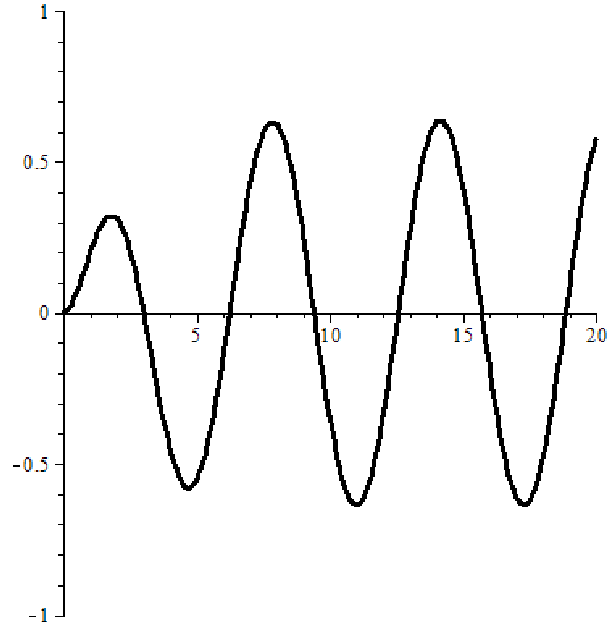

Fig. 5. Mean value of the response process $U_C(t)$.

The autocorelation of the response is shown in the Fig. 6, 7, 8, 9.

The autocorrelation of the response is the function:

$$\mathbf{R}_X(t_1,t_2)=\begin{bmatrix} R_I(t_1,t_2) & R_{IU_C}(t_1,t_2) \\ R_{U_CI}(t_1,t_2) & R_{U_C}(t_1,t_2)\end{bmatrix}. \quad (36)$$

In the same way, applying the formula (10) to the equation (27), one can obtain the autocorrelation $\mathbf{R}_X(t_1,t_2)$ of the output process:

$$\begin{cases} \dfrac{\partial R_I(t_1,t_2)}{\partial t_1}=-R_I(t_1,t_2)-R_{U_CI}(t_1,t_2)+ \\ \qquad +R_{UI}(t_1,t_2) \\ \dfrac{\partial R_{IU_C}(t_1,t_2)}{\partial t_1}=-R_{IU_C}(t_1,t_2)-R_{U_C}(t_1,t_2)+ \\ \qquad +R_{UU_C}(t_1,t_2) \\ \dfrac{\partial R_{U_CI}(t_1,t_2)}{\partial t_1}=R_I(t_1,t_2) \\ \dfrac{\partial R_{U_C}(t_1,t_2)}{\partial t_1}=R_{IU_C}(t_1,t_2) \\ R_I(0,t_2)=R_{IU_C}(0,t_2)=0 \\ R_{U_CI}(0,t_2)=R_{U_C}(0,t_2)=0 \end{cases}. \quad (37)$$

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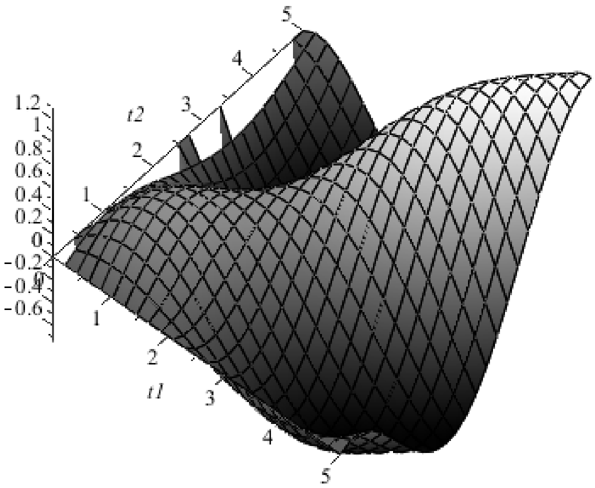

Fig. 6. Autocorrelation of the responseprocess I(t).

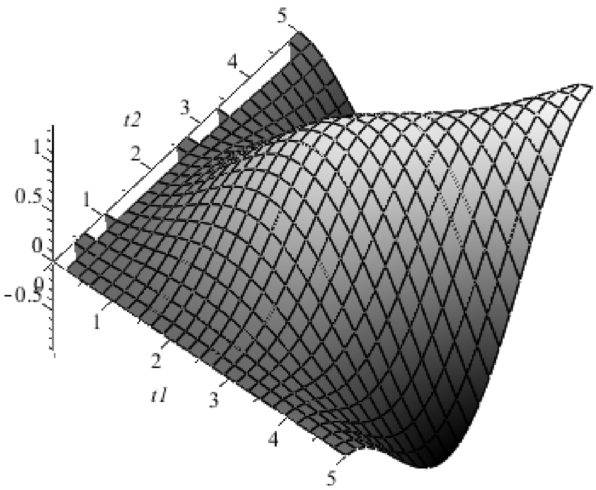

Fig. 7. Autocorrelation of the response process $U_C(t)$.

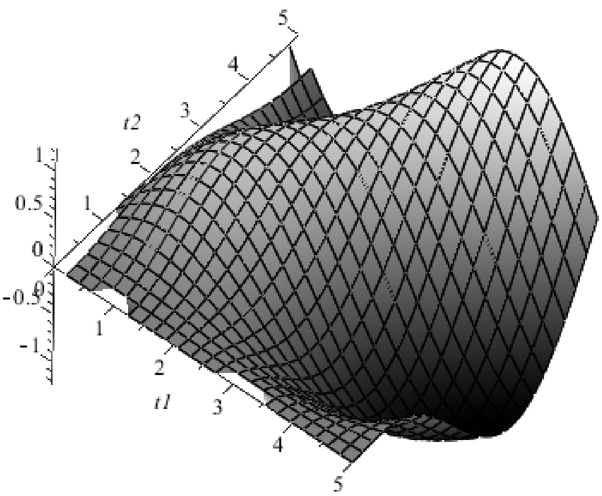

Fig. 8. Correlation of the response processes I(t) and $U_C(t)$.

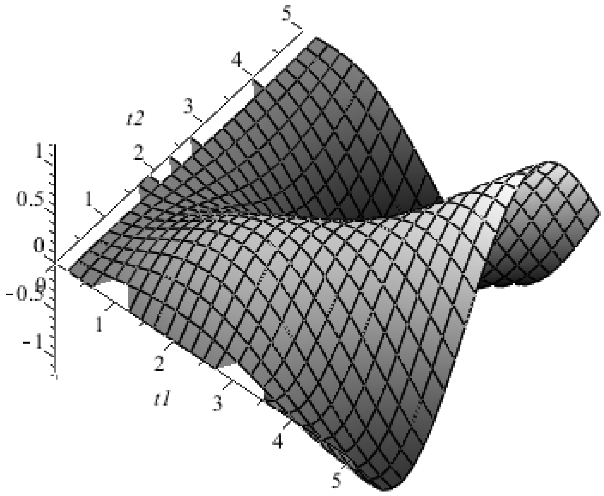

Fig. 9. Correlation of the response processes $U_C(t)$ and I(t).

5. Conclusion

In this paper the method of the problem of the moment determination for complex deterministic and dynamical systems of the n-th order with random excitations has been described. It enables converting the problem of the n-th order stochastic differential equation solving into the problem of ordinary differential equations solving. The ordinary differential equations have been defined in terms of the stochastic process moments and as a result their solution is relatively simple.

Contrary to the previous works the proposed method can be applied for systems of any order.

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ДИНАМІЧНІ ЛІНІЙНІ СИСТЕМИ N-го ПОРЯДКУ ЗА ВИПАДКОВИХ УМОВ

Северин Мазуркевич, Януш Вальчак

Розглянуто лінійну динамічну стохастичну модель n-го порядку, що описується стохастичним рівнянням стану. Показано метод визначення імовірнісних характеристик стохастичного процесу, які є розв’язком такого рівняння. Ці характеристики, як, наприклад, середні значення та кореляційні функції, визначаються додатковими умовами — детермінованими системами звичайних диференційних рівнянь.

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