Оптимальний поліс страхування життя в інвестиції тривалого часу – проблема витрат

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Ми розглядаємо домогосподарство, дохід якого покладений на конкретного члена сім’ї. Домогосподарство має стимул купити контракт на страхування життя, щоб пом’якшити ризик загибелі годувальника. Період часу інвестиції домогосподарства [0, T], де T позначає планований час виходу на пенсію цієї особи. Тобто домогосподарство розраховує отримувати дохід від зарплати на рівні $y(t)$ весь час до моменту T, який може настати до визначеного часу T у випадку неочікуваної втрати здатності заробляти (наприклад, у випадку смерті).

Таким чином, природньо припускати, що домогосподарство придбає страховку, термін дії якої закінчиться на момент T. Іншими словами, термін дії страхового покриття діє до часу T.

Ризик смертності моделюється першим настанням пуассонівського процесу $N = \{N(t);\ t \geq 0\}$ з процесом інтенсивності $\lambda = \{\lambda(t);\ t \geq 0\}$.

Ми позначаємо випадковий період часу даної події τ.

Домогосподарство купує n кількість акцій страхового полісу, оплачуючи одноразову загальну суму страхової премії $n \times p_0$ та інвестує решту наявних коштів у фінансовий ринок в час 0. Ми припускаємо, що сума ціни за акцію p_0 визначається екзогенно і n є одним із умовних змінних. Страхова компанія виплачує страхову суму $n \times X(\tau)$ в час τ у випадку $\tau \leq T$, а домогосподарство використає кошти на витрати та/чи додаткове інвестування у фінансовий ринок. З іншого боку, якщо $\tau > T$, дія страхового контракту закінчується і застрахована особа виходить на пенсію у час T. Домогосподарство намагається максимізувати загальну очікувану універсальність від витрат та кінцевого благополуччя.

Ми демонструємо, використовуючи метод опуклої подвійності, якщо певне n вирішує подвійне завдання, то n також вирішує проблему максимізації початкової універсальності. Тоді ми явно обчислюємо відповідне споживання та процеси збагачення.

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An Optimal Life Insurance Policy in the Continuous-Time Investment – Consumption Problem

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This paper considers an optimal life insurance purchase for a household subject to mortality risk. The household receives wage income continuously, which could be terminated by unexpected premature loss of earning power. In order to hedge the risk of losing income stream, the household enters a life insurance contract for the bereft members. The household may also invest their wealth into a financial market. If insurance payment is made prior to the planned time horizon, the amount shall be used for consumption and investment. Therefore, the problem is to determine an optimal insurance/investment/consumption strategy in order to maximize the expected total discounted utility from consumption and terminal wealth. We provide explicit solutions in a fairly general setup.

Keywords – Life Insurance, Investment/Consumption Model, Martingale, Convex duality, Incomplete Marke.

I. Introduction

In this paper, we consider optimal life insurance for a household in a continuous time economy. The householder's death is formulated as the first occurrence of events in a Poisson process. We also consider optimal portfolio selection for the household simultaneously. We tackle the problem by applying the auxiliary market approach in the martingale method for constrained optimal portfolio selection problems(cf. Karatzas and Shreve (1998)).

Most of previous treatments of insurance in the literature are either to assume that an insurance amount is given exogenously or to obtain an insurance premium or reserve for insurers (cf. Iwaki (2002)). However, Zhu (2007), in one-period model, performed a comprehensive study of the insurance-investment-consumption problem and analyzed effects of parameters on individuals’ insurance purchase, consumption, and stock investment decisions.

In contrast, this paper discusses an optimal insurance from the standpoint of households in the Continuous-Time Investment-Consumption Problem.

II. Model

Let us consider a complete filtered probability space $(\Omega, \mathrm{F}, (\mathrm{F}_t)_{t\in \mathrm{R}+}, \mathrm{P})$ that hosts a Brownian motion $B := \{B(t) : t \geq 0,\ B(0) = 0\}$ and a Poisson Process $N := \{N(t) : t > 0,\ N(0) = 0\}$ with the intensity process $\lambda := \{\lambda(t);\ t \geq 0\}$. The

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Brownian motion is the source of randomness other than the time τ:

$$\tau := \inf\{t > 0; N(t) = 1\},$$

which denotes the time of the insured person's loss of earning power (e.g. death). We assume that the Poisson process N and the Brownian motion B are mutually independent. Suppose that the current time is 0, and let $T > 0$ be the termination time of an insurance contract which is set to be the same as the retirement time. We consider a continuous-time economy in $[0,T]$ that consists of the insurance contract and a financial market.

For simplicity, we assume that household relies on one member's income stream: $y = \{y(t);\ t \in [0, T]\}$ (called income process hereafter) which is given exogenously until time T. To hedge the risk of loss of income flow at time $\tau < T$, the household buys an insurance policy described as follows: Once the household buys n shares of the policy by paying the insurance premium amounts $p_0 \times n$ at time 0, the insurance company makes an insurance payment in the amount of

$$n \times X(t) = n \times (1 + H(t)) \qquad (2.1)$$

at time $t = \tau \wedge T$. Here $H : [0, T] \rightarrow R_+$ is given exogenously, representing payment schedule until time T.

In case τ > T, the policy pays 1 dollar per share at time T. In order to avoid unnecessary complications, we assume that the schedule function satisfies the following assumption.

Assumption 2.1. $H : [0, T] \rightarrow R_+$ is a nonincreasing continuous function with $H(T) = 0$.

Let $c = \{c(t);\ t \in [0, T]\}$ be the consumption process to be determined by the household. It is assumed that income and consumption processes are adapted to F. In the financial market, there is a riskless security whose time t price is denoted by $S_0(t)$. The riskless security evolves according to the differential equation;

$$\frac{dS_0(t)}{S_0} = r(t)dt, t \in [0,\mathrm{T}],$$

where $r(t)$ is a positive, predictable process. The household can also invest their wealth into a risky security whose time t price is denoted by $S_1(t)$. The risky security evolves according to the stochastic differential equation (abbreviated SDE);

$$\frac{dS_1(t)}{S_1(t)} = \mu(t)dt + \sigma(t)dB(t), t \in [0,\mathrm{T}], \qquad (2.2)$$

where $\mu(t)$ and $\sigma(t)$ are progressively measurable processes.

Let $\pi(t)$ be the amount to be invested into the risky security at time t. The process $\pi = \{\pi(t);\ t \in [0, T]\}$ is referred to as a portfolio process. Now, given a portfolio process π, a consumption process c, the number of shares of the insurance policy n and an income process y, the wealth process $W = \{W(t);\ t \in [0,T]\}$ is defined by

$$W(t) := \begin{cases} W_0 - np_0 + \int_0^l \big(r(s)W(s) + y(s) - c(s)\big)ds + \int_0^l \pi(s)\big[\big(\mu(s) - r(s)\big)ds + \sigma(s)dB(s)\big] \ if\ t \in [0, \tau \cap T) \\ W(\tau -) + nX(\tau) + \int_x^l \big(r(s)W(s) - c(s)\big)ds + \int_x^l \pi(s)[\mu(s) - r(s))ds + \sigma(s)dB(s)] \ if\ t \in [\tau \cap T, T) \end{cases} \qquad (2.3)$$

Recall that the household consumes the wage income and, if any, insurance money to maximize the expected discounted utility from consumption c and terminal wealth $W(T)$. Let $U_1 : (0, \infty) \rightarrow R$ be the utility function of the household from consumption, and let $U_2 : (0, \infty) \rightarrow R$ be the utility function of the household from the terminal wealth.

Assumption 2.2. We assume that our utility functions satisfy the following:

(1) U_i ($_i$ = 1, 2) are strictly increasing, strictly concave and twice continuously differentiable with properties

$U'_i(\infty) := \lim_{x\to\infty} U'_i(x) = 0,\ \ U'_i(0+) := \lim_{x\downarrow 0} U'_i(x) = \infty, i = 1,2.$

(2) For any $c \in (0, \infty)$ there exist real numbers $a \in (0, \infty)$ and $b \in (0, \infty)$ satisfying $aU_i\,'(c) \geq U_i\,'(bc)$.

Also, in order to represent time-preference of the household, we introduce a time-discount factor

$$e^{-\int_0^t \rho(s)ds}, t \in [0, T],$$

where the process $\rho = \{\rho(t),\ t \in [0, T]\}$ is adapted to F. A natural problem for the household is as follows: Given the initial wealth W_0, the household decides how many insurance contracts to buy at time zero to protect from the risk of the Poisson event. The rest of the money $W_0 - np_0$ can be invested in the financial market. If $\tau \leq T$, the household receives the insurance money $nX(\tau)$ as in (2.1) and re-solves the optimal investment-consumption problem (2.4) by using the sum of the wealth at τ, $W(\tau-)$ and the insurance money $nX(\tau)$ as the "initial" wealth at τ . On the other hand, if $\tau > T$, the problem reduces to an ordinary investment-consumption problem from time zero to T. By keeping these possibilities in mind, the household decides on the number of insurance contract n at time zero along with the optimal consumption-investment pair to maximize the overall utility. Mathematically, it is stated as follows:

(**MP**): Given the discount process ρ and utility functions U_i (x), i = 1, 2, find an optimal triplets consisting of consumption process, portfolio process and the number of shares of the insurance policy $(\hat{c}, \hat{w}, \hat{n})$ to solve the following maximization problem:

$$\max E\left[\int_0^{\tau\wedge T} e^{-\int_0^t \rho(s)ds} U_1\big(c(t)\big)dt + e^{-\int_0^{\tau\wedge T}\rho(s)ds} V(W(\tau \wedge T))\right] \qquad (2.4)$$

with

$$V\big(W(\tau \wedge T)\big) := \max E\left[\int_{\tau\wedge T}^T e^{-\int_{\tau\wedge T}^t \rho(x)dx} U_1\big(e(t)\big)dt + e^{-\int_{\tau\wedge T}^T \rho(x)dx} U_2(W(T)) | \mathcal{F}_{\tau\wedge T}\right] \qquad (2.5)$$

where the maximum is taken over the feasible consumption and wealth pairs, (c,W) under the budget constraint (2.3).

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III. Main Results

In order to apply the martingale approach, we need to specify a state price density process first. Let $\wp$ be a class of positive and predictable stochastic processes:

$$\wp = \left\{\psi(t);\ \int_0^{\tau\wedge T} \psi(t)dt < \infty,\ \ t \in [0, \tau \wedge \mathrm{T}]\right\}$$

For each $\psi = \{\psi(t);\ t \in [0, \tau \wedge T]\} \in \wp$, the state price density process is given by

$$\chi(t) \coloneqq \beta(t)\chi^B(t)\chi^N(t), \tag{3.1}$$

Where

$$\beta(t) \coloneqq \exp\left\{-\int_0^t r(s)ds\right\}, \tag{3.2}$$

$$\chi^N(t) \coloneqq \left(\frac{\psi(\tau)}{\lambda(\tau)} l_{\{\tau\le t\}} + l_{\{\tau>t\}}\right) e^{\int_0^{t\wedge\tau}(\lambda(s)-\psi(s))ds}, \tag{3.3}$$

and

$$\chi^B(t) \coloneqq \exp\left\{-\int_0^t \xi(s)dB(s) - \frac{1}{2}\int_0^t \xi^2(s)ds\right\}, \tag{3.4}$$

with

$$\xi(t) \coloneqq \frac{\mu(t)-r(t)}{\sigma(t)}, \qquad t \in [0, \mathrm{T}] \tag{3.5}$$

Here and hereafter, we denote the conditional expectation operator given F_t under the equivalent martingale measure Q by E_t^Q with $E^Q = E_0^Q$. Furthermore, in order to make the dependence on $\psi \in \wp$ explicit, we denote the state price density by $\chi_\psi(t) = \beta(t)\chi^B(t)\chi_\psi^N(t)$, $t \in [0, T]$, where

$$\chi_\psi^N(t) = \left(\frac{\psi(\tau)}{\lambda(\tau)} l_{\{\tau\le t\}} + l_{\{\tau>t\}}\right) e^{-\int_0^{t\wedge\tau}(\psi(s)-\lambda(s))ds}, \tag{3.6}$$

See also (3.3). Similarly, Q_ψ denotes the equivalent martingale measure associated with the state price density χ_ψ, that is given by $dQ_\psi/dP = \chi_\psi(T)/\beta(T)$.

Lemma 3.1. *If a consumption and wealth pair* (c, W) *is feasible, then it satisfies the following inequalities.*

$$\begin{cases} E^{Q_\psi}\left[\int_0^{\tau\wedge T} \beta(t)\big(c(t)-y(t)\big)dt + \beta(\tau\wedge T)\big(W(\tau\wedge T) - nX(\tau\wedge T)\big)\right] \le W_0 - np_0 \\ E_{\tau\wedge T}^{Q_\psi}\left[\int_{\tau\wedge T}^{T} \beta(t)c(t)dt + \beta(T)W(T)\right] \le \beta(\tau\wedge T)W(\tau\wedge T) \end{cases} \tag{3.7}$$

for each $\psi \in \wp$.

For each utility function $U_i(x)$, $i = 1, 2$ and each (s,t) such that $s \in [0, T]$ and $t \in [s, T]$, we denote by $I_s^{(i)}(x, t)$ the inverse function of

$$\frac{d}{dx}\left[U_i(x)e^{-\int_s^t \rho(s)ds}\right]$$

with respect to x. Similarly, for the function $V(x)$ defined in (2.5), we denote by $J(x)$ the inverse function of

$$\frac{d}{dx}\left[V(x)e^{-\int_0^{\tau\wedge T} \rho(s)ds}\right]$$

with respect to x. Under Assumption 2.2, for each s, t, the functions $I_s^{(i)}(x, t)$ $(i = 1, 2)$ and $J(x)$ exist, are continuous and strictly decreasing, and map $(0, \infty)$ onto itself. For each s, t, and i, we define the Legendre transformation $\bar{u}_s(z,t)$ and $\bar{V}$ by

$$\bar{u}_s^{(t)}(z,t) = sup_{c\ge 0}\left[e^{-\int_s^t \rho(u)du}U_i(c) - zc\right],\ \ t \in [0, \mathrm{T}], \mathrm{i} = 1,2, \tag{3.8}$$

$$\bar{V}(z) = \sup_{\omega\ge 0}\left[e^{-\int_0^{\tau\wedge T}\rho(s)ds}V(\omega) - z\omega\right] \tag{3.9}$$

Then we can be readily shown that $I_s^{(i)}(0+, t) = \infty$, $I_s^{(i)}(\infty, t) = 0$, $J(0+, t) = \infty$, $J(\infty, t) = 0$, and

$$\bar{u}_s^{(t)}(z,t) = e^{-\int_s^t \rho(u)du}U_i\left(I_s^{(i)}(z,t)\right) - zI_s^{(i)}(z,t),\ \ t \in [0, \mathrm{T}], \mathrm{i} = 1,2, \tag{3.10}$$

$$\bar{V}(z) = e^{-\int_0^{\tau\wedge T}\rho(s)ds}V\big(J(z)\big) - zJ(z), \tag{3.11}$$

Now, in order to solve the problem (**MP**), we consider the following dual optimization problem:

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$$(\boldsymbol{DP}) \quad \max_{n\in R+(\varsigma^{(n)},\psi^{(n)})\in R_{++}\chi\mathcal{P}} V_0(\varsigma^{(n)},\psi^{(n)}),$$

Where

$$V_0(\varsigma,\psi) = \mathbb{E}\left[\int_0^{\tau\wedge T}\tilde{u}_0^{(1)}(\varsigma\chi_\psi(t),t)dt + \bar{V}\left(\varsigma\chi_\psi(\tau\wedge T)\right) + \varsigma\left(W(0) - np_0 + \int_0^{\tau\wedge T}\chi_\psi(t)y(t)dt + \chi_\psi(\tau\wedge T)nX(\tau\wedge T)\right)\right] \quad (3.12)$$

Proposition 3.1. *For a given* $w > 0$, *Let* $Z(w)$ *be a solution of the equation*;

$$w = E_{\tau\wedge T}\left[\int_{\tau\wedge T}^{T}\chi(\tau\wedge T,t)I_{\tau\wedge T}^{(l)}\big(Z(w)\chi(\tau\wedge T,t)\big)dt + \chi(\tau\wedge T,T)I_{\tau\wedge T}^{(2)}(Z(w)\chi(\tau\wedge T,T))\right] \quad (3.13)$$

where, by recalling (3.1),

$$\chi(\tau\wedge T,t) = \frac{\beta(t)\chi^B(t)}{\beta(\tau\wedge T)\chi^B(\tau\wedge T)}, \quad t\in[\tau\wedge T,T]$$

Suppose that Assumptions 2.1 and 2.2 hold. Let n be a solution to* (**DP**) *satisfying*

$$\mathbb{E}_{\tau\wedge T}\left[\int_{\tau\wedge T}^{T}\chi(\tau\wedge T,t)I_{\tau\wedge T}^{(1)}\left(Z(J(\varsigma^*\chi_{\psi^*}(\tau\wedge T)))\chi(\tau\wedge T,t),t\right)dt + \chi(\tau\wedge T,T)I_{\tau\wedge T}^{(2)}\left(Z(J(\varsigma^*\chi_{\psi^*}(\tau\wedge T)))\chi(\tau\wedge T,T),T\right)\right] < \infty$$

and

$$\mathbb{E}^{Q_{\psi^*}}\left[\int_0^{\tau\wedge T}\beta(t)I_0^{(1)}\left(\varsigma^*\chi_{\psi^*}(t),t\right)dt + \beta(\tau\wedge T)J(\varsigma^*\chi_{\psi^*}(\tau\wedge T))\right] < \infty$$

where (ζ, ψ*) := argminV0 (ζ (n), ψ (n)). Then, n* agrees with an optimal share* $\hat{n}$ *of the insurance policy in* (**MP**) *and an optimal consumption process* $\hat{c}$ *and the corresponding wealth process* $\widehat{W}$ *are given, respectively, by*

$$\hat{c}(t) = \begin{cases} I_0^{(1)}\left(\varsigma^*\chi_{\psi^*}(t),t\right) & t\in[0,\tau\wedge T] \\ I_{\tau\wedge T}^{(1)}\left(Z(J(\varsigma^*\chi_{\psi^*}(t)))\chi(\tau\wedge T,t),t\right) & t\in[\tau\wedge T,T] \end{cases} \quad (3.14)$$

and

$$\widehat{W}(t) = \begin{cases} \frac{1}{\beta(t)}\mathbb{E}_t^{Q_{\psi^*}}\left[\int_t^{\tau\wedge T}\beta(s)(\hat{c}(s) - y(s))ds + \beta(\tau\wedge T)(W(\tau\wedge T) - n^*X(\tau\wedge T))\right] & if\ t\in[0,\tau\wedge T) \\ \frac{1}{\beta(t)}\mathbb{E}_t^{Q_{\psi^*}}\left[\int_t^{T}\beta(s)\hat{c}(s)ds + \beta(T)W(T)\right] & if\ t\in[\tau\wedge T,T] \end{cases} \quad (3.15)$$

With $\widehat{W}(\tau\wedge T) = J(\varsigma^*\chi_{\psi^*}(\tau\wedge T)$ *. Furthermore, ζ* satisfies*

$$\mathbb{E}^{Q_{\psi^*}}\left[\int_0^{\tau\wedge T}\beta(t)I_0^{(1)}\left(S^*\chi_{\psi^*}(t),t\right)dt + \beta(\tau\wedge T)\widehat{W}(\tau\wedge T)\right] = W_0 - n^*p_0 + \mathbb{E}^{Q_{\psi^*}}\left[\int_0^{\tau\wedge T}\beta(t)y(t)dt + n^*\beta(\tau\wedge T)X(\tau\wedge T)\right] \quad (3.16)$$

Proposition 3.2. *In addition to Assumptions 2.1 and 2.2, suppose that, for each* $x \in R^+$, $I_0^{(1)}(\zeta^* x, t)x$, $t \in [0, T]$, *and* $J(\zeta^* x)x$ *are convex w.r.t. x. Then, the optimal solution (ζ*, ψ*) of the problem* (**DP**) *is given by (ζ*, λ). Especially, ψ* is uniquely given by λ.*

Proposition 3.3. *Suppose that the assumptions of Proposition 3.2 hold. The optimal share n is given* $\hat{n}$ *as follows. If*

$$\mathbb{E}[\chi_\lambda(\tau\wedge T)X(\tau\wedge T)] > p_0$$

Then $\hat{n} = W_0/p_0$.

Otherwise, if

$$\mathbb{E}[\chi_\lambda(\tau\wedge T)X(\tau\wedge T)] < p_0,$$

Then $\hat{n} = 0$ *, otherwise* $\hat{n}$ *is indefinite.*

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