Mathematical modeling of the process of the deformation of an isotropic half-space under the action of distributed load at elastic fixing of the boundary

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***Abstract* – We explore the questions of numerical realization of the analytical solution of the problem of the theory of elasticity about an axisymmetric deformation of half-space under the action of normal load distributed over a circular area. We assume that the surface of half-space is elastically fixed outside the domain of application of the load, and the shear stresses on the boundary are absent. The algorithm for calculating of the vertical displacements at the points of the boundary plane is suggested. The influence of elastic fixing of surface of half-space on the distribution of displacements at the boundary is analyzed.**

Key words – the theory of elasticity, axisymmetric problem, analytical solution, displacements, numerical analysis, regularities.

I. Introduction

Boussinesq's problem about concentrated force, applied to an elastic half-space, is used often for the study of different engineering tasks. In [1] the distribution of the radial displacements at the border of a half-space is investigated on the base of the analytical solution of the mixed problem in the case, when the surface of the half-space is elastically fixed and a concentrated force acts to it. A solution of the axisymmetric problem about deformation of an isotropic half-space with boundary elastically fixed outside the circular area of the application of the distributed load is obtained in [2]. In this paper an algorithm for calculating the vertical displacements in elastically fixed domain of the boundary is developed on the base analytical solution of the axisymmetric problem [2]. The regularities of the distribution of displacements on the boundary are studied in the case when the load of constant intensity is applied in the circular domain. The problem has practical applications in the mining and building industries.

II. The mathematical formulation of problem. Analytical formulas for the displacements

We consider the axisymmetric problem about stressed - strained state of an isotropic half-space. On its boundary the distributed load acts to a circular area *V*. Outside the domain *V* normal stresses and displacements are proportionally. The beginning of cylindrical coordinate system r, θ, z is located in the center of the domain of the load application, the z axis will send straight up.

The boundary conditions for an elastic half-space $z \ge 0$ can be written as

$$\begin{aligned} &\sigma_z(r,0) = -q(r), \qquad r < a;\\ &\sigma_z(r,0) = kw(r), \qquad r > a; \\ &\tau_{rz}(r,0) = 0, \qquad r < \infty. \end{aligned} \tag{1}$$

Here $q(r)$ - the intensity of the distributed load, a - radius of circular domain of the application of the load, k - coefficient of proportionality normal stresses σ_z and displacements $w(r)$ at an elastically fixed part of the boundary, τ_{rz} - shear stresses.

According to [2], where an analytical solution of the axisymmetric problem with boundary conditions (1) is obtained, the formula for the distribution of vertical displacements in the plane $z = 0$ has the form

$$w(r,0) = \frac{2(1-\upsilon^2)}{E}\int_0^\infty \bar{\beta}(t) J_0(rt)\frac{tdt}{t+\chi}, \tag{2}$$

here E - Young's modulus, υ - Poisson's ratio, $J_0(rt)$ - zero-order Bessel function, $\bar{\beta}(t)$ – transformanta function $\beta(r)$ at an integral transformation of Hankel [3]

$$\bar{\beta}(t) = \int_0^\infty \beta(r) r J_0(rt) dr. \tag{3}$$

In the relation (2) the parameter χ is given by the equality

$$\chi = 2k(1-\upsilon^2)/E, \tag{4}$$

unknown function

$$\beta(r) = \begin{cases} q(r) + kw(r,0), & r < a; \\ 0, & r > a \end{cases} \tag{5}$$

is the solution of the integral equation

$$\beta(r) = q(r) + \int_0^a \beta(\xi) g_{1w}(\xi,r) d\xi, \quad r < a, \tag{6}$$

the core of which is written as follows:

$$g_{1w}(\xi,r) = \chi g_w(\xi,r)$$

$$g_w(\xi,r) = \xi \int_0^\infty J_0(\xi t) J_0(rt)\frac{tdt}{t+\chi}. \tag{7}$$

We substitute the expression (3) for $\bar{\beta}(t)$ in the equality (2) and taking into account the relation (5) transform (2) to the form

$$w(r,0) = \frac{2(1-\upsilon^2)}{E}\int_0^a \beta(\xi) g_w(\xi,r) d\xi. \tag{8}$$

Thus, at the boundary of the half-space the vertical displacements are determined by the formula (8), where unknown function $\beta(\xi)$ is the solution of the integral equation (6).

III. Numerical analysis of the distribution of displacements at the boundary of the half-space

The calculation of displacements with the help of the formula (8) is connected with overcoming of significant computational difficulties. It is clear from (7) that the expression for $g_w(\xi, r)$ contains an improper integral with infinite upper limit from the alternate signs function. Before calculating of the function $g_w(\xi, r)$ the singularities are studied in improper integral and it was converted to another form.

Numerical analysis of the problem is made for the case when to the boundary of the half-space uniformly distributed load q_0 is applied.

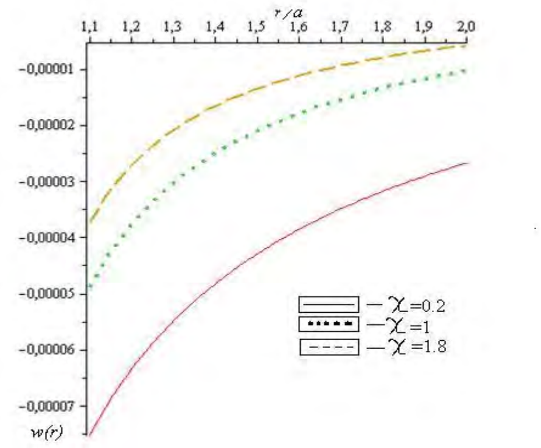
For the numerical solution of the integral equation (6) the method of successive approximations [4] is used. The values of the functions $\beta(r)$ are determined in the domain V for $0 < r < a$ with the help of the algorithm

$$\beta^{[0]}(r) = q_{0,}$$

$$\beta^{[j+1]} = q_0 + \int_0^a q_{1w}(r,\xi)\beta^{[j]}(\xi)d\xi, \quad j \ge 0.$$

Fig.1 illustrates the dependence of vertical

a)



b)

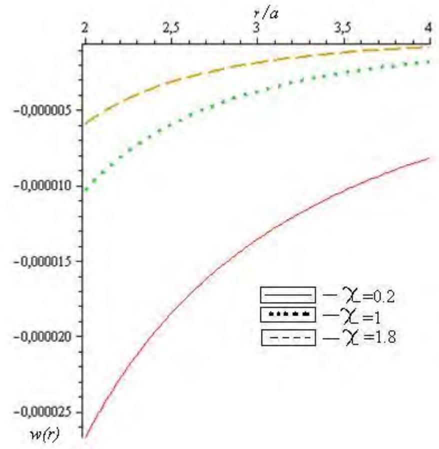


Fig. 1. Vertical displacements at the boundary of the half-space

displacements at the boundary of the half-space on the dimensionless radial coordinate r/a for various values of the parameter χ, given by formula (4).

The calculations of vertical displacements are carried out for the following values of parameters: $E = 10^4 MPa$; $\nu = 0.25$; $q_0 = 1MPa$; $a = 1m$; $\chi = 0.2; 1; 1.8 (m^{-1})$. The integral in formula (8) was calculated with the help of the trapeziums rule [5].

It is seen from fig.1 that with increasing of radial coordinate vertical displacements are reduced. Fig. 1b shows when $\frac{r}{a} > 2$ a tendency of convergence of curves is appeared. If $r/a \in [1;4]$ there is the regularity: the more χ, the less displacements $w(r)$. In the domain of elastic fixing of the border the displacements become maximum at $\frac{r}{a} = 1$. When parametr χ is changed from 0.2 to 1.8 the displacements are decreased in the point $\frac{r}{a} = 1$ approximately in 2 times, if $\frac{r}{a} = 2$ – in 4.5 times, and when $\frac{r}{a} = 4$, then in 30 times. Therefore, the influence of the parameter χ is increased with the motion away from the domain of application of the load.

Conclusion

The formula for calculation of vertical displacements at the boundary of an elastic half-space by fixing of its surface outside the domain of the load application is built. It is analyzed the influence of the parameter, characterizing of elastic properties of the half -space and fixing its surface, on the distribution of displacements at the boundary in case of a uniformly distributed load.

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