Numerical-Analitical Appoaches to the Calculation of Thermal Field in Parallelepiped Considering Inner Sources

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We compare the efficiency using indirect boundary and near-boundary elements methods for building numerical-analytical solution of three-dimensional stationary heat conduction problems. We built mathematical and discrete-continual models of problems with boundary conditions of the first kind, second kind and third kind using integral representations for the temperature.

Key words – heat field, thermal flat and three-dimensional internal sources, complex heat transfer, indirect near-boundary elements method, indirect boundary elements method.

I. Introduction

Modeling and optimization of thermal processes are essential in a variety of industries and technology particularly in instrument-making and mechanical engineering, in the design of microelectronic devices, a cover constructions and equipment by fireproof materials [7]. The basis of a mathematical model of a heat stationary process, as filtering incompressible fluids, electrostatics, serves as a differential equation in partial derivatives of elliptic type (the Laplace and the Poisson), supplemented by the boundary conditions of the first kind, the second kind, the third kind and mixed (at their combination).

Since exact analytical solutions of these problems can be obtained only for simple domains, numerical-analytical and numerical methods are used, which include finite difference and finite elements, integral and boundary integral equations, boundary and near-boundary elements, and others [1, 2, 4, 5, 9-10]. In the indirect boundary and near-boundary elements methods integrated image output differential equation written to a fold of its fundamental solution of intensities "fictitious" sources distributed on the edge of an object or external to it near-boundary region. By themselves, the intensity functions have no physical meaning, but when they are found, the value of the desired temperature inside the body can be obtained by using integration.

In this paper, by model stationary heat processes in the parallelepiped we compared indirect boundary and near-boundary elements methods. Discrete-continuous model for the intensities of the unknown source which are introduced onto the boundary or near-boundary elements and approximated by constant, is reduced to a system of linear algebraic equations (SLAE), formed as a result of satisfaction of boundary condition in collocation sence.

II. Mathematical model for finding the thermal field

We consider a homogeneous isotropic parallelepiped in Cartesian coordinate system x_1, x_2, x_3, in region

$$\Omega = \{(x_1, x_2, x_3) : a_1 < x_1 < a_2, b_1 < x_2 < b_2, \tilde{n}_1 < x_3 < \tilde{n}_2\} \quad (1)$$

with a boundary $\Gamma = \cup_{j=1}^{6} \Gamma^{(j)}$,

$$\Gamma^{(1)} = \{(x_1, x_2, x_3) : x_1 = a_1, b_1 < x_2 < b_2, c_1 < x_3 < c_2\},$$

$$\Gamma^{(2)} = \{(x_1, x_2, x_3) : x_1 = a_2, b_1 < x_2 < b_2, c_1 < x_3 < c_2\},$$

$$\Gamma^{(3)} = \{(x_1, x_2, x_3) : a_1 < x_1 < a_2, x_2 = b_1, c_1 < x_3 < c_2\},$$

$$\Gamma^{(4)} = \{(x_1, x_2, x_3) : a_1 < x_1 < a_2, x_2 = b_2, c_1 < x_3 < c_2\},$$

$$\Gamma^{(5)} = \{(x_1, x_2, x_3) : a_1 < x_1 < a_2, b_1 < x_2 < b_2, x_3 = c_1\},$$

$$\Gamma^{(6)} = \{(x_1, x_2, x_3) : a_1 < x_1 < a_2, b_1 < x_2 < b_2, x_3 = c_2\}.$$

To find the unknown temperature $\theta(x)$ we have an original equation

$$\sum_{i=1}^{n} \frac{\partial^2 \theta(x)}{\partial x_i^2} = -\psi(x)\chi_\psi, x \in \Omega \subset \mathbf{R}^3, \quad (2)$$

and boundary conditions of the first kind, second kind and third kind:

$$\theta(x) = f_\Gamma^{(1)}(x), x \in \partial\Omega^{(1)}, \quad (3)$$

$$-\lambda_0 \frac{\partial \theta(x)}{\partial \mathbf{n}(x)} = f_\Gamma^{(2)}(x), x \in \partial\Omega^{(2)}, \quad (4)$$

$$-\lambda_0 \frac{\partial \theta(x)}{\partial \mathbf{n}(x)} + \upsilon(x)\theta(x) = \upsilon(x) f_\Gamma^{(3)}(x), x \in \partial\Omega^{(3)}. \quad (5)$$

Where $\psi(x) = \tilde{\psi}(x)/\lambda_0$; $\tilde{\psi}(x)$ – the intensity sources in $\Omega_{\psi n} \subset \Omega$; $\Omega_{\psi n}$ – rectangle (n=2) or parallelepiped (n=3); χ_ψ – characteristic function in area Ω_ψ, for instence $\chi_\psi = 1$ at $x \in \chi_\psi$, $\chi_\psi = 0$ at $x \notin \chi_\psi$; $\partial\Omega^{(1)} \cup \partial\Omega^{(2)} \cup \partial\Omega^{(3)} = \Gamma$ – boundary of Ω; $f_\Gamma^{(1)}(x)$, $f_\Gamma^{(2)}(x)$ – functions describing the temperature and heat flow at the boundary; $f_\Gamma^{(3)}(x)$ – ambient temperature; λ_0, $\upsilon(x)$ – coefficient of the thermal conductivity of the material and coefficient of heat transfer from the surface of the object; $x = (x_1, x_2, x_3)$.

III. Integral representation of solution

To construct an algorithm for solving the problem (1)-(5) we use an indirect boundary (IBEM) [1] and indirect near-boundary elements methods (INBEM) [4]. According to the main provisions of these methods on the boundary of the object Γ or to the external near-boundary Ω area G we introduce unknown functions $\varphi^\gamma(x)$, $\gamma \in \{\Gamma, G\}$, describing the distribution of fictitious heat sources.

After the expansion of the function $\theta(x)$ domain on the whole R^3, equation (2) can be written as

$$\sum_{i=1}^{3} \frac{\partial^2 \theta(x)}{\partial x_i^2} = -\psi(x)\chi_\psi - \varphi^\gamma(x)\chi_\gamma, \ x \in \mathbf{R}^3, \quad (6)$$

where χ_γ – characteristic function of area γ, that is $\chi_\gamma = 1$ at $x \in \gamma$, $\chi_\gamma = 0$ at $x \notin \gamma$.

Eq. (6) $U(x,\xi)$ [3] integral representation of temperature and its normal's derivative [1, 4]:

$$\theta^\gamma(x) = \mathbf{F}^\gamma(x,U) + b(x,U), \quad (7)$$

$$-\lambda_0 \frac{\partial \theta^\gamma(x)}{\partial \mathbf{n}(x)} = \mathbf{F}^\gamma(x,Q) + b(x,Q),\ x \in \mathbf{R}^3, \quad (8)$$

where $\mathbf{F}^\gamma(x,\Phi) = \int_\gamma \Phi(x,\xi)\varphi^\gamma(\xi)d\gamma(\xi)$, $\Phi \in \{U,Q\}$,

$b(x,\Phi) = \int_{\Omega_{\psi n}} \Phi(x,\xi)\psi(\xi)d\Omega_{\psi n}(\xi)$, $\xi = (\xi_1,\xi_2,\xi_3)$,

$Q(x,\xi) = -\lambda_0 \dfrac{\partial U(x,\xi)}{\partial \mathbf{n}(x)}$, $\mathbf{n}(x) = (n_1(x), n_2(x), n_3(x))$ – outer unit vector normal to unambiguously defined boundaries Γ.

We slanted x in (7)-(8) from the middle of region Ω to boundary Γ to satisfy the boundary conditions (3)-(5), we obtain the boundary integral equation (BIE):

$$\mathbf{F}^\gamma(x,U) = f_\Gamma^{(1)}(x) - b(x,U),\ x \in \Gamma_1, \quad (9)$$

$$-\frac{1}{2}\varphi^\Gamma(x) + \mathbf{F}^\Gamma(x,Q) = f_\Gamma^{(2)}(x) - b(x,Q)$$

or

$$\mathbf{F}^G(x,Q) = f_\Gamma^{(2)}(x) - b(x,Q),\ x \in \Gamma_2, \quad (10)$$

$$-\frac{1}{2}\varphi^\Gamma(x) + \mathbf{F}^\Gamma(x, Q+\upsilon(x)U) = \upsilon(x) f_\Gamma^{(3)}(x) - b(x, Q+\upsilon(x)U)$$

or

$$\mathbf{F}^G(x, Q+\upsilon(x)U) = \upsilon(x) f_\Gamma^{(3)}(x) - b(x, Q+\upsilon(x)U),\ x \in \Gamma_3. \quad (11)$$

Due to arbitrariness of region Ω and function $\varphi^\gamma(\xi)$ and $\psi(\xi)$, it is almost impossible to perform an analytical integration in BIE (9)-(11) for applied problem. Let us perform three-dimensional discretization. Faces $\Gamma^{(j)}$ $(j = \overline{1,6})$ of the parallelepiped or corresponding near-boundary zones $G^{(j)}$ of region G we discretizat respectively on V with boundary or near-boundary elements γ_v, moreover $\cup_{v=1}^{V} \gamma_v = \gamma$; Γ_v, G_v – are flat and three-dimensional elements of the second order the 4 - s and 8-s nodes [4], they do not overlap each other. Thickness of the near-boundary elements, which are built on one boundary elements, is selected the same and equals h_j ($j = \overline{1,6}$). Then the unknown function, which describes the distribution of fictitious heat sources within the element is γ_v, is approximated with constant d_v^γ. Region $\Omega_{\psi n}$ we discretize with elements of the second order that have 4 or 8 nodes $\Omega_{\psi nq}$ $(q = 1,...,Q)$ when setting internal sources in a rectangular or parallelepiped respectively.

To satisfy the boundary conditions we use the collocation method. We choose the collocation points inside each boundary element Γ_w, $w = 1,...,V$, $\cup_{w=1}^{V} \Gamma_w = \Gamma$. After discretization the BIE (9)-(11) can be written in the form SLAE:

$$\sum_{v=1}^{V} A_v^\gamma(x^w,U)d_v^\gamma = f_\Gamma^{(1)}(x^w) - b(x^w,U),\ x^w \in \Gamma_1,$$

$$w = 1,...,V_1, \quad (12)$$

$$-\frac{1}{2}d_w^\Gamma + \sum_{v=1}^{V} A_v^\Gamma(x^w,Q)d_v^\Gamma = f_\Gamma^{(2)}(x^w) - b(x^w,Q),\ x^w \in \Gamma_2,$$

$$\text{or } \sum_{v=1}^{V} A_v^G(x^w,Q)d_v^G = f_\Gamma^{(2)}(x^w) - b(x^w,Q),\ x^w \in \Gamma_2,$$

$$w = V_1+1,...,V_2 \quad (13)$$

$$-\frac{1}{2}d_w^\Gamma + \sum_{v=1}^{V} A_v^\Gamma(x^w, Q+\nu(x^w)U)d_v^\Gamma = \nu(x^w) f_\Gamma^{(3)}(x^w) -$$

$$- b(x^w, Q+\nu(x^w)U),\ x^w \in \Gamma_3,\ \text{or}$$

$$\sum_{v=1}^{V} A_v^G(x^w, Q+\nu(x^w)U)d_v^G = \nu(x^w) f_\Gamma^{(3)}(x^w) -$$

$$- b(x^w, Q+\nu(x^w)U),\ x^w \in \Gamma_3,\ w = V_2+1,...,V, \quad (14)$$

where $A_v^\gamma(x,\Phi) = \int_{\gamma_v} \Phi(x,\xi)d\gamma_v(\xi)$.

After solving system (12)-(14), we obtain d_v^γ and use them in (7), (8) to find the temperature and heat flow:

$$\theta^\gamma(x) = \sum_{v=1}^{V} A_v^\gamma(x,U)d_v^\gamma + b(x,U), \quad (15)$$

$$-\lambda_0 \frac{\partial \theta^\gamma(x)}{\partial \mathbf{n}(x)} = \sum_{v=1}^{V} A_v^\gamma(x,Q)d_v^\gamma + b(x,Q). \quad (16)$$

IV. The numerical research

Problem 1. To compare obtained by IBEM and analytically distributions of thermal fields in the parallelepiped (1) with a set boundary conditions of the first kind:

$$\theta(x) = v_1,\ x \in \Gamma^{(1)},\ \ \theta(x) = v_2,\ x \in \Gamma^{(2)},$$

$$\theta(x) = 0,\ x \in \Gamma^{(j)},\ j = \overline{3,6}, \quad (17)$$

where $a_1 = 0, a_2 = 2$, $b_1 = 0, b_2 = 2$, $c_1 = 0, c_2 = 2$. Here and in future, all physical quantities are taken in the system SI, except for the temperature in degrees Celsius selected, $\lambda_0 = 1$ W/(m°C).

We approximate the unknown heat source by constant and compare the solution, obtained by IBEM for different numbers of boundary elements V, with the analytical solution [6]. Faces of the parallelepiped were discretizated in the same number of elements $V_\Gamma, V = 6V_\Gamma$. In the Table 1 the absolute error $\delta\theta(x) = \left|\theta^\Gamma(x) - \theta^a(x)\right|$ is shown at interior points of the parallelepiped, for $v_1 = 2$, $v_2 = 10$.

As we can see, with increasing of the number of boundary elements, the numerical solution in the internal points with the analytical one.

TABLE 1

COMPARISON OF ANALYTICAL AND INDIRECT BOUNDARY ELEMENTS METHOD SOLUTIONS

The coordinates	$\delta\theta(x)$, $V_\Gamma = 4$, V=24	$\delta\theta(x)$, $V_\Gamma = 36$, V=216
(0.5, 0.5, 0.5)	0.3746654121	0.0463688868
(0.5, 0.5, 1.0)	0.1682535106	0.0197977721
(0.5, 0.5, 1.5)	0.3746654121	0.0463688868
(0.5, 1.0, 0.5)	0.1682535106	0.0197977721
(0.5, 1.0, 1.0)	0.1331050888	0.0202940332
(0.5, 1.0, 1.5)	0.1682535106	0.0197977721
(0.5, 1.5, 0.5)	0.3746654121	0.0463688868
(0.5, 1.5, 1.0)	0.1682535106	0.0197977721
(0.5, 1.5, 1.5)	0.3746654121	0.0463688868
(1.0, 0.5, 0.5)	1.0082569230	0.1026429162
(1.0, 0.5, 1.0)	0.5805761182	0.0517733055
(1.0, 0.5, 1.5)	1.0082569230	0.1026429162
(1.0, 1.0, 0.5)	0.5805761182	0.0517733055
(1.0, 1.0, 1.0)	0.1526992651	0.0270093067
(1.0, 1.0, 1.5)	0.5805761182	0.0517733055
(1.0, 1.5, 0.5)	1.0082569230	0.1026429162
(1.0, 1.5, 1.0)	0.5805761182	0.0517733055
(1.0, 1.5, 1.5)	1.0082569230	0.1026429162
(1.5, 0.5, 0.5)	0.5834400108	0.1085786090
(1.5, 0.5, 1.0)	1.6058204488	0.2063962951
(1.5, 0.5, 1.5)	0.5834400108	0.1085786090
(1.5, 1.0, 0.5)	1.6058204488	0.2063962951
(1.5, 1.0, 1.0)	3.1731142048	0.3454613118
(1.5, 1.0, 1.5)	1.6058204488	0.2063962951
(1.5, 1.5, 0.5)	0.5834400108	0.1085786090
(1.5, 1.5, 1.0)	1.6058204488	0.2063962951
(1.5, 1.5, 1.5)	0.5834400108	0.1085786090

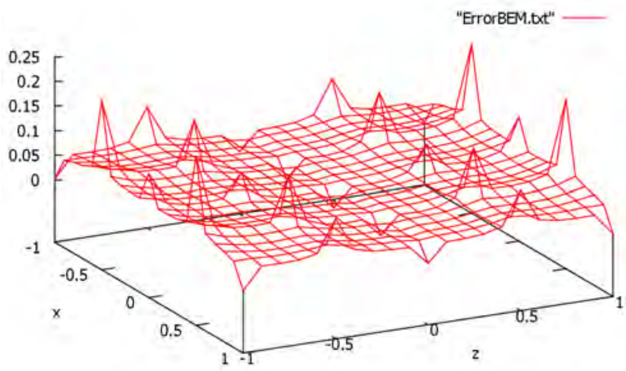
Problem 2. To compare obtained by IBEM and INBEM distributions of thermal fields in the parallelepiped (1) with boundary condition of the first kind:

$$\theta(x) = f(x) = x_2,\ x \in \Gamma, \quad (18)$$

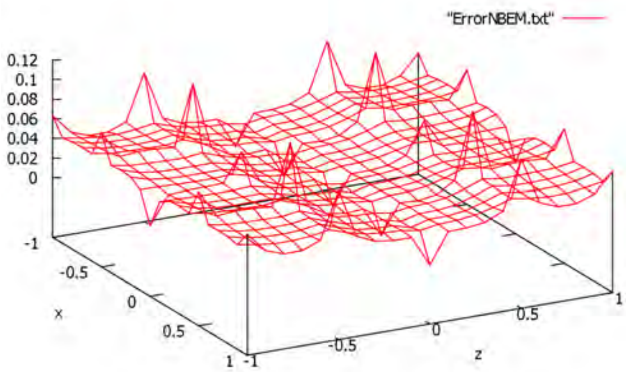
where $a_1 = -1, a_2 = 1,\quad b_1 = -1, b_2 = 1,\quad c_1 = -1, c_2 = 1$.

Since, using IBEM and INBEM we have the most errors when we approach the boundary (they are less in the middle of the solid by the maximum principle), on Fig. 1 we show the absolute error satisfying the boundary condition $\delta\theta^\gamma(x) = \left|\theta^\gamma(x) - f(x)\right|$ on the top face $\Gamma^{(4)}$ after solving this problem for $V_\Gamma = 16$, h_j=h=0.25 ($j = \overline{1,6}$).

As we can see, numerical results deteriorate in the points which are located near the boundary of the solid, also we observe growth of the error at the interface boundary and near-boundary elements (due to the choice of just one collocation point on the boundary element). When we increment number of discretization elements (corresponding to more accurate satisfying the boundary condition) we can see increased precision on the boundary and inside of the object. However, as shown in Fig. 1, INBEM allows achieves higher accuracy satisfying boundary conditions compared to IBEM, this is due to the possibility of changing the parameter h_v (thickness of near-boundary elements), which smooths the abrupt transition from the temperature on the boundary to the zero in outside.



a)



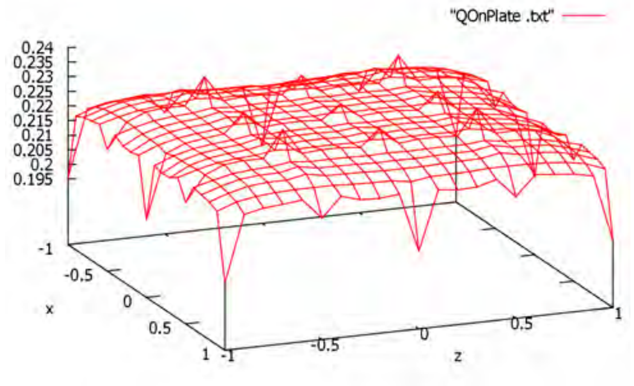
b)

Fig. 1. Error of satisfying the boundary condition on the top face of the parallelepiped in solving problem using: (a) IBEM and (b) INBEM

Problem 3. To estimate the effect of internal source, which is given in the form of a square plate $\Omega_{\psi 2} = \{(x_1, x_2, x_3): -d \le x_1 \le d,\ x_2 = 0.25,\ -d \le x_3 \le d\}$, by intensity $\tilde{\psi}(x) = \psi_g (1 + \cos(\frac{\pi x_1}{d}))(1 + \cos(\frac{\pi x_2}{d}))$, where $d = 0.5$, on the distribution of the thermal field in the parallelepiped (1) with the boundary condition of the first kind (18).

On Fig. 2 we compare distribution of thermal field of parallelepiped with a heat internal source and without one. This problem was solved by IBEM for $V_\Gamma = 16$, the number of discretization elements in internal plate is Q=4 (we note that it has no affect on dimension of the matrix of SLAE (12)).

As we can see, the temperature increases when parallelepiped is heated ($\psi_g = 0.5$) and decreases when it is cooled ($\psi_g = -0.5$) compared with the case where no internal source.



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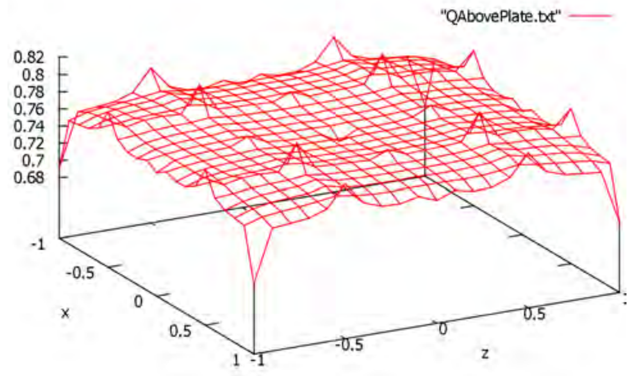

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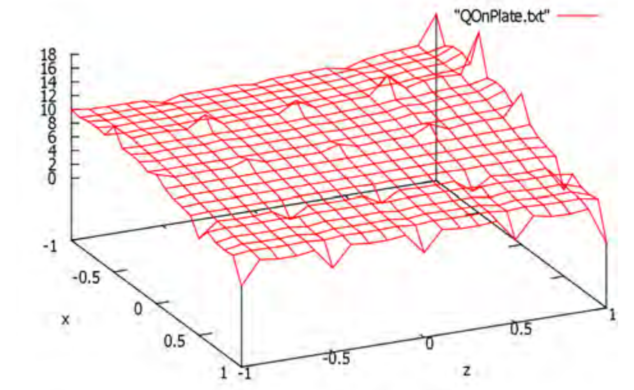

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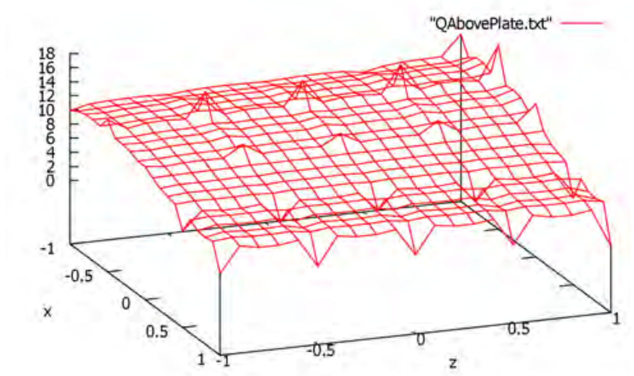

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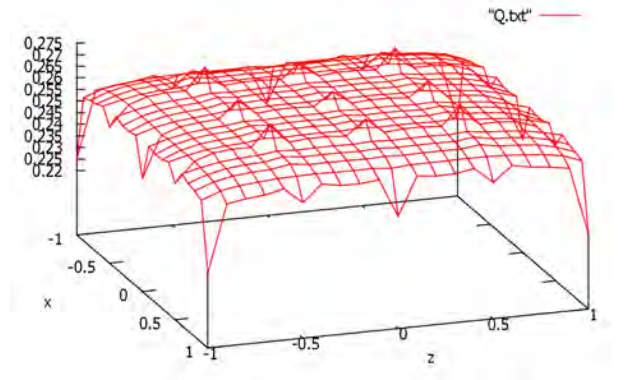

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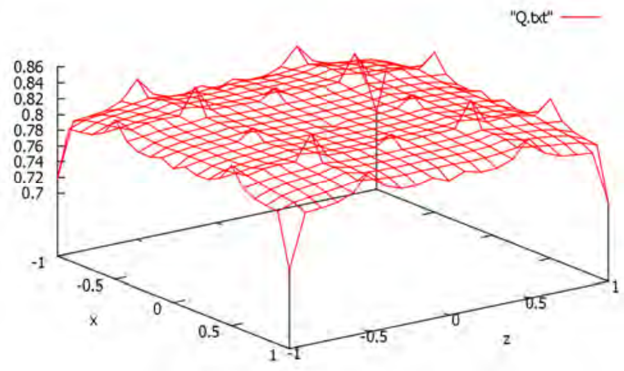


f)

Fig. 2. Temperature distribution in the parallelepiped with an internal heat source in the form of plate:for ψ_g =-0.5 (a) on the plate x_2 =0.25; (b) above the plate x_2 =0.75; for ψ_g =0.5- (c) on the plate x_2 =0.25; (d) above the plate x_2 =0.75; without internal source (e) x_2 =0.25 and (f) x_2 =0.75

Problem 4. To estimate the effect of internal source, which is given in the form of parallelepiped

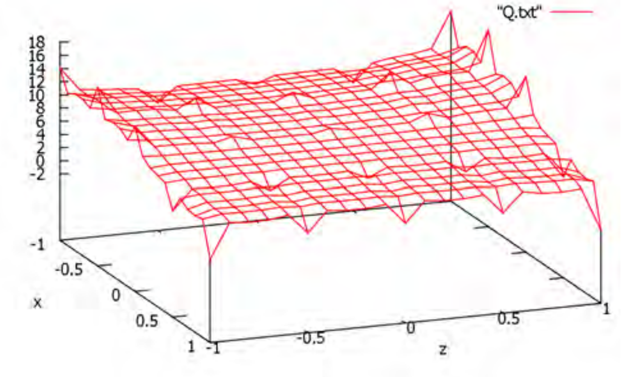
$$\Omega_{\psi 3} = \{(x_1, x_2, x_3) : -d_1 \le x_1 \le d_1, -d_2 \le x_2 \le d_2,$$
$$-d_3 \le x_3 \le d_3\}, \qquad (19)$$

with intensity

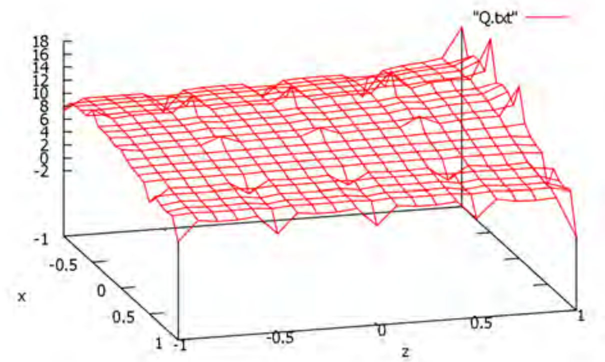
$$\tilde{\psi}(x) = \psi_g (1+\cos(\frac{\pi x_1}{d_1}))(1+\cos(\frac{\pi x_2}{d_2}))(1+\cos(\frac{\pi x_3}{d_3}), \qquad (20)$$

where $d_1 = d_2 = d_3 = 0.75$, on the distribution of the thermal field in the parallelepiped (1) with boundary condition of the first kind (18).

Fig. 3 shows the distribution of thermal field of parallelepiped with internal heat source of different intensity and without one, this problem was solved by IBEM for $V_\Gamma = 16$, the number of discretization elements in internal cube is Q=8.



a)



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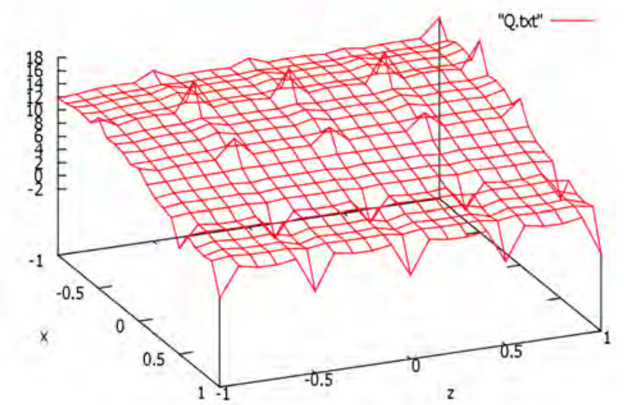

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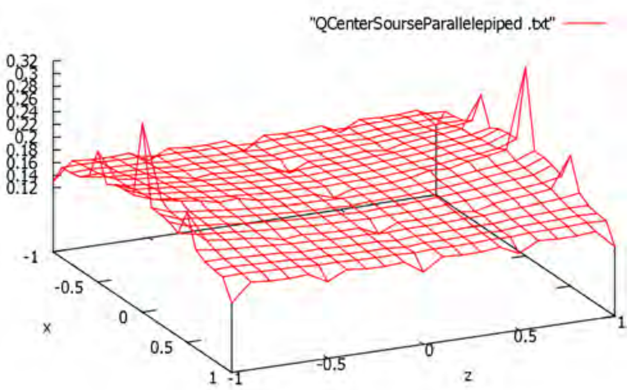

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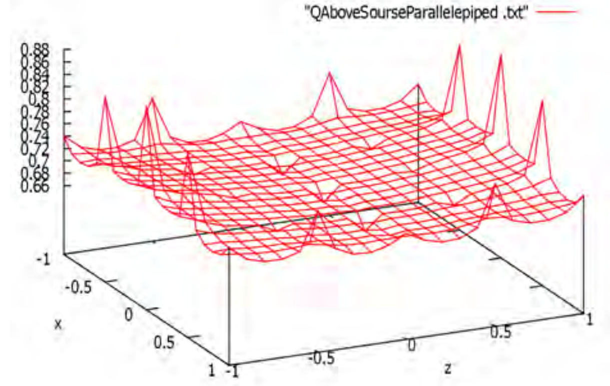

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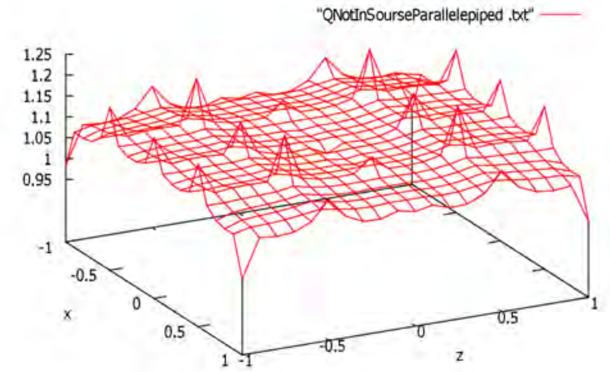


f)

Fig. 3. Temperature distribution in the parallelepiped with an internal heat source in the form of cube: for ψ_g =0.5 (a) on the plate, through the center of the source, x_2 =0.0; (b) on the plate in the middle of the source x_2 =0.5 and (c) on its boundary x_2 =0.75; for ψ_g =-0.5 (d-f) on the same planes

As we can see, temperature of the parallelepiped (ψ_g =0.5) is higher than in the previous case (when the source was flat) when volumetric source is heated, and it is lower than otherwise (ψ_g =-0.5).

Problem 5. To find the distribution of thermal field in the parallelepiped (1) with mixed boundary conditions:

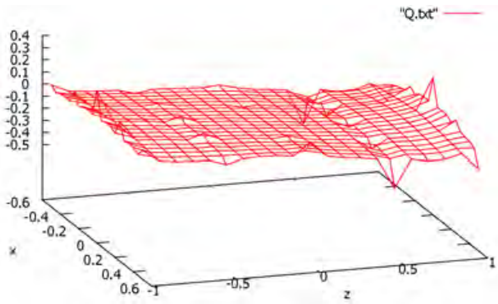
$$\theta(x)=10,\ x\in\Gamma^{(1)},\quad \theta(x)=0,\ x\in\Gamma^{(2)},$$

$$-\lambda_0\frac{\partial\theta(x)}{\partial\mathbf{n}(x)}+\upsilon_0\theta(x)=0,\ x\in\Gamma^{(j)},\ j=\overline{3,6},$$

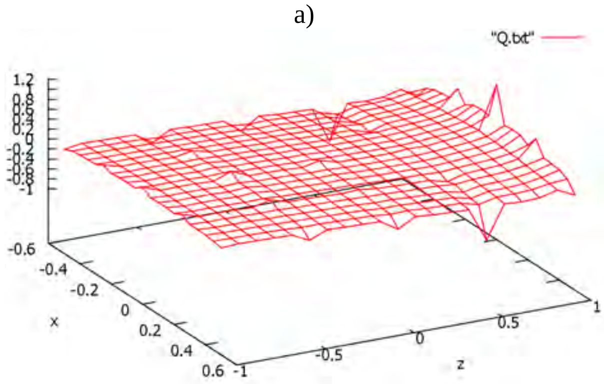
and volumetric internal source (19) with intensity (20), where

$a_1=-0.25, a_2=0.25, b_1=\tilde{n}_1=-0.75, b_2=\tilde{n}_2=0.75,$
ψ_g =-0.5, $\upsilon_0=1$ W/m² C, $d_1=0.25, d_2=d_3=0.75$.

Fig. 4 shows the distribution of thermal field after solving this problem by INBEM for $V_\Gamma=16$, $h_1=h_2=0.3$, $h_3=h_4=0.1$, $h_5=h_6=0.15$, the number of discretization elements in internal cube is Q=8.



a)



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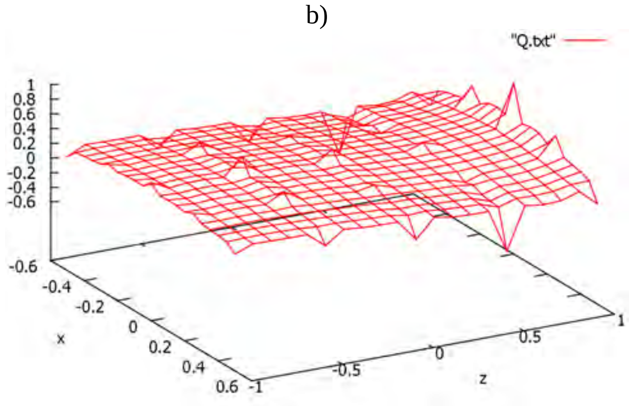

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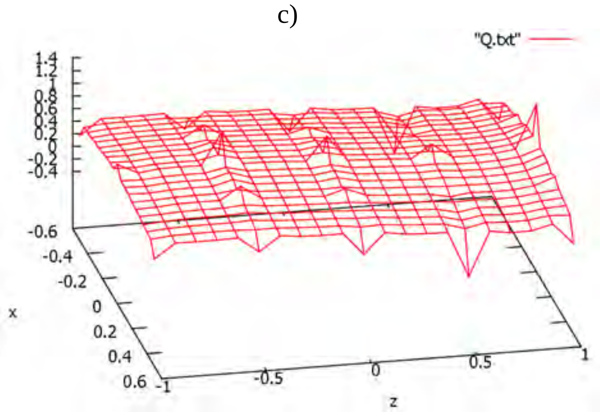

d)

Fig. 4. Temperature distribution in parallelepiped with an internal volumetric heat source on the plane, (a) through the center of the source, $x_2 = 0.0$; (b) on the planes in the middle of the source $x_2 = 0.5$, (c) on its boundary $x_2 = 0.7$, (d) beyond boundary $x_2 = 0.95$

Conclusion

We realized the approbation of the proposed approaches that are based on the combined use of the advantages of analytical and numerical methods. They include the fundamental solution of the Laplace equation, the basic idea of indirect boundary and near-boundary elements methods and collocation method. The error of satisfying the boundary conditions decreases when the number of boundary elements or near-boundary elements increases. However, complications of the procedure of numerical integration would significantly reduce the computational error, even with a smaller number of elements. Note also that the benefits of both approaches include the fact that they do not require differentiation of numerical values.

For calculation language C# was used. To visualize the results, we use Gnuplot.

The approaches can be extended for consideration of three-dimensional solid of arbitrary shape, then we must calculate integrals over 8 nodes of boundary element instead of 4 and 24 nodes of near-boundary element instead of 8.

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