The upper and lower bounds for solutions of general quadratic optimization problems

Anatolii Kosolap[1], Anastasiia Peretiatko[2]

[1]Specialized Computer Systems Department,
Ukrainian State University of Chemical Technology,
UKRAINE, Dnipropetrovsk, Gagarina Avenue 8,
E-mail: anivkos@ua.fm

[2]Specialized Computer Systems Department,
Ukrainian State University of Chemical Technology,
UKRAINE, Dnipropetrovsk, Gagarina Avenue 8,
E-mail: _nastya_@ua.fm

We consider the general problem of quadratic minimization with quadratic constraints. We are searching for the upper and lower bounds for the values of the minimized function. Semidefinite optimization is used for finding the lower bound. This lower bound is used to obtain an upper bound by interior point method. Numerical experiments often show that obtained upper bound is the exact solution of the original problem.

Key words – quadratic functions, semidefinite relaxation, semidefinite optimization, semidefinite simplex method, interior point method.

I. Introduction

Many problems in economy, finance, optimization of complex projects, planning, computer graphics, control of complex systems are converted to quadratic optimization problems in finite-dimensional space, where the objective function and constraints are defined by general quadratic functions. Such problems may contain many local minima and they are NP-hard. Feasible set of these problems can be nonconvex and discrete.

One of the common approaches for solving this class of problems is semidefinite relaxation [1-2]. In this case the quadratic function $x^T Ax$ is represented in the form Axx^T or $A \bullet X$, where A is symmetric matrix, and X is positive semidefinite matrix of rank one. This transformation allows to reduce the general quadratic problem to linear semidefinite optimization problem (SDP), in which the unknown variable is the semidefinite matrix. Semidefinite optimization problems can be effectively solved [3]. However, semidefinite relaxation is an approximate transformation (without the requirement that the rank of the matrix X is 1).

A primal-dual interior point method [3] was proposed for solving semidefinite optimization problems, but the search for more efficient algorithms is still continuing. In this paper we use semidefinite simplex method for solving semidefinite optimization problems [4].

II. Problem Statement

Let's consider the general quadratic optimization problem

$$\min\{ f_0(x) \mid f_i(x) \le 0,\, i = 1,\ldots, m,\, x \in E^n\}, \qquad (1)$$

where all functions $f_i(x) = x^T A_i x + b_i^T x - c_i$ are quadratic, b_i, x are vectors of n–dimensional Euclidean space, c_i are constants, and all of the matrices A_i are symmetric.

Let's use the semidefinite relaxation to transform the problem (1) to the form

$$\min\left\{ \overline{A}_0 \bullet X \mid \overline{A}_i \bullet X \preceq 0,\ i = 1,\ldots, m,\ X \succeq 0 \right\} \qquad (2)$$

where

$$X = \begin{pmatrix}1\\x\end{pmatrix}\begin{pmatrix}1\\x\end{pmatrix}^T = \begin{pmatrix}1 & x^T\\ x & xx^T\end{pmatrix},$$

$$x^T A_i x + b_i^T x - c_i = \begin{pmatrix}-c_i & \frac{b_i^T}{2}\\ \frac{b_i}{2} & A_i\end{pmatrix} \bullet \begin{pmatrix}1 & x^T\\ x & xx^T\end{pmatrix} =$$

$$= \begin{pmatrix}-c_i & \frac{b_i^T}{2}\\ \frac{b_i}{2} & A_i\end{pmatrix} \bullet X = \overline{A}_i \bullet X,\ \ i = 0,1,\ldots, m,$$

and $A \bullet X = \sum\sum a_{ij} x_{ij}$ defines the inner product of symmetric matrices.

The transformed problem (2) is equivalent to the problem (1) if the matrix X is semidefinite matrix of rank one. However, the condition that the matrix should be of rank one cannot be set analytically. Therefore, we solve the problem (2) without this condition. Then the solution of the problem (2) defines a lower bound for the solution of the problem (1). The solution of the problem (2) X^* determines the exact solution of the problem (1) if X^* is semidefinite one-rank matrix.

III. Problem-Solving Methods

Methods for solving the problem (2) are studied in various papers [1-3]. The best method is the interior point method [3]. However, it allows to find the solution of the problem (2) with less accuracy than a new semidefinite simplex method, which uses a local approximation of a semidefinite cone by the sum of one-rank matrices. Moreover, the interior point method solves the problem (2) with equality constraints. When you convert inequalities into equalities by putting free variables, a size of the solving problem will be increased on the number of new variables.

Unlike the usual simplex method, in semidefinite simplex method we solve a sequence of linear programming problems. At each iteration we define a new column of the constraint matrix from the solution of simple quadratic optimization problem

$$\min\{x^T Qx \mid \| x \|^2 = 1\}, \qquad (3)$$

where

$$Q = C - \sum_j C \bullet x_i x_i^T \sum_{j=1}^{m} b_{ij}^{-1} A_j,$$

b_{ij}^{-1} are elements of the basic matrix B^{-1} of simplex method. It is well known that the problem (3) is effectively solved [5]. It is obviously that the solution of the problem (3) coincides with the solution of the problem

$$\min\{x^T Qx + r(\| x \|^2 - 1) \mid \| x \|^2 = 1\}.$$

Let's choose $r > 0$ so that the matrix $Q^* = Q + rI$ is positive definite. It's enough

$$q_{ii}^* > \sum_{i \neq j} | q_{ij}^* |, \forall i,$$

where q_{ij}^* are the elements of matrix Q^*. Thus, the solution of the problem (10) reduces to finding the eigenvector of matrix Q^* that corresponds to its minimum eigenvalue. It is equivalent to the solution of the problem

$$\min\{x^T Q^* x \,|\, \|x\|^2 = 1\}$$

or the problem

$$\max\{\|x\|^2 \,|\, x^T Q^* x = 1\}. \quad (4)$$

If x^* is the solution of the problem (4) then the matrix Q is positive semidefinite if $x^{*T} Q x^* \geq 0$. In this case the problem (2) is solved; otherwise the search for the solution of the problem (2) by the simplex method will be continued.

The solution of the problem (2) we use as a starting point to solve the problem (1) by the primal-dual interior point method [6]. It was shown that this method converges to a local minimum in polynomial time [6].

Numerical experiments show that this method often can find the point of global minimum of the problem (1) if we use solution of the problem (2) as a starting point for the problem (1).

IV.Numerical Results

Software for the proposed method was developed and numerical experiments were performed.

Let's consider some test problems from [7] and [8]. At first we find a lower bound of the problems by semidefinite simplex method, and then we search the upper bound. The results of numerical experiments are presented in Table I and Table II. It is shown that we receive the exact solution of the original problem in about 90% of problems. In the problems *fp*_2_1 [7] and *g*15 [8] we couldn't find the point of global minimum.

TABLE 1

THE RESULTS OF NUMERICAL EXPERIMENTS FROM [7]

Problem's name in [7]	Dimension	Lower bound	Upper bound	Optimal solution
fp_2_1	6*7	-18,86	-16,5	-17
fp_2_2	7*9	-213	-213	-213
fp_2_4	7*12	-23.71	-11	-11
fp_3_3	7*13	-438	-310	-310
fp_3_4	3*6	-5	-4	-4
e_1	3*4	-3	-3	-3
f_a	3*5	-5,98	-1,083	-1,083
f_b	2*3	-11,99	-8,5	-8,5
f_c	5*11	-13	-13	-13
f_f	2*6	-2,828	-2,828	-2,828
s_1	3*5	0	0,74	0,74
s_1b	3*5	0	0,74	0,74
s_1c	3*5	0,69	0,74	0,74
s_1d	3*5	0,4	0,74	0,74
s_2	3*4	-1,5	-0,5	-0,5
s_2b	3*4	-1,5	-0,5	-0,5
s_2c	3*4	-0,54	-0,5	-0,5
s_2d	3*4	-0,938	-0,5	-0,5

TABLE 2

THE RESULTS OF NUMERICAL EXPERIMENTS FROM [8]

Problem's name in [8]	Dimension	Lower bound	Upper bound	Optimal solution
g01	13*22	-15	-15	-15
g04	5*11	-32232	-30665	-30665
g07	10*18	24,3064	24,3062	24,3062
g11	3*4	0,75	0,75	0,75
g15	3*2	943,985	-	961,715
g18	10*23	-0,866	-0,866	-0,866

Conclusion

In this paper we use new methods for searching the upper and lower bounds of solutions of general quadratic optimization problems. The numerical experiments for well-known test problems showed that the upper bound of the global minimum is accurate for the majority of these problems.

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