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POSITIVE FRACTIONAL AND CONE FRACTIONAL LINEAR SYSTEMS

Tadeusz Kaczorek
Bialystok University of Technology, Poland
kaczorek@isep.pw.edu.pl

Abstract: The positive fractional and cone fractional continuous-time and discrete-time linear systems are addressed. Sufficient conditions for the reachability of positive and cone fractional continuous-time linear systems are given. Necessary and sufficient conditions for the positivity and asymptotic stability of the continuous-time linear systems are established. The realization problem for positive fractional continuous-time systems is formulated and solved.

Key words: cone, continuous-time, fractional, positive, realization problem.

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear systems behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. The overview of a state-of-the-art situation in the field of positive systems is given in the monographs [8, 9]. The stability and robust stability of positive and fractional 1D linear systems has been investigated in many papers and books [1-9, 13, 23, 28]. Realization problem of a positive continuous-time and discrete-time linear system has been considered in [10, 12-15, 19, 20, 22]. Recently, the reachability, controllability and minimum energy control of positive linear discrete-time systems with time-delays have been considered in [9, 16-18, 21, 24].

The first definition of the fractional derivative was introduced by Liouville and Riemann at the end of the 19th century [50-52, 54, 55]. This idea was used by engineers for modeling different processes in the late 1960s. Mathematical fundamentals of fractional calculus are given in the monographs [23, 25-30]. The fractional order controllers were developed in [29]. Some other applications of fractional order systems can be found in [31, 32].

The main purpose of this paper is to give an overview of some recent results on positive and cone fractional continuous-time and discrete-time linear systems.

The paper is arranged as follows. In section 2 the positive fractional linear continuous-time systems are introduced. In section 3 the fractional cone systems are discussed. Sufficient conditions for the reachability are established in section 4. The realization problem for positive fractional continuous-time linear system is investigated in section 5. Positive fractional discrete-time linear systems are addressed in section 6. Sufficient conditions for the reachability of discrete-time linear systems are established in section 7. Concluding remarks are given in section 8.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ - the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n=\Re_+^{n\times 1}$, M_n - the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n - the $n\times n$ identity matrix.

2. Positive fractional continuous-time linear systems

The following Caputo definition of the fractional derivative will be used [23, 25, 27, 29]

$$\frac{d^\alpha}{dt^\alpha}f(t)=\frac{1}{\Gamma(k-\alpha)}\int_0^t\frac{f^{(k)}(\tau)}{(t-\tau)^{\alpha+1-k}}d\tau, \tag{1}$$

$$k-1<\alpha\le k\in N=\{1,2,...\}$$

where $\alpha\in\Re$ is the order of fractional derivative and $f^{(n)}(\tau)=\frac{d^k f(\tau)}{d\tau^k}$.

Consider the continuous-time fractional linear system described by the state equations

$$\frac{d^\alpha}{dt^\alpha}x(t)=Ax(t)+Bu(t),\quad 0<\alpha\le 1, \tag{2a}$$

$$y(t)=Cx(t)+Du(t), \tag{2b}$$

where $x(t)\in\Re^n$, $u(t)\in\Re^m$, $y(t)\in\Re^p$ are the state, input and output vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$, $D\in\Re^{p\times m}$.

Theorem 1. [23] The solution of equation (2.2a) is given by

$$x(t)=\Phi_0(t)x_0+\int_0^t\Phi(t-\tau)Bu(\tau)d\tau,\quad x(0)=x_0\ , \tag{3}$$

where

$$\Phi_0(t) = E_\alpha(At^\alpha) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}, \tag{4}$$

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \tag{5}$$

and $E_\alpha(At^\alpha)$ is the Mittag-Leffler matrix function, $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ is the gamma function.

Definition 1. [23] The system (2) is called the internally positive fractional system if and only if $x(t) \in \Re_+^n$ and $y(t) \in \Re_+^p$ for $t \ge 0$ for any initial conditions $x_0 \in \Re_+^n$ and all inputs $u(t) \in \Re_+^m$, $t \ge 0$.

Theorem 2. [23] The continuous-time fractional system (2) is internally positive if and only if the matrix A is a Metzler matrix and

$$A \in M_n,\ B \in \Re_+^{n\times m},\ C \in \Re_+^{p\times n},\ D \in \Re_+^{p\times m}. \tag{6}$$

3. Cone fractional systems

Following [10, 23] the definitions are recalled.

Definition 2. Let $P = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} \in \Re^{n\times n}$ be nonsingular and p_k be the k-th ($k = 1,2,\dots,n$) its row.

The set

$$\mathcal{P} := \left\{ x \in \Re^n : \bigcap_{k=1}^{n} p_k x \ge 0 \right\} \tag{7}$$

is called the linear cone generated by the matrix P.

In a similar way we may define the linear cone

$$\mathcal{Q} := \left\{ u \in \Re^m : \bigcap_{k=1}^{m} q_k u \ge 0 \right\} \tag{8}$$

generated by the nonsingular matrix $Q = \begin{bmatrix} q_1 \\ \vdots \\ q_m \end{bmatrix} \in \Re^{m\times m}$ for the inputs u, and the linear cone

$$\mathcal{V} := \left\{ y \in \Re^p : \bigcap_{k=1}^{p} v_k y \ge 0 \right\} \tag{9}$$

generated by the nonsingular matrix $V = \begin{bmatrix} v_1 \\ \vdots \\ v_p \end{bmatrix} \in \Re^{p\times p}$ for the outputs y.

Definition 3. The fractional system (2) is called $(\mathcal{P}, \mathcal{Q}, \mathcal{V})$ cone fractional system if $x(t) \in \mathcal{P}$ and $y(t) \in \mathcal{V}$, $t \ge 0$ for every $x_0 \in \mathcal{P}$, $u(t) \in \mathcal{Q}$, $t \ge 0$.

The $(\mathcal{P}, \mathcal{Q}, \mathcal{V})$ cone fractional system (2) will be shortly called the cone fractional system. Note that if $\mathcal{P} = \Re_+^n$, $\mathcal{Q} = \Re_+^m$, $\mathcal{V} = \Re_+^n$, then the $(\Re_+^n, \Re_+^m, \Re_+^p)$ cone system is equivalent to the classical positive system [18, 26].

Theorem 3. The fractional system (2) is $(\mathcal{P}, \mathcal{Q}, \mathcal{V})$ a cone fractional system if and only if

$$\bar{A} = PAP^{-1} \in \Re_+^{n\times n},\ \bar{B} = PBQ^{-1} \in \Re_+^{n\times m},$$

$$\bar{C} = VCP^{-1} \in \Re_+^{p\times n},\ \bar{D} = VDQ^{-1} \in \Re_+^{p\times m}. \tag{10}$$

Proof is given in [23, 17].

3. Reachability of positive fractional systems

Definition 4. The state $x_f \in \Re_+^n$ of the fractional system (2) is called reachable in time t_f if there exists an input $u(t) \in \Re_+^m$, $t \in [0, t_f]$ which steers the state of system (2) from zero initial state $x_0 = 0$ to the state x_f. If every state $x_f \in \Re_+^n$ is reachable in time t_f, the system is called reachable in time t_f. If for every state $x_f \in \Re_+^n$ there exists such a time t_f that the state is reachable in time t_f, the system (2) is called reachable.

A real square matrix is called monomial if and only if each its row and column contains only one positive entry and the remaining entries are zero.

Theorem 4. The continuous-time fractional system (2) is reachable in time t_f if the matrix

$$R(t_f) = \int_0^{t_f} \Phi(\tau) BB^T \Phi^T(\tau) d\tau \tag{11}$$

is a monomial matrix.

The input which steers the state of the system (2) from $x_0 = 0$ to x_f is given by the formula

$$u(t) = B^T \Phi^T(t_f - t) R^{-1}(t_f) x_f \tag{12}$$

where T denotes the transposition. A proof is given in [21].

Definition 5. A state $x_f \in \mathcal{P}$ of the cone fractional system (2) is called reachable in time t_f if there exists an input $u(t) \in \mathcal{Q}$, $t \in [0, t_f]$ which steers the state of the system from zero initial state $x_0 = 0$ to the desired state x_f, i.e. $x(t_f) = x_f$. If every state $x_f \in \mathcal{P}$ is reachable in time t_f, then the cone fractional system is called reachable in time t_f. If for every state $x_f \in \mathcal{P}$ there exists a time t_f, then the cone fractional system is called reachable.

Theorem 5. The cone fractional system (2) is reachable in time t_f if and only if the matrix

$$\bar{R}(t_f) =$$

$$= P\int_0^{t_f} \Phi(\tau)BQ^{-1}Q^{-T}B^T\Phi^T(\tau)d\tau P^T \, (Q^{-T} = (Q^{-1})^T) \quad (13)$$

is a monomial matrix. A proof is given in [21].

From Theorem 5 we have the following corollary.

Corollary 1. If $Q = I_m$, then $\bar{R}(t_f) = PR(t_f)P^T$, and the cone fractional system (2) is reachable in time t_f if the positive fractional system is reachable and P is a monomial matrix.

Example 1. Consider the cone fractional system (2) with

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},\ Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad (14)$$

The $\mathcal{P}$-cone generated by the matrix P is shown in Fig. 1.

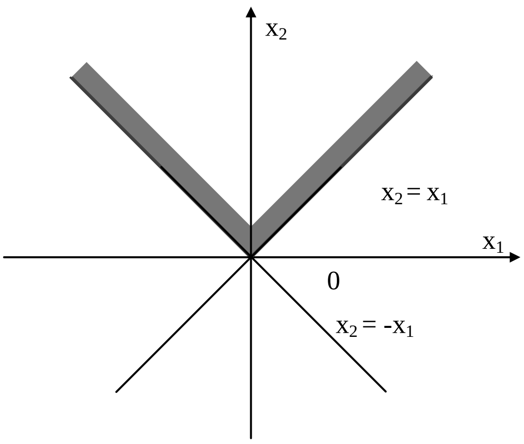

Fig. 1. $\mathcal{P}$-cone.

It is easy to show that

$$\Phi(t)B = \begin{bmatrix} 0 & \Phi_1(t) \\ \Phi_2(t) & 0 \end{bmatrix} \quad (15)$$

and

$$R(t_f) = \int_0^{t_f} \Phi(\tau)BB^T\Phi^T(\tau)d\tau = \int_0^{t_f} \begin{bmatrix} \Phi_1^2(\tau) & 0 \\ 0 & \Phi_2^2(\tau) \end{bmatrix} d\tau \quad (16)$$

where

$$\Phi_1(t) = \sum_{k=0}^{\infty} \frac{t^{(k+1)\alpha-1}}{[\Gamma(k+1)\alpha]},\quad \Phi_2(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)},\quad 0<\alpha<1 \quad (17)$$

The matrix (16) is monomial and according to Theorem 4 the positive fractional system is reachable in time t_f.

In the case $Q = I_2$ the matrix

$$\bar{R}(t_f) = PR(t_f)P^T =$$

$$= \int_0^{t_f} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \Phi_1^2(\tau) & 0 \\ 0 & \Phi_2^2(\tau) \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} d\tau =$$

$$= \int_0^{t_f} \begin{bmatrix} \Phi_1^2(\tau)+\Phi_2^2(\tau) & \Phi_2^2(\tau)-\Phi_1^2(\tau) \\ \Phi_2^2(\tau)-\Phi_1^2(\tau) & \Phi_1^2(\tau)+\Phi_2^2(\tau) \end{bmatrix} d\tau \quad (18)$$

is not monomial, since $\Phi_1^2(\tau) \neq \Phi_2^2(\tau)$.

Therefore, the sufficient condition for the reachability in time t_f of Theorem 5 is not satisfied.

From this example and comparison of (11) and (13) it follows that the sufficient condition for the reachability of the cone fractional systems is much stronger than for the positive fractional systems.

A state $x_0 \in \mathcal{P}$ of the cone fractional system (2) is called controllable to zero in time t_f if there exist an input $u(t) \in \mathcal{Q}$, $t \in [0, t_f]$ which steers the state of the system from x_0 to the zero state x_f = 0 Following [26] it is possible to extend the considerations to the controllability to zero of the cone fractional linear system.

4. Realization problem for positive fractional systems

Consider the continuous-time fractional linear system described by the state equations

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bu(t), \quad 0<\alpha\le 1 \quad (19a)$$

$$y(t) = Cx(t) + Du(t) \quad (19b)$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $y(t) \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$.

Applying the Laplace transform to (19), it is easy to show that the transfer matrix of the system is given by the formula

$$T(s) = C[I_n s^\alpha - A]^{-1}B + D. \quad (20)$$

The transfer matrix is called proper if and only if

$$\lim_{s\to\infty} T(s) = K \in \Re^{p\times m} \quad (21)$$

and it is called strictly proper if and only if $K = 0$.

From (20) we have

$$\lim_{s\to\infty} T(s) = D \quad (22)$$

since

$$\lim_{s\to\infty} [I_n s^\alpha - A]^{-1} = 0. \quad (23)$$

Definition 6. Matrices A, B, C, D are called a positive fractional realization of a given transfer matrix

$T(s)$ if they satisfy the equality (20). A realization is called minimal if the dimension of A is minimal among all realizations of $T(s)$.

The positive fractional realization problem can be stated as follows. Being given a proper transfer matrix $T(s)$, find its positive realization.

First the realization problem will be solved for single-input single-output (SISO) linear fractional systems with the proper transfer function

$$T(s) = \frac{b_n (s^\alpha)^n + b_{n-1}(s^\alpha)^{n-1} + \ldots + b_1 s^\alpha + b_0}{(s^\alpha)^n - a_{n-1}(s^\alpha)^{n-1} - \ldots - a_1 s^\alpha - a_0} \tag{24}$$

Using (22), we obtain

$$D = \lim_{s \to \infty} T(s) = b_n , \tag{25}$$

and the strictly proper transfer function has the form

$$T_{sp}(s) = T(s) - D = \\ = \frac{\overline{b}_{n-1}(s^\alpha)^{n-1} + \overline{b}_{n-2}(s^\alpha)^{n-2} + \ldots + \overline{b}_1 s^\alpha + \overline{b}_0}{(s^\alpha)^n - a_{n-1}(s^\alpha)^{n-1} - \ldots - a_1 s^\alpha - a_0} \tag{26}$$

where

$$\overline{b}_k = b_k + a_k b_n, \quad k = 0,1,\ldots,n-1 . \tag{27}$$

From (27) it follows that if $a_k \geq 0$ and $b_k \geq 0$ for $k = 0,1,\ldots,n$, then also $\overline{b}_k \geq 0$ for $k = 0,1,\ldots,n-1$.

Theorem 6. There exist positive fractional minimal realizations of the forms

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \\ a_0 & a_1 & a_2 & \ldots & a_{n-1} \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{28a}$$
$$C = \begin{bmatrix} \overline{b}_0 & \overline{b}_1 & \ldots & \overline{b}_{n-1} \end{bmatrix}, \; D = b_n ,$$

$$A = \begin{bmatrix} 0 & 0 & \ldots & 0 & a_0 \\ 1 & 0 & \ldots & 0 & a_1 \\ 0 & 1 & \ldots & 0 & a_0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & a_{n-1} \end{bmatrix}, \; B = \begin{bmatrix} \overline{b}_0 \\ \overline{b}_1 \\ \vdots \\ \overline{b}_{n-1} \end{bmatrix}, \tag{28b}$$
$$C = \begin{bmatrix} 0 & 0 & \ldots & 1 \end{bmatrix}, \; D = b_n ,$$

$$A = \begin{bmatrix} a_{n-1} & a_{n-2} & \ldots & a_1 & a_0 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{28c}$$
$$C = \begin{bmatrix} \overline{b}_{n-1} & \overline{b}_{n-2} & \ldots & \overline{b}_0 \end{bmatrix}, \; D = b_n ,$$

$$A = \begin{bmatrix} a_{n-1} & 1 & 0 & 0 & 0 \\ a_{n-2} & 0 & 1 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_1 & 0 & 0 & \ldots & 1 \\ a_0 & 0 & 0 & \ldots & 0 \end{bmatrix}, \; B = \begin{bmatrix} \overline{b}_{n-1} \\ \overline{b}_{n-2} \\ \vdots \\ \overline{b}_0 \end{bmatrix}, \tag{28d}$$
$$C = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}, \; D = b_n$$

of the transfer function (4.4) if

i) $b_k \geq 0$ for $k = 0,1,\ldots,n$,

ii) $a_k \geq 0$ for $k = 0,1,\ldots,n-2$ and $b_{n-1} + a_{n-1} b_n \geq 0$.

Proof is given in [37].

The matrices (28) are minimal realizations if and only if the transfer function (24) is irreducible.

If the conditions of Theorem 6 are satisfied then the positive minimal realizations (28) of the transfer function (24) can be computed by use of the following procedure.

Procedure 1.

Step 1. Knowing $T(s)$ and using (25), find D and the strictly proper transfer function (26).

Step 2. Using (28), find the desired realizations.

Example 1. Find the positive minimal fractional realizations (28) of the irreducible transfer function

$$T(s) = \frac{2(s^\alpha)^2 + 5 s^\alpha + 1}{(s^\alpha)^2 + 2 s^\alpha - 3} . \tag{29}$$

Using Procedure 1 and (29) we obtain the following:

Step 1. From (25) and (29) we have

$$D = \lim_{s \to \infty} \frac{2(s^\alpha)^2 + 5 s^\alpha + 1}{(s^\alpha)^2 + 2 s^\alpha - 3} = 2 \tag{30}$$

and

$$T_{sp}(s) = T(s) - D = \frac{s^\alpha + 7}{(s^\alpha)^2 + 2 s^\alpha - 3} . \tag{31}$$

Step 2. Taking into account that in this case $\overline{b}_0 = 7, \overline{b}_1 = 1$ and using (28), we obtain the desired positive minimal fractional realizations

$$A = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 7 & 1 \end{bmatrix}, \; D = 2 , \tag{32a}$$

$$A = \begin{bmatrix} 0 & 3 \\ 1 & -2 \end{bmatrix}, \; B = \begin{bmatrix} 7 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \; D = 2 , \tag{32b}$$

$$A = \begin{bmatrix} -2 & 3 \\ 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 7 \end{bmatrix}, \; D = 2 , \tag{32c}$$

$$A = \begin{bmatrix} -2 & 1 \\ 3 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 7 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \; D = 2 . \tag{32d}$$

Let's consider a multi-input multi-output (MIMO) positive fractional system (19) with a proper transfer matrix $T(s)$.

Using the formula

$$D = \lim_{s \to \infty} T(s) \tag{33}$$

we can find the matrix D and the strictly proper transfer matrix which can be written in the form

$$T_{sp}(s) = T(s) - D =$$

$$= \begin{bmatrix} \frac{N_{11}(s)}{D_1(s)} & \dots & \frac{N_{1m}(s)}{D_m(s)} \\ \dots & \dots & \dots \\ \frac{N_{p1}(s)}{D_1(s)} & \dots & \frac{N_{pm}(s)}{D_m(s)} \end{bmatrix} = N(s)D^{-1}(s), \tag{34}$$

Where

$$N(s) = \begin{bmatrix} N_{11}(s) & \dots & N_{1m}(s) \\ \dots & \dots & \dots \\ N_{p1}(s) & \dots & N_{pm}(s) \end{bmatrix},$$

$$D = \mathrm{diag}\left[D_1(s), \dots, D_m(s)\right] \tag{35}$$

$$N_{ik}(s) = c_{ik}^{d_k-1}(s^\alpha)^{d_k-1} + \dots + c_{ik}^1 s^\alpha + c_{ik}^0,$$

$$i = 1, \dots, p;\ k = 1, \dots, m$$

$$D_k(s) = (s^\alpha)^{d_k} - a_{k\,d_k-1}(s^\alpha)^{d_k-1} - \dots - a_{k1}s^\alpha - a_{k0} \tag{36}$$

Theorem 7. There exists the positive fractional realization

$$A = \text{blockdiag}\,[A_1, \dots, A_m] \in \Re^{n \times n},$$

$$A_k = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_{k,0} & -a_{k,1} & -a_{k,2} & \dots & -a_{k,d_k-1} \end{bmatrix} \in \Re_+^{d_k \times d_k},$$

$$k = 1, 2, \dots, m, \quad n = d_1 + \dots + d_m,$$

$$B = \text{blockdiag}\ [B_1, \dots, B_m] \in \Re_+^{n \times m},$$

$$B_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \Re_+^{d_k}, \quad k = 1, 2, \dots, m, \quad D = T(\infty) \in \Re_+^{p \times m},$$

$$C = \begin{bmatrix} c_{11}^0 & \dots & c_{11}^{d_1-1} & \dots & c_{1,m}^0 & \dots & c_{1,m}^{d_m-1} \\ \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ c_{p,1}^0 & \dots & c_{p,1}^{d_1-1} & \dots & c_{p,m}^0 & \dots & c_{p,m}^{d_m-1} \end{bmatrix} \in \Re_+^{p \times n}.$$

of the transfer matrix $T(s)$ if the following conditions are satisfied:

i) $T(\infty) \in \Re_+^{p \times m}$

ii) $a_{kl} \geq 0$ for $k = 1, \dots, m;$ $l = 0, 1, \dots, d_k - 2$ and $a_{k\,d_k-1}$ can be arbitrary

iii) $c_{ik}^j \geq 0$ for $i = 1, \dots, p;$ $j = 0, 1, \dots, d_k - 1;$ $k = 1, .., m$.

A proof is given in [22].

If the conditions of Theorem 7 are satisfied, then the positive fractional realization of the transfer matrix $T(s)$ can be computed by use of the following procedure.

Procedure 2.

Step 1. Knowing the proper transfer matrix $T(s)$ and using (33), compute the matrix D and the strictly proper matrix $T_{sp}(s)$.

Step 2. Find the minimal degrees $d_1, \dots, d_m$ of the denominators $D_1(s), \dots, D_m(s)$ and write the matrix $T_{sp}(s)$ in the form (34).

Step 3. Using the equality

$$D(s) = \mathrm{diag}[(s^\alpha)^{d_1}, \dots, (s^\alpha)^{d_m}] - \mathrm{diag}[a_1, \dots, a_m]S \tag{38}$$

find $a_k = [a_{k0} \quad a_{k1} \quad \dots \quad a_{k\,d_k-1}]$ for $k = 1, \dots, m$ and the matrix A.

Step 4. Knowing the matrix $N(s)$ and using

$N(s) = CS$

$$= \begin{bmatrix} c_{11}^0 & \dots & c_{11}^{d_1-1} & \dots & c_{1,m}^0 & \dots & c_{1,m}^{d_m-1} \\ \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ c_{p,1}^0 & \dots & c_{p,1}^{d_1-1} & \dots & c_{p,m}^0 & \dots & c_{p,m}^{d_m-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 \\ s^\alpha & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ (s^\alpha)^{d_1-1} & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & s^\alpha \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & (s^\alpha)^{d_m-1} \end{bmatrix} \tag{39}$$

find the matrix C.

Example 2. Find the positive fractional realization (37) of the transfer matrix

$$T(s) = \begin{bmatrix} \frac{2s^\alpha + 1}{s^\alpha} & \frac{(s^\alpha)^2 + 3s^\alpha + 2}{(s^\alpha)^2 + 2s^\alpha - 3} \\ \frac{s^\alpha + 3}{s^\alpha + 1} & \frac{2s^\alpha + 1}{(s^\alpha)^2 + 2s^\alpha - 3} \end{bmatrix} \tag{40}$$

Using the Procedure 2, we obtain the following.

Step 1. From (33), (34) and (40) we have

$$D = \lim_{s \to \infty} T(s) = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \tag{41}$$

and

$$T_{sp}(s) = T(s) - D = \begin{bmatrix} \frac{1}{s^\alpha} & \frac{s^\alpha + 5}{(s^\alpha)^2 + 2s^\alpha - 3} \\ \frac{2}{s^\alpha + 1} & \frac{2s^\alpha + 1}{(s^\alpha)^2 + 2s^\alpha - 3} \end{bmatrix} \tag{42}$$

Step 2. In this case $D_1(s)=(s^\alpha)^2+s^\alpha$, $D_2(s)=(s^\alpha)^2+2s^\alpha-3$, $d_1=d_2=2$ and the matrix (42) takes the form

$$T_{sp}(s)=\begin{bmatrix} \frac{s^\alpha+1}{(s^\alpha)^2+s^\alpha} & \frac{s^\alpha+5}{(s^\alpha)^2+2s^\alpha-3} \\ \frac{2s^\alpha}{(s^\alpha)^2+s^\alpha} & \frac{2s^\alpha+1}{(s^\alpha)^2+2s^\alpha-3} \end{bmatrix} \quad (43)$$

Step 3. Using (38) we obtain

$$\begin{bmatrix} (s^\alpha)^2+s^\alpha & 0 \\ 0 & (s^\alpha)^2+2s^\alpha-3 \end{bmatrix}=$$

$$=\begin{bmatrix} (s^\alpha)^2 & 0 \\ 0 & (s^\alpha)^2 \end{bmatrix}-\begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ s^\alpha & 0 \\ 0 & 1 \\ 0 & s^\alpha \end{bmatrix} \quad (44)$$

and

$$a_1=[a_{10} \quad a_{11}]=[0 \quad -1], \quad a_2=[a_{20} \quad a_{21}]=[3 \quad -2] \quad (45)$$

Therefore, the matrix A has the form

$$A=\text{block diag}[A_1 \quad A_2]=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & -2 \end{bmatrix} \quad (46)$$

Step 4. Using (39) we obtain

$$\begin{bmatrix} s^\alpha+1 & s^\alpha+5 \\ 2s^\alpha & 2s^\alpha+1 \end{bmatrix}=\begin{bmatrix} 1 & 1 & 5 & 1 \\ 0 & 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ s^\alpha & 0 \\ 0 & 1 \\ 0 & s^\alpha \end{bmatrix} \quad (47)$$

and

$$C=\begin{bmatrix} 1 & 1 & 5 & 1 \\ 0 & 2 & 1 & 2 \end{bmatrix} \quad (48)$$

The matrix B in this case has the form

$$B=\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad (49)$$

The desired positive fractional realization (37) of (40) is given by (41), (46), (48) and (49).

A dual approach for MIMO systems is given in [37]. Necessary and sufficient conditions for the existence of cone-realization with delays and a procedure for computation of cone-realization are given in [32].

5. Positive fractional discrete-time systems

In this paper the following definition of the fractional discrete derivative will be used

$$\Delta^\alpha x_k=\sum_{j=0}^{k}(-1)^j\binom{\alpha}{j}x_{k-j}, \quad 0<\alpha<1 \quad (50)$$

where $\alpha\in\Re$ is the order of the fractional difference, and

$$\binom{\alpha}{j}=\begin{cases} 1 & \text{for } j=0 \\ \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!} & \text{for } j=1,2,\ldots \end{cases} \quad (51)$$

Consider the fractional discrete linear system described by the state-space equations

$$\Delta^\alpha x_{k+1}=Ax_k+Bu_k, \quad k\in Z_+ \quad (52a)$$

$$y_k=Cx_k+Du_k \quad (52b)$$

where $x_k\in\Re^n$, $u_k\in\Re^m$, $y_k\in\Re^p$ are the state, input and output vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$, $D\in\Re^{p\times m}$.

Using the definition (50) we may write the equations (52) in the form

$$x_{k+1}+\sum_{j=1}^{k+1}(-1)^j\binom{\alpha}{j}x_{k-j+1}=Ax_k+Bu_k, \; k\in Z_+ \quad (53a)$$

$$y_k=Cx_k+Du_k \quad (53b)$$

Definition 7. The system (53) is called the (internally) positive fractional system if and only if $x_k\in\Re_+^n$ and $y_k\in\Re_+^p$, $k\in Z_+$ for any initial conditions $x_0\in\Re_+^n$ and all input sequences $u_k\in\Re_+^m$, $k\in Z_+$.

Theorem 8. The solution of equation (53a) is given by

$$x_k=\Phi_k x_0+\sum_{i=0}^{k-1}\Phi_{k-i-1}Bu_i \quad (54)$$

where Φ_k is determined by the equation

$$\Phi_{k+1}=(A+I_n\alpha)\Phi_k+\sum_{i=2}^{k+1}(-1)^{i+1}\binom{\alpha}{i}\Phi_{k-i+1} \quad (55)$$

with $\Phi_0=I_n$.

The proof is given in [16, 23].

Lemma 1. [16] If

$$0<\alpha\le 1 \quad (56)$$

then

$$(-1)^{i+1}\binom{\alpha}{i}>0 \quad \text{for } i=1,2,\ldots \quad (57)$$

Theorem 9. [16] Let $0<\alpha<1$. Then the fractional system (53) is positive if and only if

$$A+I_n\alpha \in \Re_+^{n\times n},\ B\in \Re_+^{n\times m},\ C\in \Re_+^{p\times n},\ D\in \Re_+^{p\times m}. \quad (58)$$

6. Reachability of positive fractional linear systems

Consider the positive fractional linear system (53).

Definition 8. A state $x_f \in \Re_+^n$ of the positive fractional system (53) is called reachable in q steps if there exist an input sequence $u_k \in \Re_+^m$, $k=0,1,\dots,q-1$ which steers the state of the system from zero ($x_0 = 0$) to the final state x_f, i.e. $x_q = x_f$.

Let e_i, $i=1,...,n$ be the i-th column of the identity matrix I_n. A column ae_i for $a > 0$ is called the monomial column.

Theorem 10. The positive fractional system (53) is reachable in q steps if and only if the reachability matrix

$$R_q := [B, \Phi_1 B, ..., \Phi_{q-1}B] \quad (59)$$

contains n linearly independent monomial columns.

Proof. Using (54) for $k = q$ and $x_0 = 0$ we obtain

$$x_f = x_q = \sum_{i=0}^{q-1}\Phi_{q-i-1}Bu_i = R_q\begin{bmatrix} u_{q-1} \\ u_{q-2} \\ \vdots \\ u_0 \end{bmatrix} \quad (60)$$

From Definition 8 and (60) it follows that for every $x_f \in \Re_+^n$ there exist an input sequence $u_i \in \Re_+^m$, $i=0,1,...,q-1$ if and only if the matrix (59) contains n linearly independent monomial columns. □

From (5.6) it follows that for positive fractional systems the coefficients a_i, $i = 0,1,\dots,k-1$ in the equality

$$\begin{aligned}\Phi_k &= (A+I_n\alpha)^k + a_{k-1}(A+I_n\alpha)^{k-1}+... \\ &+a_1(A+I_n\alpha)+a_0I_n\end{aligned} \quad (61)$$

are nonnegative.

Theorem 11. The positive fractional system (53) is reachable only if the matrix

$$[A+I_n\alpha, B] \quad (62)$$

contains at least n linearly independent monomial columns.

Proof. From the form of the matrix (59) and the equality (61) it follows that the number of linearly independent monomial columns of (59) can not be greater than of the matrix (62). □

The following example shows that the condition of Theorem 11 is necessary but not sufficient.

Example 3. It is easy to show that the positive fractional system (53) with the matrices

$$A=\begin{bmatrix} 1 & 0 \\ 0 & -\alpha \end{bmatrix},\ B=\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{for} \quad 0<\alpha<1 \quad (63)$$

is not reachable in spite of that in this case the matrix

$$[A+I_n\alpha, B] = \left[\begin{array}{cc:c} 1+\alpha & 0 & 0 \\ 0 & 0 & 1 \end{array}\right] \quad (64)$$

contains two linearly independent monomial columns.

The following example shows that for positive fractional systems the matrix (59) in Theorem 10 can not be substituted by the matrix

$$\bar{R}_q = \left[B, (A+I_n\alpha)B, \dots, (A+I_n\alpha)^{q-1}B\right] \quad (65)$$

Example 4. Consider the positive fractional system (53) with the matrices

$$A=\begin{bmatrix} 0 & 1 & 0 \\ 0 & -\alpha & 1 \\ 1 & 0 & -\alpha \end{bmatrix},\ B=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (66)$$

In this case

$$A+I_n\alpha = \begin{bmatrix} \alpha & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \in \Re_+^{3\times 3} \quad (67)$$

and the matrix (65) has the form

$$\bar{R}_3 = \left[B, (A+I_n\alpha)B, (A+I_n\alpha)^2 B\right] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad (68)$$

and it contains three linearly independent monomial columns. Using (55) for $k = 0,1$ for (66) we obtain

$$\Phi_1 = (A+I_n\alpha) = \begin{bmatrix} \alpha & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\Phi_2 = (A+I_n\alpha)\Phi_1 - \binom{\alpha}{2} I_n = \quad (69)$$

$$= \begin{bmatrix} \frac{\alpha(\alpha+1)}{2} & \alpha & 1 \\ 1 & \frac{\alpha(\alpha-1)}{2} & 0 \\ \alpha & 1 & \frac{\alpha(1-\alpha)}{2} \end{bmatrix},$$

and the matrix (59) has the form

$$R_3 = [B, \Phi_1 B, \Phi_2 B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & \frac{\alpha(\alpha-1)}{2} \end{bmatrix} \quad (70)$$

This matrix contains only two linearly independent monomial columns.

Definition 9. Let the j-th column b_j ($j = 1,\dots,m$) of the matrix B be monomial. The column $\Phi_{1j} = \Phi_1 b_j$ ($j=1,\dots,n$)

of the matrix Φ_1 is called monomial column corresponding to the j-th column of B if and only if it is monomial and linearly independent of the monomial column b_j.

In the new test for checking the reachability of the positive fractional systems a crucial role will play the following procedure [11].

Procedure 3. (finding linearly independent monomial columns).

Using Definition 9 find all monomial linearly independent columns (starting from the first column of B)

$$\Phi_{kj} = \Phi_k b_j \quad \text{for} \quad j = 1,\ldots,m\,;\; k = 2,\ldots,q-1 \tag{71}$$

of the matrix (59). Stop the procedure if the last column is not monomial or/and linearly dependent from the previous monomial columns.

Theorem 12. The positive fractional system (53) is reachable if and only if using Procedure 3 to the matrix (59) it is possible to find its n monomial linearly independent columns.

Proof. By Theorem 10 the positive fractional system (53) is reachable in q steps if and only if the reachability matrix (59) contains n monomial linearly independent columns. Thus, the system is reachable if and only if using the procedure it is possible to find n monomial linearly independent columns of the matrix (59).

Example 5. Consider the positive fractional system (53) with the matrices

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & a & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{72}$$

for $a > 0$.

It is easy to shown that for $a \neq 0$, $rank[B, AB, A^2B] = 3$ and the standard (nonpositive) system is reachable in $q = 3$ steps. Now it will be shown that the positive fractional system (53) with (72) for $a > 0$ is unreachable. Using the Procedure 3 for the matrix

$$R_3 = [B, \Phi_1 B, \Phi_2 B] = \\ = \left[B, (A + I_n\alpha)B, (A + I_n\alpha)^2 B - \binom{\alpha}{2} B\right] \tag{80}$$

we obtain only one monomial column B, since

$$(A + I_n\alpha)B = \begin{bmatrix} \alpha \\ 1 \\ 0 \end{bmatrix},$$

$$(A + I_n\alpha)^2 B - \binom{\alpha}{2} B = \begin{bmatrix} \dfrac{\alpha(\alpha+1)}{2} \\ 2\alpha \\ a \end{bmatrix} \tag{81}$$

Thus, the positive fractional system is unreachable.

Theorem 13. The positive fractional system (53) is reachable if and only if the matrix

$$[B, (A + I_n\alpha)B] \tag{82}$$

contains n monomial linearly independent columns.

Proof. From (55) for positive fractional systems we have

$$\Phi_k B = \sum_{i=0}^{k} a_{ki}(A + I_n\alpha)^i B \quad \text{for} \quad k = 1, 2, \ldots, n-1 \tag{83}$$

where $a_{ki} \geq 0$, $k = 1,\ldots,n-1$; $i = 0,1,\ldots,k$.

Note that besides the matrix B only the matrix $\Phi_1 B$ may have additional monomial linearly independent columns and the matrix (83) for $k = 2,3,\ldots n-1$ do not introduce additional monomial linearly independent columns to the matrix (59).

From Theorem 11 we have the following remark

Remark 1. If all m columns of the matrix B are monomial linearly independent columns, then the matrix (82) has n monomial linearly independent columns only if the matrix $(A + I_n\alpha)$ has at least $n - m$ monomial linearly independent columns.

Example 6. Consider the positive fractional system (53) with the matrix

$$A = \begin{bmatrix} a_{11}-\alpha & a_{12} & 1 & 0 \\ a_{21} & a_{22}-\alpha & 0 & 1 \\ a_{31} & a_{32} & -\alpha & 0 \\ a_{41} & a_{42} & 0 & -\alpha \end{bmatrix}, \quad a_{ij} \geq 0,$$

$i = 1,2,3,4;\ j = 1,2$

$$a) \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad b) \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{84}$$

Taking into account that

$$A + I_n\alpha = \begin{bmatrix} a_{11} & a_{12} & 1 & 0 \\ a_{21} & a_{22} & 0 & 1 \\ a_{31} & a_{32} & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \end{bmatrix} \tag{85}$$

in the case a) we obtain the matrix

$$[B, \Phi_1 B] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{86}$$

which has $n = 4$ monomial linearly independent columns. Therefore, in this case the system is reachable in $q = 2$ steps.

In the case b) using (59) and (85) we obtain the matrix

$$[B, \Phi_1 B, \Phi_2 B, \ldots] = \begin{bmatrix} 0 & 0 & a_{12} & \cdots \\ 0 & 1 & a_{22} & \cdots \\ 0 & 0 & a_{23} & \cdots \\ 1 & 0 & a_{24} + \dfrac{\alpha(1-\alpha)}{2} & \cdots \end{bmatrix} \quad (87)$$

with only two monomial linearly independent columns. By Theorem 10 in this case the positive fractional system is unreachable.

It is well-known that the observability is a dual notion to the reachability. All results presented in this section for the reachability of positive fractional systems can be applied for checking the observability of the positive fractional systems.

7. Concluding remarks

The positive fractional linear continuous-time systems have been addressed. The cone fractional linear systems have been introduced. Sufficient conditions for the reachability of positive fractional and cone fractional linear systems have been established. The realization problem for positive fractional linear continuous-time systems has been formulated and solved. The positive fractional discrete-time linear systems are also considered.

Extensions of these considerations for the following classes of systems are open problems

1) 1D and 2D varying positive linear systems
2) 2D hybrid systems without and with delays
3) 2D Lyapunov systems
4) 1D and 2D positive fractional switching systems.

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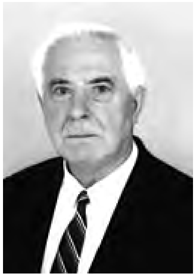
ДОДАТНІ ДРОБОВІ ТА КОНІЧНІ ДРОБОВІ ЛІНІЙНІ СИСТЕМИ

Тадеуш Качорек

У статті розглянуто додатні дробові та конічні дробові неперервні та дискретні лінійні системи. Наведено достатні умови для досяжності таких систем. Встановлено необхідні та достатні умови для додатності та асимптотичної стабільності неперервних у часі лінійних систем. Сформульовано та розв'язано проблему реалізації додатних дробових неперервних у часі систем.



Tadeusz Kaczorek – Ph.D., D.Sc., Professor, born in 1932 in Poland, received his M.Sc., Ph.D. and D.Sc. degrees in Electrical Engineering from Warsaw University of Technology, Poland, in 1956, 1962 and 1964, respectively. In the period 1968–1969 he was the dean of Electrical Engineering Faculty and in the period 1970–1973 – the pro-rector of Warsaw University of Technology, Poland. Since 1971 he has been a professor and since 1974 a full professor at Warsaw University of Technology, Poland. In 1986 he was elected a corresponding member and in 1996 full member of Polish Academy of Sciences. In the period 1988 - 1991 he was the director of the Research Centre of Polish Academy of Sciences in Rome. In June 1999 he was elected the full member of the Academy of Engineering in Poland. In May 2004 he was elected the honorary member of the Hungarian Academy of Sciences. He was awarded by the University of Zielona Gora, Poland (2002) by the title doctor honoris causa, the Technical University of Lublin, Poland (2004), the Technical University of Szczecin, Poland (2004), Warsaw University of Technology, Poland (2004), Bialystok University of Technology, Poland (2008), Lodz University of Technology, Poland (2009), Opole University of Technology, Poland (2009), Poznan University of Technology, Poland (2011), and Rzeszow University of Technology, Poland (2012).

His research interests cover the theory of systems and the automatic control systems theory, specially, singular multidimensional systems, positive multidimensional systems and singular positive 1D and 2D systems. He has initiated the research in the field of singular 2D, positive 2D linear systems and positive fractional 1D and 2D systems. He has published 25 books (7 in English) and over 1000 scientific papers.

He supervised 69 Ph.D. theses. More than 20 of these PhD students became professors in USA, UK and Japan. He is the editor-in-chief of the Bulletin of the Polish Academy of Sciences, Technical Sciences and the editorial member of about ten international journals.