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ON TRANSPORT OPERATOR WITH PRESCRIBED EIGENVALUE

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For transport equation

$$-i\mu\frac{\partial f}{\partial x}(x,\mu)+c_0(x)\int\limits_{-1}^{1}b(\mu')f(x,\mu')d\mu'=\sigma f(x,\mu),\ -\infty<x<\infty,\ x\in\mathbb{R},\ -1<\mu<1$$

explicit examples of solution $f(x,\mu)$, function $C_0(x)$, $b(\mu)$ are given. The solution $f(x,\mu)$ is presented as polynomial, or as series on Hermite polynomials, or as function of type $f(x,\mu)= = h(x)g\left(\frac{x}{\mu}\right)$. Last examples give the solutions which belong to the space $L^2(D_\pm)$, $D_\pm = \mathbb{R}_\pm\times(-1,1)$ and correspond to bounded function C_0x. For some type of coefficient $C_0(x)$, $b(\mu)$ a condition of absence of eigenvalue σ in case of the space $L^2(D)$, $D=\mathbb{R}\times(-1,1)$ is given.

Key words: transport operator, spectrum, point spectrum, eigenvalue.

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Introduction

There are many works conserning point spectrum of transport operator of different types, including the estimates for the number of eigenvalues (see for example [1]-[2], where the reader can find other references).

Does there exist the operator of given type such that point spectrum contains given set of numbers? We do not say about difficult inverse problem. Particular question conserning some part of point spectrum only may be interesting too. Recall for example addition and elimination eigen values of Sturm-Liouville operators (see [3]). Close question relating to prescribed eigenvalues was considered in [4]. In the work [5] the author supposes that some transport operator has not spectral singularities. We do not know if considered operator may have spectral singularities. The operators of like type are considered in given note.

I. Statement of the problem

We will study the existence of the solution of transport equation (see [5])

$$-i\mu\tfrac{\partial f}{\partial x}(x,\mu)+c_0(x)\int\limits_{-1}^{1}b(\mu')f(x,\mu')d\mu'=\sigma f(x,\mu),\quad -\infty<x<\infty, \tag{1}$$

where σ is a given number. Our aim is to indicate the parameters $c_0(x)$, $b(\mu)$ and give the solution of the equation (1) for given value σ. We do not use directly the notion of the operator. However, it is convenient for us to consider $f(x,\mu)$ for every $\mu=$ const as an element of some space of type $L^2_\rho(-\infty,\infty)$. We consider two methods of construction of the solution $f(x,\mu)$. In the beginning the value $\sigma\neq 0$ is arbitrary complex number.

First method consists in representation of the solution as following series

$$f(x,\mu)=\sum_{k=0}^{\infty}f_k(x)\mu^k. \tag{2}$$

Substituing in (1), we obtain formally

$$-i\mu\sum_{k=0}^{\infty}f_k'(x)\mu^k+c_0(x)\int\limits_{-1}^{1}b(\mu')f(x,\mu')d\mu'= =\sigma\sum_{k=0}^{\infty}f_k(x)\mu^k.$$

The coefficient at the powers μ^k gives

$$\mu^0: c_0(x)\int\limits_{-1}^{1}b(\mu')f(x,\mu')d\mu'=\sigma f_0(x) \tag{3}$$

$$\mu^k: -if_{k-1}'(x)=\sigma f_k(x),\ k=1,2,...$$

Last equation gives obviously

$$f_k(x)=\frac{1}{(i\sigma)^k}f_0^{(k)}(x),\ k=0,1,... \tag{4}$$

Then we rewrite (2) as

$$f(x,\mu)=\sum_{k=0}^{\infty}\left(\frac{\mu}{i\sigma}\right)^k f_0^{(k)}(x) \tag{5}$$

and (3) as

$$c_0(x)\sum_{t=0}^{\infty}\beta_k f_k(x)=\sigma f_0(x), \tag{6}$$

where

$$\beta_k = \int_{-1}^{1} b(\mu)\mu^k d\mu, \ k = 0, 1, ... \qquad (7)$$

So, this method consists in search for the function $f_0(x)$ and $b(\mu)$ such that the series (2) converges and the equality (6) permits to define the function $c_0(x)$.

Second method is given in n.4, but there σ is real value, $\sigma \neq 0, \ \infty < \sigma < \infty$.

II. Case of the space $L^2_\rho(\infty, \infty)$, $\rho(x) = \exp(-|x|)$

We search the solution $f(x, \mu)$ as polynomial on variable x. Let $f_0(x)$ be arbitrary polynomial, $\deg f_0(x) = N$. Then (see(4)) $f_k(x) \equiv 0, \ k > N$ and

$$f(x, \mu) = \sum_{k=0}^{N} f_k(x)\mu^k.$$

According to (5) we have

$$c_0(x) \sum_{k=0}^{N} \beta_k f_k(x) = \sigma f_0(x), \qquad (8)$$

where

$$\beta_k = \int_{-1}^{1} b(\mu)\mu^k d\mu, \ k = 0, 1, ...N. \qquad (9)$$

Now we choose arbitrary numbers β_k such that the polynomial $\sum_{k=0}^{N} \beta_k f_k(x)$ has not real roots. Then we define $c_0(x)$ from the equality (8) as rational function bounded on real axis. Later we choose some function $b(\mu)$ integrable on $(-1, 1)$, which satisfies the relation (9). Obviously, for every value $\mu \in (-1, 1)$ the function $f(x, \mu)$ belongs to the space $L^2_\rho(-\infty, \infty)$.

So, we have following proposition.

Proposition. Let $\sigma \neq 0$ be arbitrary complex value and the coefficients $c_0(x), \ b(\mu)$ of the equation (1) are arbitrary functions which satisfies the relations (4), (8)-(9), where $f_0(x)$, $\deg f_0(x) = N$ is arbitrary polynomial. Then the polynomial

$$f(x, \mu) = \sum_{k=0}^{N} \frac{1}{(i\sigma)^k} f_0^{(k)}(x)\mu^k$$

is a solution of the equation (1).

III. Case of the space $L^2_\rho(\infty, \infty)$, $\rho(x) = \exp\left(-\frac{x^2}{2}\right)$

We search a serie on Hermite polynomials

$$f_0(x) = \sum_{k=0}^{N} \alpha_k H_k(x), \ -\infty < x < \infty, \qquad (10)$$

(where α_k are unknown coefficients) for the function $f_0(x)$ in the relations (5)-(6). Recall that (see[6]) $H_0(x) = 1, \ H_1(x) = 2x, \ H_2(x) = 4x^2 - 2, \ ...$ The relation

$$H'_n(x) = 2nH_{n-1}(x), \ n = 1, 2, ...$$

permits to present the derivatives of $f_0(x)$ again as serie on Hermite polynomials. We have formally

$$f'_0(x) = \sum_{k=1}^{\infty} \alpha_k H'_k(x) = \sum_{k=1}^{\infty} 2k\alpha_k H_{k-1}(x) =$$

$$= \sum_{k=0}^{\infty} 2(k+1)\alpha_{k+1} H_k(x).$$

If we repeat r times this calcul, we obtain

$$f_0^{(r)}(x) = \sum_{k=0}^{N} \alpha_k^{(r)} H_k(x), \ r = 1, 2, ..., \qquad (11)$$

where

$$\alpha_k^{(r)} = 2^r (k+r)(k+r-1)...(k-1)\alpha_{k+r}. \qquad (12)$$

Substituing (11) in (5) we obtain

$$f(x, \mu) = \sum_{k=0}^{\infty} \left(\sum_{r=0}^{\infty} \left(\frac{\mu}{i\sigma}\right)^r \alpha_k^{(r)} \right) H_k(x). \qquad (13)$$

Now we are looking for the conditions on the coefficient α_k, which guarantees the convergence of the series (10), (11) and (13). We begin with the convergence of (13) for every $x = \text{const}$. Using known estimate

$$|H_k(x)| < e^{\frac{x^2}{2}} 2^{\frac{k}{2}} \sqrt{k!}, \ k \neq 1, 2, ... \qquad (14)$$

we obtain corresponding majorant

$$\sum_{k=0}^{\infty} \left(\sum_{k=0}^{\infty} \left|\frac{\mu}{\sigma}\right|^r \left|\alpha_k^{(r)}\right| \right) 2^{\frac{k}{2}} \sqrt{k!}$$

or (see(12))

$$\sum_{k=0}^{\infty} \left(\sum_{r=0}^{\infty} \left|\frac{\mu}{\sigma}\right|^r |\alpha_{k+r}| \, 2^r (k+r)...(k+1) \right) 2^{\frac{k}{2}} \sqrt{k!}.$$

As $\frac{k}{2} < k, \ \sqrt{k!} < k!$ we can rewrite the majorant

$$\sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \left|\frac{\mu}{\sigma}\right|^r |\alpha_{k+r}| \, 2^{k+r} (k+r)! \qquad (15)$$

Denote by A arbitrary number such that

$$0 < A < \min\{1, |\sigma|\},$$

such value A exists because $\sigma \neq 0$. we suppose that $|\mu| < 1$, then

$$A \left|\frac{\mu}{\sigma}\right| < 1.$$

Under the condition

$$|\alpha_k|\, 2^k k! \leq M \cdot A^k,\ k = 1, 2, ..., M = \text{const}, \qquad (16)$$

where $M > 0$ is arbitrary value, the serie (14) is convergent as

$$M\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\left|\frac{\mu}{\sigma}\right|^r A^{k+r} = M\left(\sum_{k=0}^{\infty}A^k\right)\left(\sum_{r=0}^{\infty}\left(A\left|\frac{\mu}{\sigma}\right|\right)^r\right) < \infty.$$

The series (10) is particular case of serie (11). So, we must prove the convergence of the serie (11) for every $x = \text{const}$ and $k = 0, 1, ...$ Corresponding majorant is (see(14),(12))

$$\sum_{k=0}^{\infty}\left|\alpha_k^{(r)}\right| \cdot |H_k(x)| \leq e^{\frac{x^2}{2}}\sum_{k=0}^{\infty}\left|\alpha_k^{(r)}\right| 2^{\frac{k}{2}}\sqrt{k!} \leq$$

$$\leq 2^r e^{\frac{x^2}{2}}\sum_{k=0}^{\infty}(k+r)...(k+1)\sqrt{k!}2^{\frac{k}{2}}\,|\alpha_{k+r}|\,.$$

Under the condition (16) the serie

$$2^r\sum_{k=0}^{\infty}(k+r)...(k+1)\sqrt{k!}2^{\frac{k}{2}}\frac{MA^k}{2^{k+r}(k+r)!} \leq$$

$$\leq M \cdot 2^r \cdot \sum_{k=0}^{\infty}2^{\frac{k}{2}}\frac{A^k}{2^k \cdot 2^r} = M\sum_{k=0}^{\infty}\left(\frac{A}{\sqrt{2}}\right)^k < \infty$$

converges as $A < 1$. The convergence is uniform in every finite interval $[a, b]$ of variable x. So, the function $f_0(x)$ is indefinitely derivables and its derivates are given by (11). Now we will prove that for every $\mu \in (-1, 1)$ the function $f(x, \mu)$ (see(13)) as function on x belongs to the space $L^2_\rho(-\infty, \infty)$, $\rho(x) = \exp\left(-\frac{x^2}{2}\right)$.

Recall (see()) that

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} H_n(x)H_m(x)dx = \begin{cases} 0, & m \neq n \\ \sqrt{\pi}2^n n!, & m = n \end{cases}$$

and if

$$f(x) = \sum_{k=0}^{\infty} c_k H_k(x),\ c_k \in \mathbb{C} \qquad (17)$$

then

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,|f(x)|^2\,dx = \sqrt{\pi}\sum_{k=0}^{\infty}2^k k!\,|c_k|^2\,,\ 0! = 1. \qquad (18)$$

If we compare the series (17) and (13), we see that it is sufficient to prove the convergence the serie (18), where

$$c_k = \sum_{r=0}^{\infty}\left(\frac{\mu}{i\sigma}\right)^r \alpha_k^{(r)}$$

and (see(12))

$$|c_k|^2 \cdot 2^k k! \leq \left(\sum_{r=0}^{\infty}\left|\frac{\mu}{\sigma}\right|^r |\alpha_{k+r}| \cdot 2^r (k+r)...(k+1)\right)^2 \left(2^{\frac{k}{2}}\sqrt{k!}\right)^2 \leq$$

$$\leq \left(\sum_{r=0}^{\infty}\left|\frac{\mu}{\sigma}\right|^r |\alpha_{k+r}| \cdot 2^{k+r}(k+r)!\right)^2.$$

In view of the condition (16), we have

$$|c_k|^2 \cdot 2^k k! \leq M^2\left(\sum_{r=0}^{\infty}\left|\frac{\mu}{\sigma}\right|^r A^{k+r}\right)^2 =$$

$$= M^2 A^{2k}\left(\sum_{r=0}^{\infty}\left|\frac{\mu}{\sigma}\right|^r A^r\right)^2 \leq M_1 A^{2k}$$

for $k = 0, 1, ...$ Therefore

$$\sum_{k=0}^{\infty}|c_k|^2\, 2^k k! \leq M_1\sum_{k=0}^{\infty}A^{2k} < \infty$$

as $A < 1$. So, the serie (18) converges and

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,|f(x, \mu)|^2\,dx < \infty.$$

We proved the following theorem

Theorem 1. Let $\sigma \neq 0$ be arbitrary complex value and the coefficients $c_0(x)$, $b(\mu)$ of the equation (1) are arbitrary functions which satisfy the relations (4), (6)–(7), where

$$f_0(x) = \sum_{k=0}^{\infty}\alpha_k H_k(x),\ -\infty < x < \infty$$

and the coefficients α_k satisfy the condition (16).

Then the function (see(12))

$$f(x, \mu) = \sum_{k=0}^{\infty}\left(\sum_{r=0}^{\infty}\left(\frac{\mu}{i\sigma}\right)^r \alpha_k^{(r)}\right) H_k(x)$$

is a solution of the equatian (1). The solution $f(x, \mu)$ as a function on x every $\mu \in (-1, 1)$ belongs to the space $L^2_\rho(-\infty, \infty)$ with $\rho(x) = \exp\left(-\frac{x^2}{2}\right)$.

IV. Case of the spaces $L^2(-\infty, 0)$ and $L^2(0, \infty)$

We consider the equation (1) for real value σ, $-\infty < \sigma < \infty$. Our second method consists in searching the solution as following product of two unknown functions

$$f(x, \mu) = h(x)g\left(\frac{x}{\mu}\right). \qquad (19)$$

Substituing (19) with $\frac{\partial f}{\partial x}(x, \mu) = h'(x)g\left(\frac{x}{\mu}\right) + h(x)g'\left(\frac{x}{\mu}\right) \cdot \frac{1}{\mu}$ in the equation (1) we have

$$-i\mu h'(x)g\left(\frac{x}{\mu}\right) - ih(x)g'\left(\frac{x}{\mu}\right) +$$

$$+c_0(x)\int_{-1}^{1} b(\mu')h(x)g\left(\frac{x}{\mu'}\right)d\mu' = \sigma h(x)g\left(\frac{x}{\mu}\right)$$

and dividing by $h(x)$ we obtain

$$-i\mu\frac{h'(x)}{h(x)}g\left(\frac{x}{\mu}\right) - ig'(x)\left(\frac{x}{\mu}\right) +$$

$$+c_0(x)\int_{-1}^{1} b(\mu')g\left(\frac{x}{\mu'}\right)d\mu' = \sigma g\left(\frac{x}{\mu}\right).$$

Denote

$$\frac{x}{\mu} = t$$

then

$$-i\left(\mu t\frac{h'(\mu t)}{h(\mu t)}\right)\frac{g(t)}{t} - ig'(t) +$$

$$+\left(c_0(\mu t)\int_{-1}^{1} b(\mu')g\left(\frac{\mu t}{\mu'}\right)d\mu'\right) = \sigma g(t)$$

Evidently, both expressions in brackets are constant with respect to μt. Therefore

$$x\frac{h'(x)}{h(x)} = k,\ k = \text{const} \qquad (20)$$

$$c_0(x)\int_{-1}^{1} b(\mu')g\left(\frac{x}{\mu'}\right)d\mu' = ik_1,\ k_1 = \text{const} \qquad (21)$$

$$-ik\frac{g(t)}{t} - ig'(t) + ik_1 = \sigma g(t). \qquad (22)$$

As a solution of the equation (20) we take

$$h(x) = |x|^k,\ k = -\frac{1}{2} - \epsilon,\ 0 < \epsilon < \frac{1}{2},\ x \neq 0. \qquad (23)$$

We consider linear differential equation (22) in two cases, where respectively $t \in (-\infty, 0)$ or $(0, \infty)$.

The equation

$$g'(t) + \left(\frac{k}{t} - i\sigma\right)g(t) = k_1 \qquad (24)$$

has corresponding homogeneous equation

$$g'(t) = -\left(\frac{k}{t} - i\sigma\right)g(t),$$

from which we obtain

$$(\ln g(t))' = -\frac{k}{t} + i\sigma$$

or $g(t) = C|t|^{-k}e^{i\sigma t},\ C = \text{const}$. So, we search the solution of the equation (24) under the form

$$g(t) = C(t)|t|^{-k}e^{i\sigma t}. \qquad (25)$$

As $(|t|)' = \pm 1$ then $\left(|t|^{-k}\right)' = -k\frac{|t|^{-k}}{|t|}\cdot(\pm 1) = -k\frac{|t|^{-k}}{t}$. Therefore

$$g'(t) = C'(t)|t|^{-k}e^{i\sigma t} - \frac{k}{t}g(t) + i\sigma g(t).$$

Then it follows from (24) that

$$C'(t)|t|^{-k}e^{i\sigma t} = k_1$$

so

$$C(t) = k_1\int_0^t |\tau|^k e^{-i\sigma\tau}d\tau + C_\pm,\ t \in (-\infty, 0)\cup(0, \infty).$$

The conditions $C(+\infty) = C(-\infty) = 0$ give evidently the representations

$$C_+(t) = -k_1\int_t^\infty |\tau|^k e^{-i\sigma\tau}d\tau,\ t > 0 \qquad (26)$$

$$C_-(t) = k_1\int_{-\infty}^t |\tau|^k e^{-i\sigma\tau}d\tau,\ t < 0, \qquad (27)$$

where $C_+(t)$ defines $C(t)$ for $t > 0$ and $C_-(t)$ respectively for $t > 0$. The solution $g(t)$ (see(25)) of the equation (22) in both intervals $(-\infty, 0), (0, \infty)$ is

$$g(t) = \begin{cases} C_+(t)|t|^{-k}e^{i\sigma t}, & t > 0 \\ C_-(t)|t|^{-k}e^{i\sigma t}, & t < 0 \end{cases} \qquad (28)$$

and the solution $f(x,\mu)$ (see(19), (23)) is

$$f(x,\mu) = \begin{cases} |\mu|^k C_+\left(\frac{x}{\mu}\right)e^{i\sigma\frac{x}{\mu}}, & x\mu > 0 \\ |\mu|^k C_-\left(\frac{x}{\mu}\right)e^{i\sigma\frac{x}{\mu}}, & x\mu > 0 \end{cases} \qquad (29)$$

Note, that $g(+0) = g(-0) = 0$. Now we precise asymptotic behaviour when $t \to \pm\infty$. Integrating par part we obtain

$$\int_t^\infty \tau^k e^{-i\sigma\tau}d\tau = \frac{t^k}{i\sigma}e^{-i\sigma t} + \frac{k}{i\sigma}\int_t^\infty \tau^{k-1}e^{-i\sigma\tau}d\tau,\ t > 0.$$

The change $k \to k-1$ gives

$$\int_t^\infty \tau^k e^{-i\sigma\tau}d\tau = \frac{t^k}{i\sigma}e^{-i\sigma t} + O\left(t^{k-1}\right),\ t \to +\infty \qquad (30)$$

and by analogy

$$\int_{-\infty}^t |\tau|^k e^{-i\sigma\tau}d\tau = \frac{|t|^k}{i\sigma}e^{-i\sigma t} + O\left(|t|^{k-1}\right),\ t \to -\infty.$$

Then it follows from (26)-(28) that $g(t) = -\frac{k_1}{i\sigma} + O\left(\frac{1}{t}\right),\ |t| \to \infty$. The function $g(t)$ is bounded if $t \to \infty$ therefore

$$g(t) = -\frac{k_1}{i\sigma} + r(t),\ -\infty < t < \infty,$$

where

$$|r(t)| \le \frac{C}{1+|t|},\ C = \text{const}.$$

Substituing $g(x)$ in (21) i.e.

$$c_0(x)\int\limits_{-1}^{1} b(\mu)\left[-\frac{k_1}{i\sigma}+r\left(\frac{x}{\mu}\right)\right]d\mu = ik_1,$$

where $\left|r\left(\frac{x}{\mu}\right)\right| \le \frac{C}{1+r\left|\frac{x}{\mu}\right|} < C\left|\frac{\mu}{x}\right|$ for $x\neq 0$ we obtain

$$c_0(x) = \frac{\sigma}{M_0} + O\left(\frac{1}{x}\right),\ |x|\to\infty \tag{31}$$

if

$$M_0 \equiv \int\limits_{-1}^{1} b(\mu)d\mu \neq 0$$

Now we consider the estimate of the solution (29) in the case $x>0,\ \mu>0$.

According to (26), (30)

$$C_+(t) = -k_1\left[\frac{t^k}{i\sigma}e^{-i\sigma t} + O\left(t^{k-1}\right)\right],\ t\to+\infty$$

threrefore

$$|C_+(t)| \le \begin{cases} M(\sigma)t^k, & t>1 \\ M_1(\sigma), & 0<t<1\end{cases}$$

for some $M(\sigma),\ M_1(\sigma) = \text{const}$ independent from k. Then it follows from (29)

$$|f(x,\mu)| \le \begin{cases} M(\sigma)x^k, & x>\mu \\ M_1(\sigma)\mu^k, & 0<x<\mu.\end{cases}$$

As consequence

$$\int\limits_0^\infty |f(x,\mu)|^2\,dx = \left(\int\limits_0^\mu + \int\limits_\mu^\infty\right)|f(x,\mu)|^2\,dx \le$$

$$\le M_1(\sigma)^2\mu^{2k}\int\limits_0^\mu dx + M(\sigma)^2\int\limits_\mu^\infty x^{2k}dx$$

or

$$\int\limits_0^\infty |f(x,\mu)|^2\,dx \le \frac{C(\sigma)}{\mu^{2\epsilon}},\ 0<\epsilon<\frac{1}{2},\ \mu\neq 0. \tag{32}$$

So, $f(\cdot,\mu)\in L^2(0,\infty),\ \mu\neq 0$. Denote $R_+ = (0,\infty)$, $D_+ = R_+\times(-1,1)$. Taking into account (32) we have $f\in L^2(D_+)$. In the case $x\in(-\infty,0)$ the same estimate (32) holds, respectively $f\in L^2(D_-)$. So, we write explicit form of the solution (see(26)-(27)) and (29)

$$f(x,\mu) = \begin{cases} -k_1\,|\mu|^k\,e^{-i\sigma\frac{x}{\mu}}\int\limits_{\frac{x}{\mu}}^{\infty}|\tau|^k\,e^{-i\sigma\tau}d\tau, & \frac{x}{\mu}>0 \\ k_1\,|\mu|^k\,e^{i\sigma\frac{x}{\mu}}\int\limits_{-\infty}^{\frac{x}{\mu}}|\tau|^k\,e^{-i\sigma\tau}d\tau, & \frac{x}{\mu}<0\end{cases} \tag{33}$$

and we can formulate following theorem.

Theorem 2. Let $\sigma\in(-\infty,\infty),\ \sigma\neq 0,\ k_1$ be arbitrary complex number and

$$k = -\frac{1}{2}-\epsilon,\ 0<\epsilon<\frac{1}{2}.$$

Then the function $f(x,\mu)$ defined by (33) is a solution in the intervals $(-\infty,0),\ (0,\infty)$ of the equation (1) where $b(\mu)$ is arbitrary function integrable in $(-1,1)$ and the function $c_0(x)$ is defined by the relations (21), (28). The solution $f(x,\mu)$ in the case $x>0$ or $x<0$ belongs respectively to the space $L^2(D_+)$ or $L^2(D_-)$. If the condition

$$M_0 = \int\limits_{-1}^{1} b(\mu)d\mu \neq 0$$

holds, then the function c_0(x) is in $(-\infty,\infty)$ bounded and

$$c_0(x) = \frac{\sigma}{M_0} + O\left(\frac{1}{x}\right),\ |x|\to\infty.$$

V. Absence of eigenvalues

We consider the equation (1), where $\sigma\in(-\infty,\infty)$ function $\frac{b(\mu)}{\mu}$ is bounded and integrable in the interval $(-1,1)$ and $c_0\in L^2(-\infty,\infty)\cap L^1(-\infty,\infty)$.

We suppose that there exists the solution $f(x,\mu)\neq 0$ of the equation (1) which belongs to the space $L^2(D),\ D=\mathbb{R}^2\times(-1,1)$.

At the beginning we verify that the function

$$F(x) = -c_0(x)\int\limits_{-1}^{1} b(\mu)f(x,\mu)d\mu \tag{34}$$

is integrable in $(-\infty,\infty)$. Really,

$$|F(x)| \le |c_0(x)|\left(\int\limits_{-1}^{1}|b(\mu)|^2\,d\mu\right)^{\frac{1}{2}}\left(\int\limits_{-1}^{1}|f(x,\mu)|^2\,d\mu\right)^{\frac{1}{2}}$$

therefore

$$\int\limits_{-\infty}^{\infty}|F(x)|\,dx \le M\int\limits_{-\infty}^{\infty}|c_0(x)|\left(\int\limits_{-1}^{1}|f(x,\mu)|^2\,d\mu\right)^{\frac{1}{2}}dx \le$$

$$\le M\left(\int\limits_{-\infty}^{\infty}|c_0(x)|^2\,dx\right)^{\frac{1}{2}}\left(\int\limits_{-\infty}^{\infty}\int\limits_{-1}^{1}|f(x,\mu)|^2\,d\mu dx\right)^{\frac{1}{2}}<\infty, \tag{35}$$

where $M = \left(\int\limits_{-1}^{1}|b(\mu)|^2\,d\mu\right)^{\frac{1}{2}}$.

Now we define the function $C(x,\mu)$ by the relation

$$f(x,\mu) = C(x,\mu)e^{i\frac{\sigma}{\mu}x}. \tag{36}$$

Taking into account the notation (34) we write (1) as following equation

$$-i\mu\frac{\partial f}{\partial x}(x,\mu) = \sigma f(x,\mu) + F(x).$$

Substituing (36) we obtain

$$\frac{\partial C(x,\mu)}{\partial x} = \frac{i}{\mu} e^{-i\frac{\sigma}{\mu}x} F(x)$$

therefore

$$C(x,\mu) = \frac{i}{\mu} \int_{-\infty}^{x} e^{-i\frac{\sigma}{\mu}y} F(y)dy + C_1(\mu). \qquad (37)$$

It follows from (35) that there exist the limit $\lim_{x\to+\infty} C(x,\mu) = A$. But right side if (36) belongs to $L^2(\mathbb{R})$ therefore $A = 0$. Then from (37) it results that

$$C_1(\mu) = -\frac{i}{\mu} \int_{-\infty}^{\infty} e^{-i\frac{\sigma}{\mu}y} F(y)dy$$

So, substituing in (37) we have

$$C(x,\mu) = -\frac{i}{\mu} \int_{x}^{\infty} e^{-i\frac{\sigma}{\mu}y} F(y)dy. \qquad (38)$$

Using again (35) we see that

$$\sup_{x,\mu} |\mu C(x,\mu)| < \infty$$

then

$$M = \sup_{x,\mu} |\mu f(x,\mu)| < \infty. \qquad (39)$$

Substituing (38), (34) in (36) we obtain

$$f(x,\mu) = \frac{i}{\mu} e^{i\frac{\sigma}{\mu}x} \int_{x}^{\infty} e^{-i\frac{\sigma}{\mu}y} \left(c_0(y) \int_{-1}^{1} b(\mu') f(y,\mu') d\mu' \right) dy$$

and later

$$|\mu f(x,\mu)| \le \int_{-\infty}^{\infty} \left(|c_0(y)| \int_{-1}^{1} \left| \frac{b(\mu')}{\mu'} \right| \cdot |\mu' f(y,\mu')| \, d\mu' \right) dy.$$

According to the definition (39) of the value M we have

$$M \le M \int_{-\infty}^{\infty} |c_0(y)| \, dy \cdot \int_{-1}^{1} \left| \frac{b(\mu')}{\mu'} \right| d\mu'.$$

If $f(x,\mu) \ne 0$ then $M > 0$ and we obtain obligatory

$$1 \le \int_{-\infty}^{\infty} |c_0(y)| \, dy \cdot \int_{-1}^{1} \left| \frac{b(\mu)}{\mu} \right| d\mu.$$

As consequence we obtain following theorem.

Theorem 3. Let $\sigma \in (-\infty,\infty)$, $\frac{b(\mu)}{\mu} \in L^1(-1,1)$ and $c_0 \in L^2(-\infty,\infty) \cap L^1(-\infty,\infty)$.

If

$$\int_{-1}^{1} \left| \frac{b(\mu)}{\mu} \right| d\mu \cdot \int_{-\infty}^{\infty} |c_0(x)| \, dx < 1$$

then the equation (1) has in the space $L^2(D)$ trivial solution $f(x,\mu) \equiv 0$ only.

Instead of conclusion note that items n. 2,3,5 belongs to Iv. and n.4 to Cher.

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О ТРАНСПОРТНОМ ОПЕРАТОРЕ ИЗ ЗАДАННЫМ СОБСТВЕННЫМ ЗНАЧЕНИЕМ

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Для некоторого транспортного оператора

$$-i\mu\frac{\partial f}{\partial x}(x,\mu)+c_0(x)\int\limits_{-1}^{1}b(\mu')f(x,\mu')d\mu'=\sigma f(x,\mu),\ -\infty<x<\infty,\ x\in\mathbb{R},\ -1<\mu<1$$

представлено явные примеры решения $f(x,\mu)$, функции $C_0(x)$, $b(\mu)$. Решение $f(x,\mu)$ представлено как многочлен, или, как серия полиномов Эрмита, или как функция типа $f(x,\mu)=h(x)g\left(\frac{x}{\mu}\right)$. В последних примерах представлены решения, которые принадлежат пространству $L^2(D_\pm)$, $D_\pm=\mathbb{R}_\pm\times(-1,1)$ и соответствует ограниченной функции C_0x. Для некоторых типов коэффициентов $C_0(x)$, $b(\mu)$ подано условие отсутствия собственного значения σ в случае пространства $L^2(D)$, $D=\mathbb{R}\times(-1,1)$.

Ключевые слова: транспортный оператор, спектр, точечный спектр, собственное значение.

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ПРО ТРАНСПОРТНИЙ ОПЕРАТОР ІЗ ЗАДАНИМ ВЛАСНИМ ЗНАЧЕННЯМ

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Для деякого транспортного оператора

$$-i\mu\frac{\partial f}{\partial x}(x,\mu)+c_0(x)\int\limits_{-1}^{1}b(\mu')f(x,\mu')d\mu'=\sigma f(x,\mu),\ -\infty<x<\infty,\ x\in\mathbb{R},\ -1<\mu<1$$

подано явні приклади розв'язання $f(x,\mu)$, функції $C_0(x)$, $b(\mu)$. Розв'язок $f(x,\mu)$ представлено як многочлен, або, як серія поліномів Ерміта, або як функція типу $f(x,\mu)= h(x)g\left(\frac{x}{\mu}\right)$. В останніх прикладах подано розв'язки, які належать простору $L^2(D_\pm)$, $D_\pm=\mathbb{R}_\pm\times(-1,1)$ і відповідають обмеженій функції C_0x. Для деяких типів коефіцієнтів $C_0(x)$, $b(\mu)$ подано умову відсутності власного значення σ у випадку простору $L^2(D)$, $D=\mathbb{R}\times(-1,1)$.

Ключові слова: транспортний оператор, спектр, точковий спектр, власне значення.

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