Оцінка нестійкості Кернела

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Похідне оцінювання є класичною проблемою фінансової математики [1,2]. Загальнопоширена стратегія вирішення цієї проблеми є вираження похідної рівноваження чи рівноважної ціни за допомогою певної фази змінної, наприклад за допомогою головної оцінки активів, не ризикованого рівня повернення, ціни реалізації опціону, часу завершеності тощо. Нажаль, існуюча теорія не враховує параметри оцінки чи непараметричний вибір застосованої нестійкості $\hat{s}$, яка є необхідною для which is necessary реплікації набору, що вміщує похідні високого рангу (опціони на майбутнє, опціони на опціони тощо), для розрахування стохастичного дисконтного фактору чи крайньої ставки підстановки так само й складання плану тривалої, нейтральної до ризику вирогідності щильності при використанні цін на похідні цінні папери. Також варто зазначити, що прийняти визначення такої інтенсивності як постійні функції з експериментальних цін X в більшості випадків доказують, що це є дастатьон складне завдання, через, по-перше, значень X, значень які визначаються переривчастою поведінкою, по-друге, вони є віддалені один від одного, відстань охоплює 2-5% ціни на основні активи, інтенсивність має задежати від сприйнятливості інвестора нести втрати від знецінення активів [3], іншими словами, від вразливості до ризику.

Цей абсолютний, відносний або усовний коефіцієнт є важливою характеристикою в управлінні активами за умов ринкової невизначеності. Значення такого коефіцієнту допоможе інвестору приймати рішення щодо фінансового інвестування при ризикованих або не ризикованих активах на короткий чи тривалий період, а також зробить можливим розрізняти нейтральних до ризику, ризико- сприймаючих та ризико-заперечуючих професійний біржевий учасників біржі, що в свою чергу перетворить впливи ліквідності та об'єм біржевої торгівлі.

Ця робота присвячена непараметричним оцінкам застосованої нестійкості при використанні функцій Кернела [4] з шириною частоти параметру h, залежно від ризику відхилення зі сторони інвестора. [5,6]. Обчислена $\hat{s}$ використовується для визначення справедливих цін для похідних російської фондової біржі.

Переклад зроблено Горьковою Н.Г., центр іноземних мов «Universal Talk», www.utalk.com.ua

Kernel estimation of the volatility[1]

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This study is devoted to nonparametric estimation of implied volatility, using kernel functions with bandwidth parameter h, depending on investor's risk aversion. The calculated $\hat{s}$ is used to find equitable prices for Russian stock market derivatives.

Keywords – Volatility, bandwidth, kernel estimation, option fee to futures.

I. Introduction

Derivative evaluation is a classical problem of financial mathematics [1,2]. Common strategy to solve this problem is the expression of derivative equilibrium or equitable price through some phase variable, for example through capital assets price, riskless rate of return, exercise price, time to maturity, etc. To our regret, the existing theory does not mention parametric estimation or nonparametric selection of implied volatility $\hat{s}$, which is necessary for replication of portfolio containing higher-degree derivatives (options on futures, options on option, etc.), for calculation of stochastic discount factor or marginal rate of substitution, as well as for plotting of continuous risk-neutral probability density using known prices for the derivatives. It should be noted that acceptable definition of such densities as continuous functions from exercise prices X in most cases proves to be quite a difficult task, because, firstly, X values are characterized by discrete behaviour, secondly, they are far from each other, the distance comprising 2-5% of capital assets price, and at last, thirdly, densities have to depend on investor's receptivity to bear losses from assets depreciation [3], in other words, from risk aversion. This absolute, relative or conditional coefficient is an important characteristic in assets management in conditions of the market uncertainty. Its value helps an investor to make a decision on financial investment in risk or riskless assets for a short-term or long-term period, and makes it possible to differentiate between risk-neutral, risk-preferring and risk-denying professional stock market participants, what in its turn influences liquidity or stock trading volume.

II. Kernel estimation of the volatility

Let's consider the portfolio π obtained by selling two call options struck at X and buying one struck at $X - \varepsilon$ and one at $X + \varepsilon$, where ε is infinitesimal quantity. The payoff

[1] This work was partially made under a financial support of FTP programm П691, 2010.

function of two last options pays nothing outside the interval $[X - \varepsilon; X + \varepsilon]$. Letting ε tend to zero, the payoff function tends to Dirac delta function with mass at *X*. As the call-option price $C(S_T, X, \tau)$ with underlying asset price S_T, strike price *X* and time to maturity τ is

$$C(S_T, X, \tau) = E^*\left(\frac{f_T}{e^{r\tau}}\right),$$

where *r* is risk free interest rate, E^* is risk free mathematical expectation with probability density function *f**. The limits of its price as ε tend tend to zero should therefore be equal to

$$C(S_T, X, \tau) = \exp(-r\tau) f^*(X).$$

On the other hand price the same option is

$$\frac{1}{e^2}\left[-2C(S_T, X, t) + C(S_T, X - e, t) + C(S_T, X + e, t)\right] \to C''_{XX}(S_T, X, t).$$

Therefore we have

$$f^*(X) = e^{r\tau} C''_{XX}(S_T, X, \tau). \quad (1)$$

It is worse to recall, that under the hypotheses of Black, Scholes and Merton, the date *t* price *C* of a call option maturing at date $T = t + \tau$, with strike price *X* written on a stock with date-*t* price S_t, dividend yield $\delta_{t,\tau}$ and risk free interest rate $r_{t,\tau}$ and risk free density function $f_{BS}{}^*$ is giving by following formula [1]:

$$C_{BS}(S_T, X, t, r_{t,t}, d_{t,t}, s) = e^{-r_{t,t} t} \int_R \max[S_T - X, 0] f_{BS}{}^*(S_t)\, dS_T,$$

or

$$C_{BS}(S_T, X, t, r_{t,t}, d_{t,t}, s) = S_t \Phi(d_1) - X e^{-r_{t,t} t} \Phi(d_2), \quad (2)$$

where

$$d_1 = \frac{\ln S_t - \ln X + (r_{t,\tau} - \delta_{t,\tau} + \sigma^2/2)\tau}{\sqrt{\tau}\,\sigma},$$

$$d_2 = d_1 - \sigma\sqrt{\tau}.$$

In this case corresponding risk free probability density $f_{BS}{}^*$ of the assets price S_t is a log normal density with mean $((r_{t,t} - d_{t,t}) - s^2/2)t$ and variance $s^2 t$:

$$f^*_{BS}(S_T) = e^{r_{t,t} t} \left|\frac{\partial^2 C_{BS}}{\partial X^2}\right|_{X=S_T} = \frac{1}{S_T\sqrt{2ps^2 t}} \exp\left[-\frac{\left[\ln(S_T/S_t) - (r_{t,t} - d_{t,t} - s^2/2)t\right]^2}{2s^2 t}\right]$$

Let's notice, that in most cases the probability density *f** cannot be represented in form of explicit analytical function. In [3] it has been proposed to estimate this nonparametric. For this purpose is used equation (1). In this method is held market-prices of option with different strike prices and times to maturity and linear regression is made. However, this approach is inappropriate, that it is necessary to recover continuous function *f**, when we are knowing the value of derivative $C''_{XX}(S_T, X, \tau)$ only in several points *X* (according to specification of contract strike prices are fixed). Therefore, in currently paper for making *f** we are going to use continuous prices of futures $F_{t,t}$. Let's denote vector of derivative characteristic or vector of repressors as $Y \equiv [F_{t,t}, X, t, r_{t,t}]$.

Supposing that the call pricing function is given by parametric Black-Scholes formula (2), except the implied volatility parameter for that option is nonparametric function $s(X/F_{t,t}, t)$:

$$C(S_t, X, t, r_{t,t}, d_{t,t}) = C_{BS}(F_{t,t}, X, t, r_{t,t}, d_{t,t}, s(X/F_{t,t}, t)).$$

We assume that the function $C(S_t, X, \tau, r_{t,\tau}, \delta_{t,\tau})$ defined by equation (2) satisfies all the requirement condition to be rational option-pricing formula in the sense of Merton.

Let's use Nadaraya-Watson kernel estimator for estimating $s(X/F_{t,t}, t)$ [7, 8]:

$$\hat{s}(X/F_{t,t}, t) = \frac{\sum_{i=1}^{n} k_{X/F}\left(\frac{X/F_{t,t} - X_i/F_{t_i,t_i}}{h_{X/F}}\right) k_t\left(\frac{t - t_i}{h_t}\right) s_i}{\sum_{i=1}^{n} k_{X/F}\left(\frac{X/F_{t,t} - X_i/F_{t_i,t_i}}{h_{X/F}}\right) k_t\left(\frac{t - t_i}{h_t}\right)}, \quad (4)$$

where σ_i is volatility implied by the option price C_i, *i*=1,2,3,…, and the univariate kernel functions $k_{X/F}$, k_t are chosen to optimize the asymptotic properties of the second derivative $\hat{C}''_{XX}(S_T, X, t)$, i. e. getting in *X/F* and τ, with different bandwidth parameter $h_{X/F}$, h_t and order $q_{X/F}$, q_τ respectively. We would use kernel function $k(x)$ of order $q = 2$ and $q = 4$ for further calculation

$$k_{(2)}(z) = \frac{1}{\sqrt{2}} e^{-\frac{z^2}{2}}, \quad k_{(4)}(z) = \frac{1}{\sqrt{8p}}(3 - z^2) e^{-\frac{z^2}{2}}.$$

Let's choose bandwidth parameter [3]:

$$h_{X/F} = \gamma_{X/F}\, s_{X/F}\, n^{-1/(8+2q)} / \ln n,$$

$$h_\tau = \gamma_\tau\, s_\tau\, n^{-1/(2+2q)} / \ln n, \quad (5)$$

where

$$n = \sum_{i=1}^{J} N_i$$

N_i is number of observation for *i*-th object, *J* is number of option for viewing period, $s_{X/F}$ and s_τ are unconditional standard deviation for nonparametric regressors *X/F* and τ respectively; $\gamma_{X/F}$, γ_τ are constant value of the risk aversion.

III. Calculation of the volatility for the LUKOIL share

Intraday prices of the LUKOIL share, respective futures and options for period 13.12.2010 to 01.03.2011 are underlain the analysis. In the condition of Russian market it is impossible to find any assets, which has many futures and options correspond with period. In case of LUKOIL share it is impossible to find more than two, therefore the order of summation for formula (4) will be 2, that is *J = 2*, and than *n* = 165. $\gamma_{X/F}$ had been calculated on the basis of assets price dynamic, than it has been averaged at ensemble. The diagram of risk aversion is shown at picture. Γ_τ has been taken as one for simplicity of statement. The input data are given in table 1. in spite of lack of data has been obtained satisfactory results. These results are given in table 2.

TABLE I

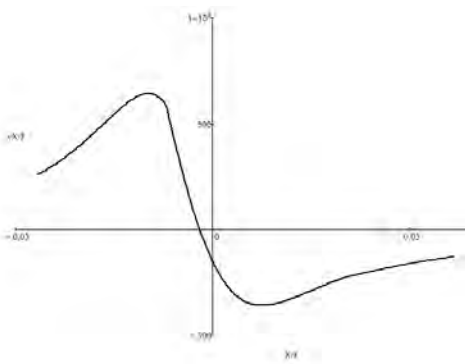
Input data

Assets	Date	τ	*T*	*C*	*S*	*F*	*X*	σ	*N*
LKOH3.11	01.02.11	41	14.03.11	1215	1904	18765	18000	27.76	41
LKOH6.11	21.02.11	113	14.06.11	431	1956.71	19284	19500	18.47	113

TABLE II

Theoretical and implied volatility for LKOH 3.11 option with strike price *X* = 19000 and for LKOH 6.11 with strike price *X* = 19500

LKOH 3.11 *X* = 19000			LKOH 6.11 *X* = 19500		
Date	σ	$\hat{\sigma}$	Date	σ	$\hat{\sigma}$
02.03.2011	28.48342	23.874	02.03.2011	20.82048	21.86
03.03.2011	29.80361	24.752	03.03.2011	21.30841	22.079
04.03.2011	35.76987	22.874	04.03.2011	20.25472	22.021
05.03.2011	36.87873	23.495	05.03.2011	21.60334	22.128
09.03.2011	42.70936	23.809	09.03.2011	23.61018	22.232
10.03.2011	31.84388	23.579	10.03.2011	24.37756	21.989



Values of risk aversion coefficient for X/F.

IV. Calculation of option price for futures under a LUKOIL share

Prior to calculate option price for futures, we have derived Black-Scholes formula for price of this option. Let's prove the theorem for this purpose.

Theorem.

Let it have got the portfolio with the same type and the same time to maturity futures. The futures price in moment of *t*, and time to maturity τ, and underlying asset price S_t is $F_{t,t} = S_t e^{rt}$, where *r* is a risk free interest rate, $S_t e^{rt}$ is discounted at another moment value underlying asset price. Then the Black-Scholes formula is obtained by simple replacement S_t by $S_t = F_{t,t} e^{-rt}$, then Black-Scholes formula has a following form:

$$C\left(F_{t,t}, t\right) = F_{t,t}\, e^{-rt}\Phi(d_1) - X e^{-rt}\Phi(d_2),$$

where

$$d_1 = \frac{\ln\frac{F_{t,t}}{X} + \frac{s^2}{2}t}{\sqrt{t}\; s},\quad d_2 = d_1 - s\sqrt{t},$$

and $X = F_{t,t}$.

Proving

Let's sell call option $C\left(F_{t,t}, t\right)$ for Δ futures with time to maturity $\tau = (T\text{-}t)$. Let price evolution of the underlying assets for futures with price $F_{t,t}$ be assigned by stochastic differential equation

$$dS_t = mS_t dt + s\, S_t dW$$

The futures price for S_t share has equitable price $F_{t,t} = S_t e^{rt}$, that also is stochastic variable. Let'apply for $F_{t,t}$ Ito's formula:

$$dF_{t,t} = \frac{\partial F_{t,t}}{\partial t}dt + \frac{1}{2}\frac{\partial^2 F_{t,t}}{\partial S_t^2}\left(dS_t\right)^2 + \frac{\partial F_{t,t}}{\partial S_t}dS_t .$$

As $\frac{\partial F_{t,t}}{\partial t} = e^{rt}$ and $\frac{\partial^2 F_{t,t}}{\partial S_t^2} = 0$, therefore

$$dF_{t,t} = \frac{\partial F_{t,t}}{\partial t}dt + e^{rt}\, dS_t = \left(-rS_t dt + dS_t\right)e^{rt} .$$

And so $\left(dF_{t,t}\right)^2 = \left(dS_t\right)^2 e^{2rt}$, as $\left(dS_t\right)^2 = s^2 S_t^2 dt$, then

$$\left(F_{t,t}\right)^2 = s^2 S_t^2 e^{2rt}\, dt = s^2 F_{t,t}^2 dt .$$

Let's apply for $C\left(F_{t,t}, t\right)$ Ito's formula:

$$dC = \frac{\partial C}{\partial t}dt + \frac{1}{2}\frac{\partial^2 C}{\partial F_{t,t}^2}\left(dF_{t,t}\right)^2 + \frac{\partial C}{\partial F_{t,t}}dF_{t,t} =$$

$$= \frac{\partial C}{\partial t}dt + \frac{1}{2}\frac{\partial^2 C}{\partial F_{t,t}^2}s^2 F_{t,t}^2\, dt + \frac{\partial C}{\partial F_{t,t}}dF_{t,t} .$$

Let's consider portfolio with option for futures and Δ futures:

$$\Pi = C - \Delta F_{t,t},\quad d\Pi = dC - \Delta dF_{t,t} .$$

This implies

$$d\Pi = \frac{\partial C}{\partial t}dt + \frac{1}{2}\frac{\partial^2 C}{\partial F_{t,t}^2} s^2 F_{t,t}^2 dt + \frac{\partial C}{\partial F_{t,t}} dF_{t,t} - \Delta dF_{t,t} \cdot$$

The two last summand determine stochastic part, that it need to nullify. Let $\Delta = \frac{\partial C}{\partial F_{t,t}}$, then we get risk free portfolio Π and

$$d\Pi = \frac{\partial C}{\partial t}dt + \frac{1}{2}\frac{\partial^2 C}{\partial F_{t,t}^2} s^2 F_{t,t}^2 dt \cdot$$

We reduce this task to derivation of the Black-Scholes equation and Black-Scholes formula for option for S_t. assets. For calculation this option price is correct Black-Scholes formula:

$$C_{BS}(S_T, t) = S_t \Phi(d_1) - Xe^{-rt}\Phi(d_2).$$

However, we have now $S_t = F_{t,t}e^{-rt}$:

$$C(F_{t,t}, t) = F_{t,t}e^{-rt}\Phi(d_1) - Xe^{-rt}\Phi(d_2),$$

where

$$d_1 = \frac{\ln\frac{S_t}{X} + (r + s^2/2)t}{\sqrt{t}\, s} = \frac{\ln\frac{S_t e^{-rt}}{X} + \ln e^{-rt} + rt + s^2/2t}{\sqrt{t}\, s} = $$

$$= \frac{\ln\frac{F_{t,t}}{X} + \frac{s^2}{2}t}{\sqrt{t}\, s}$$

by analogy $d_2 = d_1 - s\sqrt{t}$. That will be the Black-Scholes formula for option for the futures.

Quod erat demonstrandum

Let's calculate this option price for futures LKOH 3.11 with τ = 12, 11, 10, 9, 5, 4 day and strike price X = 19000. And for the case of LKOH 6.11 with τ = 104, 103, 102, 101, 97, 96 day and strike price X = 19500 by following way:

$$C(F_{t,t}, X, t, r_{t,t}, s) = F_{t,t}e^{-r_{t,t}t}\Phi(d_1) - Xe^{-r_{t,t}t}\Phi(d_2). \quad (6)$$

The deriving results are given in table 3.

TABLE III

The value of otion price, that has been calculated by formula (6) for futures LKOH 3.11 with strike price X = 19000 and LKOH 6.11 with strike price X = 19500

LKOH 3.11 X = 19000		LKOH 6.11 X = 19500	
Date	Option price	Date	Option price
02.03.2011	6095	02.03.2011	0.604
03.03.2011	6823	03.03.2011	0.68
04.03.2011	7501	04.03.2011	0.746
05.03.2011	8355	05.03.2011	0.832
09.03.2011	1252	09.03.2011	1.247
10.03.2011	1354	10.03.2011	1.347

Conclusion

This study is devoted to nonparametric estimation of implied volatility, using kernel functions [4] with bandwidth parameter *h*, depending on investor's risk aversion [5,6]. The calculated $\hat{s}$ is used to find equitable prices for Russian stock market derivatives.

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