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Computation of positive stable realizations of fractional continuous-time linear systems

The positive stable realization problem for continuous-time linear systems has been formulated and partly solved in [1, 2]. In [3] the positive realization problem of fractional 2D discrete-time linear systems has been investigated. Sufficient conditions will be established for the existence of positive stable realizations with system Metzler matrix of fractional continuous-time linear systems and a procedure for computation of the realizations of proper transfer matrices will be proposed.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ - the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, $\Re^{n\times m}(s)$ - the set of $n\times m$ polynomial matrices in s with real coefficients, M_n - the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n - the $n\times n$ identity matrix.

Consider the continuous-time linear system

$$ {}_0D_t^\alpha x(t) = Ax(t) + Bu(t),\ y(t) = Cx(t) + Du(t),\ 0<\alpha<1 \quad (1)$$

where $x(t)\in\Re^n$, $u(t)\in\Re^m$, $y(t)\in\Re^p$ are the state, input and output vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$, $D\in\Re^{p\times m}$,

$$ {}_0D_t^\alpha x(t) = \frac{d^\alpha x(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\dot{x}(\tau)}{(t-\tau)^\alpha}d\tau,\ \dot{x}(\tau) = \frac{dx(\tau)}{d\tau}.$$

is the Caputo definition of $\alpha\in\Re$ order derivative and $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$ is the Euler gamma function. The fractional system (1) is called positive if $x(t)\in\Re_+^n$, $y(t)\in\Re_+^p$, $t\ge 0$ for any initial conditions $x(0)=x_0\in\Re_+^n$ and all inputs $u(t)\in\Re_+^m$, $t\ge 0$. The fractional system (1) is positive if and only if

$$A\in M_n,\ B\in\Re_+^{n\times m},\ C\in\Re_+^{p\times n},\ D\in\Re_+^{p\times m}. \quad (2)$$

The transfer matrix of the fractional system (2.1) is given by

$$T(s^\alpha) = C[I_n s^\alpha - A]^{-1}B + D = C[I_n\lambda - A]^{-1}B + D,\ s^\alpha = \lambda \quad (3)$$

The transfer matrix is called proper if $\lim_{\lambda\to\infty} T(\lambda) = K\in\Re^{p\times m}$ and it is called strictly proper if $K=0$. Matrices (2) are called a positive realization of transfer matrix $T(\lambda)\in\Re^{p\times m}(\lambda)$ if they satisfy the equality (3). The realization is called minimal if the dimension of A is minimal among all realizations of $T(\lambda)$. The realization is called (asymptotically) stable if the matrix A is a (asymptotically) stable Metzler matrix (Hurwitz Metzler matrix). Given a rational matrix $T(\lambda)\in\Re^{p\times m}(\lambda)$, find a positive stable realization with system Metzler matrix A of $T(\lambda)$, i.e.

$$A\in M_{nS},\ B\in\Re_+^{n\times m},\ C\in\Re_+^{p\times n},\ D\in\Re_+^{p\times m} \quad (4)$$

where M_{nS} is the set of $n\times n$ stable Metzler matrices.

Consider a stable positive continuous-time linear system (1) with a given proper transfer matrix of the form

$$T(\lambda)\in\Re^{p\times m}(\lambda) \quad (5)$$

where $\Re^{p\times m}(\lambda)$ is the set of proper rational real matrices. The matrix D can be found by the use of the formula

$$D = \lim_{\lambda\to\infty} T(\lambda) \quad (6)$$

and the strictly proper transfer matrix

$$T_{sp}(\lambda) = T(\lambda) - D \quad (7)$$

which can be written in the form

$$T_{sp}(\lambda) = \frac{N(\lambda)}{d(\lambda)}\in\Re^{p\times m}(\lambda) \quad (8)$$

where $N(\lambda)\in\Re^{p\times m}[\lambda]$ (the set of $p\times m$ polynomial matrices) and the polynomial $d(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0$ has only distinct real negative zeros $\lambda_1, \lambda_2, ..., \lambda_n$. In this case the transfer matrix (8) can be written in the form

$$T_{sp}(\lambda) = \sum_{i=1}^n \frac{T_i}{\lambda - \lambda_i} \quad (9)$$

where

$$T_i = \lim_{s\to s_i}(\lambda - \lambda_i)T_{sp}(\lambda) = \frac{N(\lambda_i)}{\prod_{j=1, j\ne i}^n(\lambda_i - \lambda_j)},\ i=1,...,n \quad (10)$$

Let $\operatorname{rank} T_i = r_i \le \min(p,m)$. It is easy to show that

$$T_i = C_iB_i,\ \operatorname{rank} C_i = \operatorname{rank} B_i = r_i,\ i=1,...,n \quad (11)$$

where

$$C_i = [C_{i,1}\ \ C_{i,2}\ \ \dots\ \ C_{i,r_i}]\in\Re^{p\times r_i},\ B_i = \begin{bmatrix} B_{i,1} \\ B_{i,2} \\ \vdots \\ B_{i,r_i}\end{bmatrix}\in\Re^{r_i\times m}.$$

The desired positive stable realization with system Metzler matrix has the form

$$A = \operatorname{blockdiag}[I_{r_1}\lambda_1\ \ \dots\ \ I_{r_n}\lambda_n],\ B = \begin{bmatrix} B_1 \\ \vdots \\ B_n\end{bmatrix},\ C = [C_1\ \ \dots\ \ C_n] \quad (12)$$

There exists a positive stable realization (11) and (6) of (5) if the following conditions are satisfied:

1) The poles of $T(s)$ are distinct real and negative $\lambda_i \ne \lambda_j$ for $i\ne j$, $\lambda_i < 0$, $i=1,...,n$.
2) $T_i\in\Re_+^{p\times m}$ for $i=1,...,n$,
3) $T(\infty)\in\Re_+^{p\times m}$.

Procedure.

Step 1. Using (6) find the matrix D and the strictly proper transfer matrix (7) and write it in the form (8).

Step 2. Find the real zeros $\lambda_1, \lambda_2, ..., \lambda_n$ of the polynomial $d(\lambda)$.

Step 3. Using (10) find the matrices $T_1, ..., T_n$ and their decomposition (11).

Step 4. Using (12) find the matrices A, B, C.

REFERENCES

[1] Kaczorek T., *Realization problem for fractional continuous-time systems*, Archives of Control Sciences, vol. 18, no. 1, 2008, 43-58.

[2] Kaczorek T., *Computation of positive stable realizations for linear continuous-time systems*, Submited to European Conference on Circuits Theory and Design (ECCTD 2011), Linköping, Sweden, August 29-31, 2011.

[3] Sajewski Ł., *Positive realization of SISO 2D different orders fractional discrete-time linear systems*, Acta Mechanica et Automatica, 2011.

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