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SYMMETRY OF DIFFERENTIAL EQUATIONS OF ELECTROMAGNETIC FIELD IN IMMOVABLE MEDIUM

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Abstract: In the paper it is shown that differential equations describing an electromagnetic field in solid immovable media based on the vectors of intensities of electric or magnetic fields in most cases practically coincide with those based on the vector-potential of the electromagnetic field. This special feature essentially simplifies practical analysis, beginning with the determination of border conditions. The expressions for static and differential parameters of the medim are given.

Key words: differential equations, vectors and vector-potential of electromagnetic field, non-linearity, anisotropy, static and differential parameters, symmetry.

Introduction

The analysis of electromagnetic processes in a solid medium is based on partial differential equations describing vectors or potentials of the electromagnetic field. The choice of equations in every given case depends on a minimum number of differential equations which are to be integrated, or on the method of determining border conditions, or, sometimes, on the possibility of determining medium parameters. In the last case, as a rule, we must resort to a linear coordinate transformation [16]. The choice of differential equations sometimes depends on the degree of detalization of physical properties of the medium, for example, non-linearity, anisotropy etc. In the paper it is shown that differential equations of electromagnetic field in the linear isotropic medium written using different vectors or potentials fully coincide with each other. In non-linear or isotropic media they can differ only in matrixes of differential or static medium parameters. Taking this symmetry into consideration, we can considerably simplify the analysis not only on the stage of forming a specific boundary problem of space-time electrodynamics but in the main while creating algorithms and applied computer programs. In that case the algorithm of integration of differential equations set for some vector may be fully used for integration of differential equations set for another vector or potential. The proposed method of analysis has been thoroughly investigated during the computer simulation.

We obtain theoretical results step by step: at first for a linear isotropic medium, later on the results are given in full form for a non-linear isotropic and anisotropic media as well as in form of quasi-stationary approximation.

Linear isotropic medium. Maxwell's equations, which belong to the main differential equations of linear media electrodynamics, can be written using vectors of an electromagnetic field,

$$\nabla \times \mathbf{H} = \gamma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t}; \qquad (1)$$

$$\nabla \times \mathbf{E} = -\frac{1}{\nu} \frac{\partial \mathbf{H}}{\partial \mathrm{t}}, \qquad (2)$$

where **E**, **H** – vectors of the intensities of electric and magnetic fields; γ – conductivity; ε – permittivity; ν – reluctivity.

We obtain calculating equations by excluding the vector **E** or the vector **H**.

So, having applied the operator $\nabla\times$ to (2), we get as follows

$$\nabla \times \nabla \times \mathbf{E} = -\frac{1}{\nu} \frac{\partial}{\partial t} (\nabla \times \mathbf{H}). \qquad (3)$$

Having substituted the right part of (3) for its expression (1), we receive the calculating differential equation for the vector of electric field intensity

$$\frac{\varepsilon}{\nu} \frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{\gamma}{\nu} \frac{\partial \mathbf{E}}{\partial t} = -\nabla \times \nabla \times \mathbf{E}. \qquad (4)$$

Having obtained the spatial-time distribution of the vector **E** as a result of integration of (4) at given initial and boundary conditions, we get the vector **H** as a result of time integration of (2). The vectors of electric induction **D** and magnetic induction **B** are obtained according to the expressions as follows

$$\mathbf{B} = \mathbf{H} / \nu; \qquad (5)$$

$$\mathbf{D} = \varepsilon \mathbf{E}. \qquad (6)$$

Now we apply the operator $\nabla\times$ to (1):

$$\nabla \times \nabla \times \mathbf{H} = \gamma \nabla \times \mathbf{E} + \frac{\partial}{\partial t} (\nabla \times \mathbf{E}). \qquad (7)$$

Having substituted the operator in the right part of (7) for its expression (1), we obtain a calculating differential equation for the vector **H**:

$$\frac{\varepsilon}{\nu}\frac{\partial^2 \mathbf{H}}{\partial t^2}+\frac{\gamma}{\nu}\frac{\partial \mathbf{H}}{\partial t}=-\nabla\times\nabla\times\mathbf{H}. \quad (8)$$

In this case we can find the vector **E** according to (1), and other vectors – according to (5), (6).

Using the vector **A** and the scalar potentials φ of electromagnetic field, we should avoid applying Lorenz's gauge which does not simplify the analysis in non-linear media, but, on the contrary, complicates it. In this case it is reasonable to accept the gauge [8–10]

$$\varphi = 0 . \quad (9)$$

Then we can find the vectors **B** and **H** according to the spatial-time distribution of the vector **A**:

$$\mathbf{B} = \nabla\times\mathbf{A} ; \quad (10)$$

$$\mathbf{E} = -\,\partial\mathbf{A}/\partial t. \quad (11)$$

Having substituted the vectors in (1) for the expressions (5), (10), and (11), we get the calculating differential equation of the vector potential

$$\frac{\varepsilon}{\nu}\frac{\partial^2 \mathbf{A}}{\partial \mathrm{t}^2}+\frac{\gamma}{\nu}\frac{\partial \mathbf{A}}{\partial \mathrm{t}}=-\nabla\times\nabla\times\mathbf{A}. \quad (12)$$

As we can see, calculating differential equations of electromagnetic field describing vectors (4), (8) and potential (12) are identical. By the way, (4) and (12) coincide in form not only in a vector entry, but also being applied in this or that coordinate system, what is the consequence of the spatial colinearity of vectors **A** and **E** according to (11). Equation (8) in this or that coordinate system may differ from the first two in the consequence of a probably different spatial orientation of the vector **H** in comparison with the orientation of the vectors **A** and **E**.

Non-linear anisotropic medium. In a non-linear anisotropic medium the expressions (5), (6) are more complicated [10, 11]

$$\mathbf{H} = \mathrm{N}\mathbf{B}; \quad (13)$$

$$\mathbf{D} = \Xi\mathbf{E}, \quad (14)$$

where N is the matrix of static reluctivities; Ξ is the matrix of static permittivities.

Taking into consideration (13), (14), the equations (1), (2) are more complicated as well

$$\nabla\times\mathbf{H} = \Gamma\mathbf{E}+\Xi_{\mathrm{d}}\frac{\partial\mathbf{E}}{\partial \mathrm{t}}; \quad (15)$$

$$\nabla\times\mathbf{E} = -\mathrm{N}_{\mathrm{d}}^{-1}\frac{\partial\mathbf{H}}{\partial t}, \quad (16)$$

where Γ is the matrix of static conductivities; Ξ_{d} is the matrix of differential permittivities; N_{d} is the differential reluctivities matrix.

Having substituted the vectors in (15) for the expressions (10), (11), (13), we obtain an analogous differential equation for the vector potential **A**

$$\Xi_{\mathrm{d}}\frac{\partial^2\mathbf{A}}{\partial \mathrm{t}^2}+\Gamma\frac{\partial\mathbf{A}}{\partial \mathrm{t}} = -\nabla\times\mathrm{N}\nabla\times\mathbf{A}. \quad (17)$$

Thereby, in formula (17) we can see matrix N of static reluctivities, matrix Γ of static conductivities and matrix Ξ_{d} of differential permittivities.

Unfortunately, analogous differential equations for vectors of electric and magnetic fields cannot be obtained in acceptable form using formula (17). They are quite complicated and do not have desirable symetry. By the way, the solutions here are simpler for vectors **B**, **D**, but it seems to be meaningless to give here these expressions. It is possible to be done for vector-potential **A** only, because it alone is the fundamental vector of electromagnetism. The field vectors **D**,**E**,**B**,**H** are spatial-time derivatives of **A** (10), (11), (13), (14). It is confirmed by the recent investigations [3, 17], which are made on the basis of the leading principles of energy variation. It can be shown that by using the Hamilton-Ostrogradsky principle we can obtain only equation (17).

If gyro effects are not taken into consideration, the matrixes of static parameters S in main anisotropy axes, as a rule, are diagonal [5]

$$S = \mathrm{N} = \mathrm{diag}(s_x, s_y, s_z). \quad (18)$$

The elements of matrix N (18) are found using the B-H curve in the main axes of magnetization [8]

$$H^{(i)} = H^{(i)}(B),\ \ i = x, y, z, \quad (19)$$

where B is a module of **B**

$$B = \sqrt{B_x^2+B_y^2+B_z^2}, \quad (20)$$

as such derivatives

$$s_i = H^{(i)}(B)/B,\ \ i = x, y, z. \quad (21)$$

Matrix S loses its diagonal form only by rotating main coordinate axes by the angle φ. It is done on grounds of linear transformation matrixes [17].

The formulae of forth and back coordinate transformations have a canonical form

$$\boldsymbol{\lambda}_{\Pi} = \Pi\boldsymbol{\lambda},\quad \boldsymbol{\lambda} = \Pi^{-1}\boldsymbol{\lambda}_{\Pi}, \quad (22)$$

where $\boldsymbol{\lambda}$ is a vector; Π is a transformation matrix. In 2D-dimensional space we have

$$\Pi = \begin{vmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{vmatrix}; \ \Pi^{-1} = \begin{vmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{vmatrix}, \quad (23)$$

It is important that $\Pi^{-1} = \Pi_t$.

So, by rotating around the *z*-axe this matrix takes the form [17]

$$N' = \begin{vmatrix} s_x\cos^2\varphi + s_y\sin^2\varphi & (s_x - s_y)\sin\varphi\cos\varphi & \\ (s_x - s_y)\sin\varphi\cos\varphi & s_x\sin^2\varphi + s_y\cos^2\varphi & \\ & & s_z \end{vmatrix} \quad (24)$$

The determination of matrixes of differential parameters *D* is a vastly difficult problem as opposed to matrixes of static parameters *S*. It is convenient to use there an expression, which connects matrixes *S* and *D* [7]

$$D(X) = \frac{\partial S(X)}{\partial X} X + S(X), \quad (25)$$

where $X = (B_x, B_y, B_z)_t$ is a column of arguments.

The matrix N_d according to (21), (25) is always filled [10, 19]

$$D{=}N_d = \begin{vmatrix} s_x + (d_x - s_x)B_x^2/B^2 & (d_x - s_x)B_xB_y/B^2 & (d_x - s_x)B_xB_z/B^2 \\ (d_y - s_y)B_yB_x/B^2 & s_y + (d_y - s_y)B_y^2/B^2 & (d_y - s_y)B_yB_z/B^2 \\ (d_z - s_z)B_zB_x/B^2 & (d_z - s_z)B_zB_y/B^2 & s_z + (d_z - s_z)B_z^2/B^2 \end{vmatrix}, \quad (26)$$

where

$$d_i = dH^{(i)}(B)/dB, \ \ i = x, y, z. \quad (27)$$

The elements (21) are main static reluctivities, the elements (27) are main differential reluctivities of the anisotropic medium in main axes of ferromagnetic magnetization. In an isotropic medium $s_x = s_y = s_z = s; \ \ d_x = d_y = d_z = d$ and matrixes (18) and (26) are simpler.

In **an isotropic non-linear medium** the differential equations (17) are much more simpler, since the matrixes of static parameters Γ, N are reduced to an identity matrix multiplied by scalar values γ, ν, and differential parameters Ξ_d (26) are simplified to E_d, because

$$s_x = s_y = s_z = s; \ \ d_x = d_y = d_z = d \ .$$

Then we get

$$E_d \frac{\partial^2 \mathbf{A}}{\partial t^2} + \gamma \frac{\partial \mathbf{A}}{\partial t} = -\nabla \times \nu \nabla \times \mathbf{A}. \quad (28)$$

Quasistationary approach. The largest interest concerning the differential equations in the non-linear anisotropic media is taken in a quasistationary approach $(\partial \mathbf{D}/\partial t = 0)$. In such a case we obtain

$$\frac{\partial \mathbf{A}}{\partial t} = -P\nabla \times N\nabla \times \mathbf{A}. \quad (29)$$

where $P = \Gamma^{-1}$ is the matrix of static resistivity.

In the quasi-stationary approach there is a possibility to get differential equations for vectors of electric and magnetic fields analogous to (29).

Solving (15) with respect to **E**, we receive

$$\mathbf{E} = P\nabla \times \mathbf{H}. \quad (30)$$

Having expressed (16) in a natural form

$$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t \quad (31)$$

and having put into this expression eqn. (30) and eqn. (13), we get

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times P\nabla \times \mathbf{H}; \quad \mathbf{H} = N\mathbf{B}. \quad (32)$$

The equations (32) are irreplaceable in practical computations, but according to (29) we may get another form

$$\frac{\partial \mathbf{H}}{\partial t} = -N_d \nabla \times P\nabla \times \mathbf{H}. \quad (33)$$

Having differentiated (30) with respect to *t*, we obtain

$$\frac{\partial \mathbf{E}}{\partial t} = P_d \nabla \times \frac{\partial \mathbf{H}}{\partial t}. \quad (34)$$

Having solved (31) concerning the vector **H** and substituting the obtained result into (34) we get finally

$$\frac{\partial \mathbf{E}}{\partial t} = -P_d \nabla \times N_d \nabla \times \mathbf{E}; \quad (35)$$

Naturally, the equations (29), (32), (35) in non-linear isotropic media will be simpler.

In practical analysis we prefer to apply the equation of vector-potential (29) and the equation of vector magnetic induction (32), because they contain matrixes of static medium parameters, which are simpler than analogous matrixes of differential parameters in equation (35).

The results obtained present not only formal mathematical exercising in electromagnetic field theory, but they give the possibility to grasp the idea of nature of electromagnetic phenomena in solid media from a different point of view. They also help interested persons get their bearing in the complicated labyrinth of numerical analysis applied to programming and computation.

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СИМЕТРІЯ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ЕЛЕКТРОМАҐНІТНОГО ПОЛЯ В НЕРУХОМОМУ СЕРЕДОВИЩІ

В. Чабан

Показано, що диференціальні рівняння електромаґнетного поля в суцільному нерухомому середовищі, записані стосовно векторів напружености електричного й маґнетного полів чи стосовно вектор-потенціалу електромаґнетного поля, у переважній більшості випадків збігаються за формою одне з одним. Ця особливість суттєво спрощує практичний аналіз, перш за все під кутом зору визначення крайових умов. Подаються вирази статичних і диференційних параметрів середовища.

Vasyl Tchaban – Full Professor of Lviv Polytechnic National University, member of The New York Academy of Science, member of National Association of Ukrainian Writers, Editor in chief of Technical News journal.

Born in Ukraine in 1940, he graduated with honors from Lviv Polytechnic National University.

V. Tchaban is the author of 28 books, about 500 scientific papers, over 500 surrealistic short-stories, over 1000 short aphorisms. He carries out research concerning electromagnetic circuits with concentrated and distributed parameters, an electromagnetic field theory related to nonlinear isotropic and anisotropic media, the theory of electric machines, the theory of independent electromechanical systems, dynamics of machines, thermodynamics, natural magnetic levitation, vector analysis, history of science. Special attention is given to the investigations of complex interdependent electrical, mechanical and thermal processes.

Prof. V. Tchaban is known for his innovations in scientific theory. He is skilled in formulating accurate mathematical models of analysis in temporal domain of physical transient and steady-state processes, static stability and parametric sensitivity. He has provided basics of three essentially new modes (circuit, circuit-field and pure field mode) of mathematical models of electric devices.

Resently he has started to attack the problem of building a common energetic principle of the mathematical simulation of physical processes of different nature.