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COMPUTATION OF POSITIVE STABLE REALIZATIONS FOR DISCRETE-TIME LINEAR SYSTEMS

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Abstract: Sufficient conditions for the existence of positive stable realizations for given proper transfer matrices are established. Two methods are proposed for determination of the positive stable realizations for given proper transfer matrices. The effectives of the proposed procedures is demonstrated on numerical examples.

Key words: computation, initial condition, observability, positive, discrete-time, linear, system

1. Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in the monographs [2, 6]. Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

An overview on the positive realization problem is given in [1, 2, 6]. The realization problem for positive continuous-time and discrete-time linear systems has been considered in [3, 4, 7, 10, 11] and the positive minimal realization problem for singular discrete-time systems with delays in [12]. The realization problem for fractional linear systems has been analyzed in [8, 15] and for positive 2D hybrid systems in [9]. The existence of positive stable realizations with system Metzler matrices for continuous-time linear systems has been investigated in [13, 14].

In this paper sufficient conditions will be established for the existence of positive stable realizations of discrete-time linear systems and two methods for determination of the positive stable realizations of proper transfer matrices will be proposed.

The paper is organized as follows. In section 2 some definitions and theorems concerning positive discrete-time linear systems are recalled and the problem formulation is given. In section 3 two methods for determinations of positive stable realizations for given transfer functions are proposed. An extension of the method 1 to multi-input multi-output discrete-time linear systems is proposed in section 4. Concluding remarks and open problems are given in section 5.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, $\Re^{n\times m}(z)$ – the set of $n\times m$ real matrices in z with real coefficients, $\Re^{n\times m}[z]$ – the set of $n\times m$ polynomial matrices in z with real coefficients, I_n - the $n\times n$ identity matrix.

2. Preliminaries

Consider the linear discrete-time systems

$$x_{i+1} = Ax_i + Bu_i, \quad i \in Z_+ = \{0,1,...\}, \tag{2.1a}$$

$$y_i = Cx_i + Du_i, \tag{2.1b}$$

where $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$ are the state, input and output vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$, $D\in\Re^{p\times m}$.

Definition 2.1. The system (2.1) is called (internally) positive if $x_i \in \Re_+^n$, $y_i\in\Re_+^p$, $i\in Z_+$ for any initial conditions $x_0\in\Re_+^n$ and all inputs $u_i\in\Re_+^m$, $i\in Z_+$.

Theorem 2.1. [2, 3] The system (2.1) is (internally) positive if and only if

$$A\in\Re_+^{n\times n},\quad B\in\Re_+^{n\times m},\quad C\in\Re_+^{p\times n},\quad D\in\Re_+^{p\times m}. \tag{2.2}$$

The transfer matrix of the system (2.1) is given by

$$T(z) = C[I_n z - A]^{-1}B + D \tag{2.3}$$

The transfer matrix is called proper if

$$\lim_{z\to\infty} T(z) = K \in \Re^{p\times m} \tag{2.4}$$

and it is called strictly proper if $K = 0$.

Definition 2.2. Matrices (2.2) are called a positive realization of transfer matrix $T(z)$ if they satisfy the equality (2.3).

The positive system (2.1) is called asymptotically stable (shortly stable) if

$$\lim_{i\to\infty} x_i = 0 \text{ for all initial conditions } x_0 \in \Re_+^n. \quad (2.5)$$

Theorem 2.2. [1, 6] The positive system (2.1) is asymptotically stable if and only if all coefficients of the polynomial

$$p_A(z) = \det[I_n(z+1) - A] = z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0 \quad (2.6)$$

are positive, i.e. $a_k > 0$ for $k = 0,1,...,n-1$.

The problem under considerations can be stated as follows.

Given a rational matrix $T(z) \in \Re^{p\times m}(z)$, find a positive stable its realization (2.2).

In this paper sufficient conditions for the existence of positive stable realization (2.2) of a given $T(z) \in \Re^{p\times m}(z)$ will be established and procedures for determination of positive stable realizations (2.2) will be proposed.

3. Problem solution for single-input single-output systems

A. Method 1

Consider the positive single-input single-output (SISO) system (2.1) with the transfer function

$$T(z) = \frac{b_n z^n + b_{n-1}z^{n-1} + ... + b_1 z + b_0}{z^n - a_{n-1}z^{n-1} - ... - a_1 z - a_0}. \quad (3.1)$$

For given (3.1) we can find the matrix *D* by the use of the formula [6]

$$D = \lim_{z\to\infty} T(z) = b_n \quad (3.2)$$

and the strictly proper transfer function

$$T_{sp}(z) = T(z) - D = C[I_n z - A]^{-1}B = \frac{\bar{b}_{n-1}z^{n-1} + ... + \bar{b}_1 z + \bar{b}_0}{z^n - a_{n-1}z^{n-1} - ... - a_1 z - a_0} = \frac{n(z)}{d(z)} \quad (3.3)$$

where

$$\bar{b}_k = b_k + a_k b_n, \; k = 0,1,...,n-1. \quad (3.4)$$

Note that if $b_k \ge 0$ for $k = 0,1,...,n$ and $a_k \ge 0$ for $k = 0,1,...,n-1$ then $\bar{b}_k \ge 0$ for $k = 0,1,...,n-1$.

Theorem 3.1. There exists a positive stable realization of the form

$$A = \begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 1 \\ a_0 & a_1 & a_2 & ... & a_{n-1} \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$, \; C = \begin{bmatrix} \bar{b}_0 & \bar{b}_1 & \bar{b}_2 & ... & \bar{b}_{n-1} \end{bmatrix}, \; D = [b_n]. \quad (3.5a)$$

of (3.1) if the following conditions are satisfied:

1) $a_k \ge 0$ for $k = 0,1,...,n-1$, (3.6a)

2) $b_k \ge 0$ for $k = 0,1,...,n$, (3.6b)

3) $a_0 + a_1 + ... + a_{n-1} < 1$. (3.6c)

Proof. If the conditions (3.6a) and (3.6b) are met then $A \in \Re_+^{n\times n}$, $B \in \Re_+^n$, $C \in \Re_+^{1\times n}$, $D \ge 0$. By Theorem 2.2 the realization (3.5) is stable if and only if all coefficients of the polynomial

$$\begin{aligned} &\det[I_n(z+1) - A] = (z+1)^n - a_{n-1}(z+1)^{n-1} - ... \\ &- a_1(z+1) - a_0 = z^n + (n - a_{n-1})z^{n-1} + ... \\ &+ (n - a_{n-1}(n-1) - ... - a_1)z + (1 - a_{n-1} - ... - a_1 - a_0) \end{aligned} \quad (3.7)$$

are positive, i.e.

$$\begin{aligned} &1 - a_{n-1} - ... - a_1 - a_0 > 0 \\ &n - a_{n-1}(n-1) - ... - a_1 > 0 \\ &\vdots \\ &n - a_{n-1} > 0 \end{aligned} \quad (3.8)$$

Note that the conditions (3.8) are satisfied if (3.6c) is met. Using (3.5) and (2.3) we obtain

$$\begin{aligned} T(z) &= C[I_n z - A]^{-1}B + D \\ &= \begin{bmatrix} \bar{b}_0 & \bar{b}_1 & \bar{b}_2 & ... & \bar{b}_{n-1} \end{bmatrix} \begin{bmatrix} z & -1 & 0 & ... & 0 \\ 0 & z & -1 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & -1 \\ -a_0 & -a_1 & -a_2 & ... & z - a_{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} + b_n \\ &= \begin{bmatrix} \bar{b}_0 & \bar{b}_1 & \bar{b}_2 & ... & \bar{b}_{n-1} \end{bmatrix} \frac{1}{z^n - a_{n-1}z^{n-1} - ... - a_1 z - a_0} \begin{bmatrix} 1 \\ z \\ \vdots \\ z^{n-1} \end{bmatrix} + b_n \\ &= \frac{\bar{b}_{n-1}z^{n-1} + ... + \bar{b}_1 z + \bar{b}_0}{z^n - a_{n-1}z^{n-1} - ... - a_1 z - a_0} + b_n = \frac{b_n z^n + b_{n-1}z^{n-1} + ... + b_1 z + b_0}{z^n - a_{n-1}z^{n-1} - ... - a_1 z - a_0} \end{aligned} \quad (3.9)$$

This completes the proof. □

If the conditions (3.6) are met then the positive stable realization (3.5) can be computed by the use of the following procedure.

Procedure 3.1.

Step 1. Using (3.2) find the matrix *D* and the strictly proper transfer function (3.3).

Step 2. Using (3.5) find the matrices $A \in \Re_+^{n\times n}$, $B \in \Re_+^n$, $C \in \Re_+^{1\times n}$.

Example 3.1. Using Procedure 3.1 find the positive stable realization of the transfer function

$$T(z) = \frac{2z^3 + 3z^2 + z + 2}{z^3 - 0.7z^2 - 0.1z - 0.08}. \quad (3.10)$$

The transfer function (3.10) satisfy the conditions (3.6) since $a_0 = 0.08$, $a_1 = 0.1$, $a_2 = 0.7$, $b_3 = 3$, $b_2 = 2$, $b_1 = 1$, $b_0 = 2$ and $a_0 + a_1 + a_2 = 0.88 < 1$.

Step 1. Using (3.2) for (3.10) we obtain

$$D = \lim_{z\to\infty} T(z) = 2 \tag{3.11}$$

and the strictly proper transfer function

$$T_{sp}(z) = T(z) - D = \frac{2z^3 + 3z^2 + z + 2}{z^3 - 0.7z^2 - 0.1z - 0.08} - 2 = \frac{4.4z^2 + 1.2z + 2.16}{z^3 - 0.7z^2 - 0.1z - 0.08} \tag{3.12}$$

Step 2. The realization (3.5a) of (3.10) has the form

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.08 & 0.1 & 0.7 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$C = [2.16 \quad 1.2 \quad 4.4],\ D = [2]. \quad (3.13$$

The realization (3.13) is asymptotically stable since the condition (3.6c) is met. the poles of (3.10) are: $z_1 = 0.9074$, $z_2 = -0.1037 + j0.282$, $z_3 = -0.1037 - j0.282$.

Remark 3.1. In a similar way as in the proof of Theorem 3.1 it can be shown that the following matrices are also the positive stable realizations of (3.1):

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 & a_0 \\ 1 & 0 & \dots & 0 & a_1 \\ 0 & 1 & \dots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & a_{n-1} \end{bmatrix},\ B = \begin{bmatrix} \bar{b}_0 \\ \bar{b}_1 \\ \bar{b}_2 \\ \vdots \\ \bar{b}_{n-1} \end{bmatrix},$$

$$C = [0 \quad 0 \quad \dots \quad 0 \quad 1],\ D = [b_n] \tag{3.14}$$

$$A = \begin{bmatrix} a_{n-1} & a_{n-2} & \dots & a_1 & a_0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

$$C = [\bar{b}_{n-1} \quad \dots \quad \bar{b}_2 \quad \bar{b}_1 \quad \bar{b}_0],\ D = [b_n] \tag{3.15}$$

$$A = \begin{bmatrix} a_{n-1} & 1 & 0 & \dots & 0 \\ a_{n-2} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & 0 & 0 & \dots & 1 \\ a_0 & 0 & 0 & \dots & 0 \end{bmatrix},\ B = \begin{bmatrix} \bar{b}_{n-1} \\ \bar{b}_{n-2} \\ \vdots \\ \bar{b}_1 \\ \bar{b}_0 \end{bmatrix},$$

$$C = [1 \quad 0 \quad \dots \quad 0 \quad 0],\ D = [b_n] \tag{3.16}$$

B. Method 2.

Let the distinct poles $z_1, z_2, \dots, z_n$ of the transfer matrix (3.1) are real positive and satisfy the conditions

$$z_k < 1 \text{ for } k = 1, 2, \dots, n. \tag{3.17}$$

Theorem 3.2. There exists a positive stable realization of the form

$$A = \text{blockdiag}[z_1 \quad \dots \quad z_n],\quad B = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \tag{3.18}$$

$$C = [c_1 \quad \dots \quad c_n], \qquad D = [b_n]$$

of (3.1) if the following conditions are satisfied:

1) The real positive poles $z_1, z_2, \dots, z_n$ of the transfer matrix (3.1) satisfy the condition (3.17).

2) The strictly proper transfer function (3.3) can be decomposed in the form

$$T_{sp}(z) = \sum_{k=1}^{n} \frac{c_k}{z - z_k} \tag{3.19a}$$

where

$$c_k = \lim_{z\to z_k} (z - z_k) T_{sp}(z) = \frac{n(z_k)}{\prod_{i=1, i\neq k}^{n} (z_k - z_i)} \tag{3.19b}$$

with $c_k \geq 0$, k = 1,2,…,n.

3) The coefficient b_n of (3.1) is nonnegative.

Proof. If the condition 1) is met then $A \in \Re_+^{n\times n}$ is asymptotically stable. If by assumption 2) $c_k \geq 0$, k = 1,2,…,n then $C \in \Re_+^{1\times n}$ and the assumption 3) implies $D \in \Re_+$.

Using (3.18) we obtain

$$C[I_n z - A]^{-1} B + D =$$

$$[c_1 \quad \dots \quad c_n] \begin{bmatrix} z - z_1 & 0 & \dots & 0 \\ 0 & z - z_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & z - z_n \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + b_n$$

$$= [c_1 \quad \dots \quad c_n] \begin{bmatrix} \frac{1}{z - z_1} & 0 & \dots & 0 \\ 0 & \frac{1}{z - z_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{z - z_n} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + b_n = \tag{3.20}$$

$$= \sum_{k=1}^{n} \frac{c_k}{z - z_k} + b_n = T(z)$$

Therefore, the conditions 1), 2) and 3) are satisfied and there exists a positive stable realization of the transfer function (3.1). □

If the assumptions of Theorem 3.2 are satisfied then the positive stable realization (3.18) can be computed by the use of the following procedure.

Procedure 3.2.

Step 1. Using (3.2) find the matrix D and the strictly proper transfer function (3.3).

Step 2. Using formula (3.19) find c_k for $k = 1,2,\dots,n$.

Step 3.Find the desired realization (3.18).

Example 3.2. Find a positive stable realization (3.18) of the transfer function

$$T(z) = \frac{4z^2 - 3.9z + 0.94}{z^3 - 1.5z^2 + 0.74z - 0.12} \tag{3.21}$$

The poles of (3.21) are $z_1 = 0.4$, $z_2 = 0.5$, $z_3 = 0.6$ and satisfy the condition (3.17). Using Procedure 3.2 we obtain the following.

Step 1. By (3.2) $D = [0]$ since the transfer function (3.21) is strictly proper.

Step 2. Using (3.19) we obtain

$$c_1 = \frac{n(z_1)}{(z_1 - z_2)(z_1 - z_3)} = \frac{4(0.4)^2 - 3.9(0.4) + 0.94}{(-0.1)(-0.2)} = 1 \quad (3.22a)$$

$$c_2 = \frac{n(z_2)}{(z_2 - z_1)(z_2 - z_3)} = \frac{4(0.5)^2 - 3.9(0.5) + 0.94}{(0.1)(-0.1)} = 1 \quad (3.22b)$$

$$c_3 = \frac{n(z_3)}{(z_3 - z_2)(z_3 - z_1)} = \frac{4(0.6)^2 - 3.9(0.6) + 0.94}{(0.1)(0.2)} = 2 \quad (3.22c)$$

Step 3. The desired positive stable realization (3.18) of the transfer function (3.21) has the form

$$A = \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.6 \end{bmatrix},\quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\quad C = [1 \quad 1 \quad 2] \tag{3.23}$$

Note that the transfer function (3.21) does not satisfy the conditions of Theorem 3.1 and the desired realization can not be determined by the use of Procedure 3.1.

The following examples show how the considerations can be extended to multiple poles of the transfer function (3.1).

Example 3.3. It will be shown that there exists a positive stable realization of the form

$$A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix},\quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\quad C = [b_0 - b_1 a \quad b_1] \tag{3.24}$$

of the transfer function

$$T_{sp}(z) = \frac{b_1 z + b_0}{(z - a)^2} \tag{3.25}$$

if

$$0 < a < 1 \text{ and } b_0 \ge b_1 a\,. \tag{3.26}$$

Using (3.24) we obtain

$$T_{sp}(z) = C[I_n z - A]^{-1} B = [b_0 - b_1 a \quad b_1] \begin{bmatrix} z - a & -1 \\ 0 & z - a \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$= [b_0 - b_1 a \quad b_1] \frac{1}{(z - a)^2} \begin{bmatrix} z - a & 1 \\ 0 & z - a \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} =$$
$$= [b_0 - b_1 a \quad b_1] \frac{1}{(z - a)^2} \begin{bmatrix} 1 \\ z - a \end{bmatrix} = \frac{b_1 z + b_0}{(z - a)^2}$$

If the condition (3.26) is satisfied then the matrix $A \in \Re_+^{2\times 2}$ is asymptotically stable $B \in \Re_+^2$ and $C \in \Re_+^{1\times 2}$. Therefore, the matrices (3.24) are a positive stable realization of (3.25).

Example 3.4. Compute a positive stable realization of the transfer function

$$T(z) = \frac{2z^2 + 0.2z + 0.02}{z^3 - 0.5z^2 + 0.08z - 0.004} \tag{3.27}$$

Note that the poles of (3.27) are $z_1 = 0.1$, $z_2 = z_3 = 0.2$ and it can be decomposed as follows

$$T_{sp}(z) = \frac{2}{z - 0.1} + \frac{1}{(z - 0.2)^2} \tag{3.28}$$

Using the result of Example 3.3 we obtain the following positive stable realization

$$A = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.2 & 1 \\ 0 & 0 & 0.2 \end{bmatrix},\quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\quad C = [2 \quad 1 \quad 0] \tag{3.29}$$

4. Problem solution for multi-input multi-output systems

Consider the proper transfer matrix of the form

$$T(z) = \begin{bmatrix} T_{11}(z) & \dots & T_{1,m}(z) \\ \vdots & \dots & \vdots \\ T_{p,1}(z) & \dots & T_{p,m}(z) \end{bmatrix} \in \Re^{p\times m}(z)$$

$$T_{i,j}(z) = \frac{n_{i,j}(z)}{d_{i,j}(z)},\ i = 1,\dots,p\,;\ j = 1,\dots,m \tag{4.1}$$

where $\Re^{p\times m}(z)$ is the set of proper rational real matrices. The matrix D of desired positive stable realization can be computed by the use of the formula

$$D = \lim_{z\to\infty} T(z) \tag{4.2}$$

The strictly proper transfer matrix

$$T_{sp}(z) = T(z) - D$$

can be written in the form

$$T_{sp}(z) = \begin{bmatrix} \frac{N_{11}(z)}{d_1(z)} & \dots & \frac{N_{1,m}(z)}{d_m(z)} \\ \vdots & \dots & \vdots \\ \frac{N_{p,1}(z)}{d_1(z)} & \dots & \frac{N_{p,m}(z)}{d_m(z)} \end{bmatrix} = N(z)D^{-1}(z) \quad (4.3a)$$

where

$$N(z) = \begin{bmatrix} N_{11}(z) & \dots & N_{1,m}(z) \\ \vdots & \dots & \vdots \\ N_{p,1}(z) & \dots & N_{p,m}(z) \end{bmatrix} \in \Re^{p\times m}[z], \quad (4.3b)$$

$$D(z) = \mathrm{diag}[d_1(z) \quad \dots \quad d_m(z)] \in \Re^{m\times m}[z],$$

$$N_{i,j}(s) = c_{i,j}^{d_j-1} z^{d_j-1} + \dots + c_{i,j}^1 z + c_{i,j}^0, \quad i = 1,\dots,p; \quad j = 1,\dots,m \quad (4.3c)$$

$$d_j(z) = z^{d_j} - a_{j,d_j-1} z^{d_j-1} - \dots - a_{j,1} z - a_{j,0}, \quad j = 1,\dots,m \quad (4.3d)$$

Theorem 4.1. There exists a positive stable realization of the form

$$A = \mathrm{blockdiag}[A_1 \quad \dots \quad A_m],$$

$$A_j = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ a_{j,0} & a_{j,1} & a_{j,2} & \dots & a_{j,d_j-1} \end{bmatrix}, \quad j = 1,\dots,m;$$

$$B = \mathrm{blockdiag}[b_1 \quad \dots \quad b_m], \quad b_j = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \Re^{d_j}, \quad j = 1,\dots,m; \quad (4.4)$$

$$C = \begin{bmatrix} c_{11}^0 & c_{11}^1 & \dots & c_{11}^{d_1-1} & \dots & c_{1,m}^0 & c_{1,m}^1 & \dots & c_{1,m}^{d_m-1} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ c_{p,1}^0 & c_{p,1}^1 & \dots & c_{p,1}^{d_1-1} & \dots & c_{p,m}^0 & c_{p,m}^1 & \dots & c_{p,m}^{d_m-1} \end{bmatrix}$$

of the transfer matrix (4.1) if the following conditions are satisfied

1) $a_{j,k} \ge 0$ for $j = 1,\dots,m$ $k = 0,1,\dots,d_j - 1$, (4.5a)

2) $c_{i,j}^k \ge 0$ for $i = 1,\dots,p$, $j = 1,\dots,m$ $k = 0,1,\dots,d_j - 1$, (4.5b)

3) $a_{j0} + a_{j1} + \dots + a_{j,d_j-1} < 1$ for $j = 1,\dots,m$. (4.5c)

4) $\lim_{z\to\infty} T(z) = T(\infty) \in \Re_+^{p\times m}$. (4.5d)

Proof. If the conditions (4.5a), (4.5b) and (4.5d) are met then from (4.4) we have $A \in \Re_+^{n\times n}$, $B \in \Re_+^n$, $C \in \Re_+^{1\times n}$ and $D \in \Re_+^{p\times m}$. By Theorem 2.2 the matrix Aj $j = 1,\dots,m$ is asymptotically stable if the condition (4.5c) is satisfied and this implies the asymptotic stability of the matrix (4.4a). From (4.3d) and (4.3c) we have

$$d_j(z) = z^{dj} - [a_{j0} \quad a_{j1} \quad \dots \quad a_{j,dj-1}]Z_j, \quad j = 1,\dots,m \quad (4.6)$$

and

$$N_{i,j}(z) = [c_{i,j}^0 \quad c_{i,j}^1 \quad \dots \quad c_{i,j}^{dj-1}]Z_j, \quad i = 1,\dots,p; \quad j = 1,\dots,m \quad (4.7)$$

where $Z_j = [1 \quad z \quad \dots \quad z^{d_j-1}]^T$, $j = 1,\dots,m$.

Knowing $d_j(z)$ and using (4.6) we may find the matrix A_j $j = 1,\dots,m$ and the matrix (4.4a). Using (4.7) we may find the matrix C since

$$N(z) = CZ \quad (4.8)$$

where $Z = \mathrm{blockdiag}\,[Z_1, Z_2, \dots, Z_n]$.

We shall show that the matrices (4.4a), (4.4b) and (4.4c) are the realization of the strictly proper transfer matrix (4.3a). It is easy to verify that

$$b_j d_j(z) = [I_n z - A_j]Z_j, \quad j = 1,\dots,m$$

and

$$BD(z) = [I_n z - A]Z \quad (4.9)$$

where

$$D(z) = \mathrm{diag}\,[z^{d_1},\dots,z^{d_m}] - \overline{A}_m Z$$

and

$$\overline{A}_m = \mathrm{diag}\,[a_0, a_1, \dots, a_m], \quad a_j = [a_{j0} \quad \dots \quad a_{j,dj-1}], \quad j = 1,\dots,m \quad (4.10)$$

Premultiplying (4.9) by $C[I_n z - A]^{-1}$ and postmultiplying by $D^{-1}(z)$ and using (4.8) we obtain

$$C[I_n z - A]^{-1}B = CZD^{-1}(z) = = N(z)D^{-1}(z) = T_{sp}(z)$$

This completes the proof. □

If the conditions of Theorem 4.1 are satisfied then the positive stable realization of the transfer matrix (4.1) can be found by the use of the following procedure.

Procedure 4.1.

Step 1. Using (4.2) we find the matrix D.

Step 2. Find the common denominators $d_j(z)$ $j=1,...,m$ and write the strictly proper transfer matrix in the form (4.3a).

Step 3. Using (4.6) find the matrices $A_1,\ldots,A_n$ and the matrix (4.4a).

Step 4. Using (4.7) or (4.8) find the matrix C.

Example 4.1. Find a positive stable realization of the transfer matrix

$$T(z)=\begin{bmatrix} \frac{z^2+0.8z+0.2}{z^2-0.2z-0.1} & \frac{2z^2+0.4z+0.2}{z^2-0.3z-0.2} \\ \frac{2z+0.2}{z^2-0.2z-0.1} & \frac{z^2+0.7z+0.4}{z^2-0.3z-0.2} \end{bmatrix}. \quad (4.11)$$

The proper transfer matrix (4.11) satisfies the conditions of Theorem 4.1. Using Procedure 4.1 we obtain the following.

Step 1. Applying (4.2) to (4.11) we obtain

$$D=\lim_{z\to\infty} T(z)=\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad (4.12)$$

Step 2. The strictly proper transfer matrix has the form

$$T_{sp}(s)=T(s)-D=\begin{bmatrix} \frac{z+0.3}{z^2-0.2z-0.1} & \frac{z+0.6}{z^2-0.3z-0.2} \\ \frac{2z+0.2}{z^2-0.2z-0.1} & \frac{z+0.6}{z^2-0.3z-0.2} \end{bmatrix}= \quad (4.13a)$$

$$=N(z)D^{-1}(z)$$

where

$$N(z)=\begin{bmatrix} z+0.3 & z+0.6 \\ 2z+0.2 & z+0.6 \end{bmatrix},$$

$$D(z)=\begin{bmatrix} z^2-0.2z-0.1 & 0 \\ 0 & z^2-0.3z-0.2 \end{bmatrix} \quad (4.13b)$$

Step 3. Using (4.4a), (4.6) and (4.13b) we obtain

$$A_1=\begin{bmatrix} 0 & 1 \\ 0.1 & 0.2 \end{bmatrix},\quad A_1=\begin{bmatrix} 0 & 1 \\ 0.2 & 0.3 \end{bmatrix},$$

$$A=\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.1 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0.2 & 0.3 \end{bmatrix} \quad (4.14)$$

Step 4. Using (4.8) and (4.13b) we obtain

$$N(z)=\begin{bmatrix} z+0.3 & z+0.6 \\ 2z+0.2 & z+0.6 \end{bmatrix}=\begin{bmatrix} 0.3 & 1 & 0.6 & 1 \\ 0.2 & 2 & 0.6 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ z & 0 \\ 0 & 1 \\ 0 & z \end{bmatrix}=CZ$$

and

$$C=\begin{bmatrix} 0.3 & 1 & 0.6 & 1 \\ 0.2 & 2 & 0.6 & 1 \end{bmatrix} \quad (4.15)$$

The matrix B defined by (4.4b) has the form

$$B=\begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}. \quad (4.16)$$

The desired positive stable realization of the transfer matrix (4.11) is given by (4.14), (4.16), (4.15) and (4.12).

Remark 4.1. The presented method can be considered as an extension of the Method 1 to positive stable multi-input multi-output discrete-time linear systems. Using the idea of Gilbert method [5, 16] it is also possible to extend the Method 2 to multi-input multi-output positive stable discrete-time linear systems.

The strictly proper transfer matrix $T_{sp}(z)$ can be also written in the form

$$T_{sp}(z)=\begin{bmatrix} \frac{\bar{N}_{11}(z)}{\bar{d}_1(z)} & ... & \frac{\bar{N}_{1,m}(z)}{\bar{d}_1(z)} \\ \vdots & ... & \vdots \\ \frac{\bar{N}_{p,1}(z)}{\bar{d}_p(z)} & ... & \frac{\bar{N}_{p,m}(z)}{\bar{d}_p(z)} \end{bmatrix}=\bar{D}^{-1}(z)\bar{N}(z) \quad (4.17a)$$

where

$$\bar{N}(z)=\begin{bmatrix} \bar{N}_{11}(z) & ... & \bar{N}_{1,m}(z) \\ \vdots & ... & \vdots \\ \bar{N}_{p,1}(z) & ... & \bar{N}_{p,m}(z) \end{bmatrix}\in\Re^{p\times m}[z], \quad (4.17b)$$

$$\bar{D}(z)=\mathrm{diag}[\bar{d}_1(z) \quad ... \quad \bar{d}_p(z)]\in\Re^{m\times m}[z],$$

$$\bar{N}_{i,j}(s)=\bar{b}_{i,j}^{\bar{d}_i-1}z^{\bar{d}_i-1}+...+\bar{b}_{i,j}^{1}z+\bar{b}_{i,j}^{0},\quad i=1,...,p;\quad j=1,...,m \quad (4.17c)$$

$$\bar{d}_j(z)=z^{\bar{d}_j}-\bar{a}_{j,\bar{d}_j-1}z^{\bar{d}_j-1}-...-\bar{a}_{j,1}z-\bar{a}_{j,0},\quad j=1,...,p \quad (4.17d)$$

Theorem 4.2. There exists a positive stable realization of the form

$$\bar{A}=\mathrm{blockdiag}[\bar{A}_1 \quad ... \quad A_p],$$

$$\bar{A}_j=\begin{bmatrix} 0 & 0 & ... & 0 & \bar{a}_{j,0} \\ 1 & 0 & ... & 0 & \bar{a}_{j,1} \\ 0 & 1 & ... & 0 & \bar{a}_{j,2} \\ \vdots & \vdots & \ddots & ... & \vdots \\ 0 & 0 & ... & 1 & \bar{a}_{j,\bar{d}_j-1} \end{bmatrix},\quad j=1,...,p; \quad (4.18)$$

$$C = \begin{bmatrix} \bar{b}_{11}^0 & \bar{b}_{12}^0 & \dots & \bar{b}_{1,m}^0 \\ \bar{b}_{11}^1 & \bar{b}_{12}^1 & \dots & \bar{b}_{1,m}^1 \\ \vdots & \vdots & \dots & \vdots \\ \bar{b}_{11}^{\bar{d}_1-1} & \bar{b}_{12}^{\bar{d}_1-1} & \dots & \bar{b}_{1,m}^{\bar{d}_1-1} \\ \bar{b}_{21}^0 & \bar{b}_{22}^0 & \dots & \bar{b}_{2,m}^{\bar{d}_0-1} \\ \vdots & \vdots & \dots & \vdots \\ \bar{b}_{p,1}^{\bar{d}_p-1} & \bar{b}_{p,2}^{\bar{d}_p-1} & \dots & \bar{b}_{p,m}^{\bar{d}_p-1} \end{bmatrix},$$

$$C = \text{blockdiag}[c_1 \quad \dots \quad c_p],$$

$$c_j = [0 \quad \dots \quad 0 \quad 1] \in \Re^{1\times\bar{d}_j}, \quad j = 1, \dots, p$$

of the transfer matrix (4.1) if the following conditions are satisfied:

1) $\bar{a}_{j,k} \ge 0$ for $j = 1,\dots,p$ $k = 0,1,\dots,\bar{d}_j - 1$, (4.19a)

2) $\bar{b}_{i,j}^k \ge 0$ for $i = 1,\dots,p$,

$j = 1,\dots,m$ $k = 0,1,\dots,\bar{d}_j - 1$, (4.19b)

3) $\bar{a}_{j0} + \bar{a}_{j1} + \dots + \bar{a}_{j,\bar{d}_j - 1} < 1$ for $j = 1,\dots,p$. (4.19c)

4) $\lim_{z\to\infty} T(z) = T(\infty) \in \Re_+^{p\times m}$. (4.19d)

Proof is similar (dual) to the proof of Theorem 4.1.

If the conditions of Theorem 4.2 are satisfied then the positive stable realization of the transfer matrix (4.1) can be found by the use of the following procedure.

Procedure 4.2.

Step 1. Using (4.2) compute the matrix D and the strictly proper transfer matrix.

Step 2. Compute the common row denominators $\bar{d}_j(z)$ $j = 1,\dots,p$ and write the strictly proper transfer matrix in the form (4.17).

Step 3. Using the equality

$$\bar{d}_j(z) = z^{\bar{d}_j} - \bar{Z}_j \begin{bmatrix} \bar{a}_{j,0} \\ \bar{a}_{j,1} \\ \vdots \\ \bar{a}_{j,\bar{d}_j-1} \end{bmatrix},$$

$$\bar{Z}_j = [1 \quad z \quad \dots \quad z^{\bar{d}_j-1}], \quad j = 1,\dots,p \tag{4.20}$$

compute the matrices $\bar{A}_1,\dots,\bar{A}_p$ and $\bar{A}$ (defined by (4.18a))

Step 4. Using the equality

$$\bar{N}(z) = \bar{Z}B, \quad \bar{Z} = \text{blockdiag}[\bar{Z}_1,\dots,\bar{Z}_p] \tag{4.21}$$

compute the matrix B.

Example 4.2. Compute the positive stable realization of the strictly proper transfer matrix

$$T_{sp}(z) = \begin{bmatrix} \frac{z+0.2}{z^2-0.3z-0.2} & \frac{z+0.3}{z^2-0.3z-0.2} \\ \frac{z+0.5}{z^2-0.4z-0.3} & \frac{z+0.1}{z^2-0.4z-0.3} \end{bmatrix}. \tag{4.22}$$

The proper transfer matrix (4.22) satisfies the conditions of Theorem 4.2. Using Procedure 4.2 we obtain the following.

Step 1. In this case

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{4.23}$$

since the transfer matrix (4.22) is strictly proper.

Step 2. The transfer matrix (4.22) has already the desired form

$$\bar{N}(z) = \begin{bmatrix} z+0.2 & z+0.3 \\ z+0.5 & z+0.1 \end{bmatrix}, \quad D(z) = \begin{bmatrix} z^2-0.3z-0.2 & 0 \\ 0 & z^2-0.4z-0.3 \end{bmatrix}. \tag{4.24}$$

Step 3. Using (4.20) and (4.24) we obtain

$$z^2 - 0.3z - 0.2 = z^2 - [1 \quad z]\begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix},$$

$$z^2 - 0.4z - 0.3 = z^2 - [1 \quad z]\begin{bmatrix} 0.3 \\ 0.4 \end{bmatrix}$$

and

$$\bar{A} = \begin{bmatrix} \bar{A}_1 & 0 \\ 0 & \bar{A}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0.2 & 0 & 0 \\ 1 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0.3 \\ 0 & 0 & 1 & 0.4 \end{bmatrix} \tag{4.25}$$

Step 4. Using (4.21) and (4.24) we obtain

$$\begin{bmatrix} z+0.2 & z+0.3 \\ z+0.5 & z+0.1 \end{bmatrix} = \begin{bmatrix} 1 & z & 0 & 0 \\ 0 & 0 & 1 & z \end{bmatrix} \begin{bmatrix} 0.2 & 0.3 \\ 1 & 1 \\ 0.5 & 0.1 \\ 1 & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0.2 & 0.3 \\ 1 & 1 \\ 0.5 & 0.1 \\ 1 & 1 \end{bmatrix}. \tag{4.26}$$

From (4.18c) we have

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.27}$$

The desired positive stable realization of the transfer matrix (4.22) is given by (4.25), (4.26) and (4.27).

Remark 4.2. The well-known Gilbert method [16, 5] can be also applied to compute a positive stable realization of the transfer matrix (4.1). With slight modifications the methods presented in [13, 14] for continuous-time linear systems can be also applied for computation of a positive stable realization of the transfer function (4.1).

5. Concludingremarks

Sufficient conditions for the existence of positive stable realizations of discrete-time linear systems have been established. Two different methods for determination of the positive stable realizations for given proper transfer functions have been proposed. The proposed procedures for determination of the positive realizations have been demonstrated on numerical examples. An open problem is formulation of necessary and sufficient conditions for existence of positive stable realizations of the linear systems. An open problem is an extension of the proposed method for positive fractional linear systems [15].

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РОЗРАХУНОК ПОЧАТКОВИХ УМОВ ТА ВХІДНИХ ДАНИХ ЗА ЗАДАНИМИ ВИХІДНИМИ ДЛЯ КЛАСИЧНИХ ТА ДОДАТНІХ ДИСКРЕТНИХ СИСТЕМ

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Встановлені достатні умови для існування позитивних стабільних реалізацій даних матриць з належним перетворенням. Пропонуються два методи для визначення позитивних стабільних реалізацій даних матриць з належним перетворенням. Ефективність запропонованих методик демонструється на числових прикладах.

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