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CONSTRUCTION OF INVARIANT DIFFERENCE AND DISCRETE ANALOGUES OF DIFFERENTIAL OPERATORS OF HIGH ORDER DISCRETIZATION ERROR

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Abstract: The paper offers the algorithm of construction of difference and discrete analogues of differential operators on the basis of invariant approximations methodology. The reciprocal Taylor's matrix of the 6-th degree has been calculated in exact figures what allows us to construct difference and discrete analogues of high order discretization errors.

Key words: invariant approximation, difference analogue, discrete analogue, discretization error.

1. Statement of the problem

To solve boundary problems of mathematical physic by means of finite-difference or finite elements methodology a regular triangular grid is usually employed. The expressions of difference analogues of differential operators used in the equations forming the mathematical model of a problem under consideration are given in appropriate literature and applied by a researcher to his/her specific task.

However, there is a problem of choice because literature claims different analogues to possess the same order of discretization error. Therefore, the act of making choice is a voluntary one what makes the solution of the task dependent on the choice of corresponding analogues.

Our article describes the algorithm of construction of difference and discrete analogues (DA) on the basis of invariant approximations methodology. Non-traditional character of our approach to the problem is accounted for by the requirement to preserve the tensor features of the original differential operator (DO), i. e. independence of its value from the co-ordinate system by means of which this value was calculated. This independence is reflected in the identity of DO's representations in arbitrary Cartesian co-ordinate systems (CCS) $Oxyz$ and $O'x'y'z'$. Invariance of equations of mathematical physics is their fundamental property since it provides independence of solution from the co-ordinate system in which this solution was obtained (in our case – from the orientation of the CCS axes and the location of origin of co-ordinates). Therefore, it is natural to demand that during the numeral solving of a boundary value problem this fundamental property be not lost on the stage of replacement of DO-s by their DA-s. The practical value of observance of this requirement is obvious: the application of difference and discrete analogues that are not invariant, in the course of solving of a boundary value problem means that the simulation results will depend on an accidental (subjective) factor – on the application of a specific DA (i.e. in the final count, on the chosen CCS in which this DA has been calculated) and, consequently, the scientific value of this result as relying on an accidental (subjective) factor remains problematic.

2. Application of invariant approximation methodology

The methodology of invariant approximations proposes to use for interpolation of unknown sought function U an abridged Taylor series. The power of the series depends on the tolerable order of discretization error. Since we are discussing the analogues of high order, we make use of Taylor's polynomial of the 6-th degree as follows

$$U = \vec{T}[x,y]\cdot\vec{u} = \vec{T}[x,y]\mathbf{T}^{-1}\vec{U}, \qquad (1)$$

where $\vec{T}[x,y] = (1, x, y, x^2/2!, xy, y^2/2!, x^3/3!, x^2y/2!, xy^2/2!, y^3/3!, x^4/4!, x^3y/3!, x^2y^2/(2!2!), xy^3/3!, y^4/4!, x^5/5!, x^4y/4!, x^3y^2/(3!2!), x^2y^3/(2!3!), xy^4/4!, y^{5!}/5!, x^6/6!, x^5y/5!, x^4y^2/(4!2!), x^3y^3/(3!3!), x^2y^4/(2!4!), xy^{5!}/5!, y^6/6!)$ is Taylor's vector for a 6-th order mesh; $\cdot\vec{u} = (u_1,\ldots,u_{28})$ is a corresponding derivatives vector; $\vec{U} = (U_1,\ldots,U_{28})$ is the nodal vector of the mesh comprising the values of a sought function in the nodes of the mesh; $\mathbf{T}$ is so-called Taylor's matrix whose rows are Taylor's vectors for the mesh nodes; $\mathbf{T}^{-1}$ is a matrix reciprocal to the Taylor's matrix. To secure the existence of the reciprocal Taylor's matrix the nodes of the mesh may not belong to one curve of the 6-th order; such mesh is called non-singular. The mesh is shown on the Fig. 1.

The nodes of the chosen mesh form an equilateral triangle with 7 nodes on every side and 10 internal no-

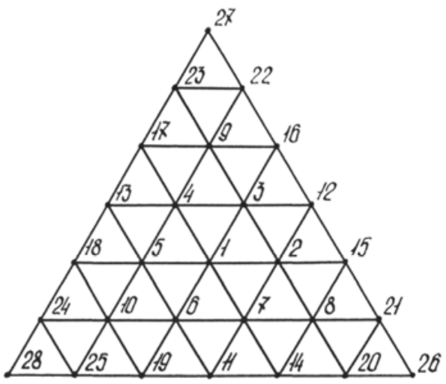

Fig. 1. The 6-th order mesh

des. The origin of coordinates coincides with the node 1; the distance between the nodes is assumed to be of one length unit marked as h; the nodes are numbered counterclockwise. For this mesh we have calculated the coordinates of every node and, correspondingly, the values of appropriate Taylor's vectors.

The DA of any differential operator for any node of the mesh is given by the expression $\vec{T}[x,y]f(\overline{\mathbf{N}})\mathbf{T}^{-1}$, e.g.

$$DA_{\nabla} = \vec{T}[x,y]\cdot\overline{\mathbf{N}}\mathbf{T}^{-1}, \qquad (2)$$

$$DA_{\nabla^2} = \vec{T}[x,y]\cdot\overline{\mathbf{N}}^2\mathbf{T}^{-1}, \qquad (3)$$

$$DA_{\nabla^4} = \vec{T}[x,y]\cdot\overline{\mathbf{N}}^4\mathbf{T}^{-1} \qquad (4)$$

where $\overline{\mathbf{N}}$ is the discrete analogue of Hamilton's operator and has been assigned the name of Hamilton's matrix [1].

3. The computation of reciprocal Hamilton's matrix

The computation results for the reciprocal matrix are following:

$\vec{\tau}_1$ =(1,0, ..., 0);

$\vec{\tau}_2$ =(0,2/3,0,0,-2/3,-2/3,2/3,-1/12,0,0,0,0,0,1/12,1/3, 1/12,0,-1/12,-1/3,1/15,1/15,-1/60,0,0,0,1/60,-1/15,-1/15);

$\vec{\tau}_3/\sqrt{3}$ =(0,-2/9,4/9,4/9,-2/9, -2/9,-2/9,1/3,0,-1/18,-2/9,-1/18, 0,1/36,1/9,1/36,0,1/36,1/9,-1/45,-1/45,1/180,2/45,-1/90,2/45, 1/180,-1/45,-1/45);

$\vec{\tau}_4$ =(-5/2,4/3,0,0,4/3,0,0,-1/12,0,0,0,0,0,-1/12,0,1/6,-2/9, 1/6,0,-1/15,0,1/90,0,0,0,1/90,0,1/15);

$\vec{\tau}_5/\sqrt{3}$ =(0,0,4/9,-4/9,0,4/9,-4/9,1/18,2/27,-1/12,0,1/12, 2/27,1/18,0,-1/36,0,1/36,0,0,1/45,-1/270,-1/45,0,1/45, 1/270, -1/45,0);

$\vec{\tau}_6$ =(-5/2,-4/9,8/9,8/9,-4/9,8/9,8/9,5/36,-4/27,1/18,0, 1/18,-4/27,5/36,0,-1/9,2/27,-1/9,0,1/45,-2/45,1/270,-2/45,2/135,-2/45,1/270,-2/45,1/45);

$\vec{\tau}_7$ =(0,-1,0,0,1,5,-5,1/2,0,0,0,0,0,-1/2,-5/2,-5/8,0,5/8, 5/2,-1/2,-1/2, 1/8,0,0,0,-1/8,1/2,1/2);

$\vec{\tau}_8/\sqrt{3}$ =(-3/2,1,-2/9,-2/9,1,5/9,5/9,-1/4,2/9,-1/18,1/9,-1/18, 2/9,-1/4,-13/18,-17/72,2/9,-17/72,-13/18,1/6,1/6,-1/24,0,0, 0,-1/24,1/6, 1/6);

$\vec{\tau}_9$ =(0,-19/9,16/9,-16/9,19/9,-1/9,1/9,2/9,0,-1/18,0,1/18, 0,-2/9,-5/6,-11/72,0,11/72,5/6,-1/6,-1/6,1/24,0,0,0,-1/24, 1/6,1/6);

$\vec{\tau}_{10}/\sqrt{3}$ = (3/2,11/9,-22/9,-22/9,11/9,-1/9,-1/9,-1/36,-2/9, 5/9,19/9,5/9,-2/9,-1/36,-7/18,1/72,-2/9,1/72,-7/18, 1/18,1/18, -1/72,-4/9,-1/9,-4/9,-1/72,1/18,1/18);

$\vec{\tau}_{11}$ =(6,-4,0,0,-4,0,0,1,0,0, 0,0,0,1,0,-5/2,10/3,-5/2,0,1,0,-1/6,0,0,0,-1/6,0,1);

$\vec{\tau}_{12}/\sqrt{3}$ =(0,-1/3,-2/3,2/3,1/3,-1/3,1/3,1/6,2/9,0,0,0,-2/9, -1/6,-1/6,-1/6,0,1/6,1/6,-1/6,-1/6,1/18,0,0,0,-1/18,1/6, 1/6);

$\vec{\tau}_{13}$ =(2,-2/9,-8/9,-8/9,-2/9,-2/9,-2/9,-7/9,8/9,-1/9,2/9,-1/9, 8/9,-7/9,-1/9,7/18,-2/3,7/18,-1/9,0,1/3,-1/18,0,0,0,-1/18, 1/3,0);

$\vec{\tau}_{14}/\sqrt{3}$ =(0,1/9,-2/3,2/3,-1/9,-7/9,7/9,1/2,-22/27,7,9,0, -7/9,22/27,-1/2,1/18,5/18,0,-5/18,-1/18,1/18,-1/6,1/54, 2/9,0,-2/9, -1/54,1/6,-1/18);

$\vec{\tau}_{15}$ =(6,8/9,-16/9,-16/9,8/9,-28/9,-28/9,-7/9,32/27,-10/9, -4/9,-10/9,32/27,-7/9,2/9,7/18,26/27,7/18,2/9,-1/9,2/9, -1/54,8/9, 8/27, 8/9,1/54,2/9,-1/9);

$\vec{\tau}_{16}$ =(0,0,0,0,0,-20,20,0,0,0,0,0,0,0,0,10,5/2,0,-5/2,-10,2,2, -1/2,0,0,0,1/2,-2,2);

$\vec{\tau}_{17}/\sqrt{3}$ (6,-4,0,0,-4,-4/3,-4/3,1,0,0,0,0,0,1,2,7/6,-4/3,7/6, 2,-2/3,-2/3,1/6,0,0,0,1/6,-2/3,-2/3);

$\vec{\tau}_{18}$ =(0,8/3,-4,4,-8,3,-8/3,8/3,-4/3,4/3,0,0,0,-4/3,4/3,2, 5/6, 0,-5/6,-2,2/3,2/3,-1/6,0, 0,0,1/6,-2/3,-2/3);

$\vec{\tau}_{19}/\sqrt{3}$ =(2,-4/3,4/9,4/9,-4/3,-8/9,-8/9,1/3,-4/9,4/9,-8/9, 4/9,-4/9,1/3,10/9,-1/18,4/9,-1/18,10/9,-2/9,-2/9,1/18,0,0, 0,1/18,-2/9,-2/9);

$\vec{\tau}_{20}$ =(0,16/3,-40/9,40/9,-16/3,20/9,-20/9,0,-8/9,8/9,0, -8/9, 8/9,0,2,-11/18 ,0,11/18,-2,2/9,2/9,-1/18,0,0,0,1/18, -2/9,-2/9);

$\vec{\tau}_{21}/\sqrt{3}$ =(-10,-140/27,280/27,280/27,-140/27,100/27, 100/27,-5/27,40/27,-80/27,-80/9,-80/27,40/27,-5/27, 10/9,-25/54,-20/27,-25/54,10/9,-2/27,-2/27,1/54,64/27, -16/27,320/135,1/54,-2/27,-2/27);

$\vec{\tau}_{22}$ =(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,15,-20,15,0,-6,0,1,0, 0,0, 1,0,-6);

$\vec{\tau}_{23}/\sqrt{3}$ =(0,0,0,0,0,-20/3,20/3,0,0,0,0,0,0,0,10/3,5/3,0, -5/3, -10/3,4/3,2/3,-1/3,0,0,0,1/3,-2/3,-4/3);

$\vec{\tau}_{24}$ =(8,-16/3, 0,0,-16/3,-8/3,-8/3,4/3,0,0,0,0,0,4/3,4,-1/3,4/3, -1/3,4,-2/3,-4/3,1/3,0,0,0,1/3,-4/ 3,-2/3);

$\vec{\tau}_{25}/\sqrt{3}$ =(0,8/3,-8/3,8/3,-8/3,4/3,-4/3,-4/3,8/9,0,0,0, -8/9, 4/3,2/3,-1/3, 0,1/3,-2/3,0,2/3,-1/9,0,0,0,1/9,-2/3,0);

$\vec{\tau}_{26}$ =(-16/3,0,32/9,32/9,0,16/9,16/9,8/3,-32/9,16/9,

-32/9,16/9,-32/9,8/3,-8/9,-1/9,-4/9,-1/9,-8/9,2/9,-8/9, 1/9,0,0,0,1/9,-8/9,2/9);

$\vec{\tau}_{27}/\sqrt{3}$ *=(0,-80/27,80/27,-80/27,80/27,-20/27,20/27, -40/27,80/27, -80/27,0,80/27,-80/27,40/27,-10/9,5/27,0, -5/27,10/9,-4/27,10/27,-1/27,-32/27,0,32/27,1/27,-10/27, 4/27);*

$\vec{\tau}_{28}$ *=(40/3,80/9,-160/9,-160/9,80/9,-40/9,40/9,20/9, -160/27,80/9,160/9,80/9,-160/27,20/9,-20/9,5/9,20/27, 5/9, -20/9,2/9, -4/9,1/27,-64/9,64/27,-64/9,1/27,-4/9,2/9).*

Hamilton's matrix in the 4-th degree for the 6-th order Taylor's polynomial (1) has such form:

$$\overline{\mathbf{N}}^4 = \frac{\begin{matrix} 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,2\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,2\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,2\,0\,1\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,2\,0\,2\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,2\,0\,1\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,2\,0\,2\,0\,1 \end{matrix}}{0} \qquad (5)$$

4. Construction of discrete analogues

Having got the value of the reciprocal matrix one can construct any invariant difference analogue of any differential operator for any node of the mesh. The schematic image of biharmonic operator difference analogue for the central node was presented in [2]. Let us show the application of the formula (2) to construction of DA for 6 other characteristic nodes (i. e. for the nodes with the numbers 2, 8, 11, 14, 20, 26).

Taylor's vector for the node #2:

$$\vec{T}_2 = (1, h, 0, h^2/2!, 0, 0, h^3/3!, 0, 0, 0, h^4/!, 0, 0, 0, 0, h^5/5!, 0, 0, 0, 0, 0, 0, h^6/6!, 0, 0, 0, 0, 0, 0, 0) \qquad (6)$$

Correspondingly, DA of biharmonic operator for the node # 2 looks as follows:

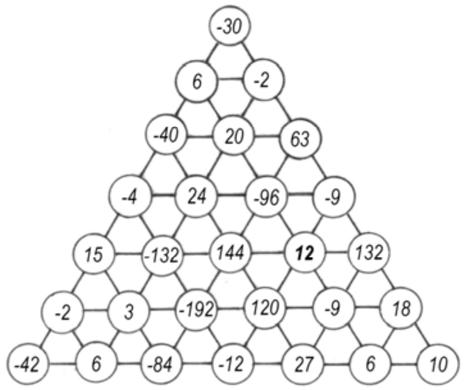

Fig. 2. Difference analogue of biharmonic operator $27/4h^4\nabla^4$ *for the 2-nd node*

Taylor's vector for the node #8:

$$\vec{T}_8 = (1, 3h/2, -h\sqrt{3}/2, 9h^2/8, -3\sqrt{3}h^2/4, 3h^2/8, 9h^3/16, -9\sqrt{3}h^3/169h^3/16, -\sqrt{3}h^3/16, 27h^4/144, -9\sqrt{3}h^4/32, 27h^4/64, -3\sqrt{3}h^4/32, 3h^4/144, 243h^5/(5!32), -81\sqrt{3}h^5/(4!32), 27h^5/128, -9\sqrt{3}h^5/128, 9h^5/256, -9\sqrt{3}h^5/(5!32), 729h^6/(6!64), -243\sqrt{3}h^6/(5!64), 81h^6/1024, -9\sqrt{3}h^6/256, 27h^6/1024, -27\sqrt{3}h^6/(5!64), 27h^6/(6!64)) \qquad (7)$$

Correspondingly, DA of biharmonic operator for the node # 8 looks as follows:

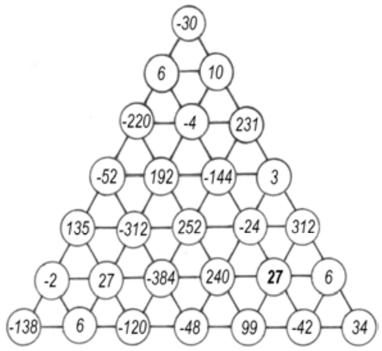

Fig. 3. Difference analogue of biharmonic operator $27/4h^4\nabla^4$ *for the 8-th node*

Taylor's vector for the node #11:

$$\vec{T}_{11} = (1, 0, -h\sqrt{3}, 0, 0, 3h^2/2!, 0, 0, 0, -h^3\sqrt{3}/2!, 0, 0, 0, 0, 9h^4/4!, 0, 0, 0, 0, 0, 0, -9\sqrt{3}h^5/5!, 0, 0, 0, 0, 0, 0, 0, 27h^6/6!) \qquad (8)$$

Correspondingly, DA of biharmonic operator for the node # 11 looks as follows:

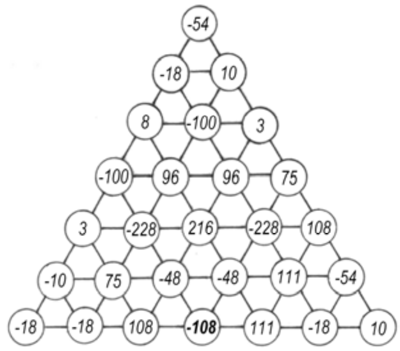

Fig. 4. Difference analogue of biharmonic operator $27/4h^4\nabla^4$ *for the 11-th node*

Taylor's vector for the node #14:

$$\vec{T}_{14} = (1, h, -h\sqrt{3}, h^2/2, -\sqrt{3}h^2, 3h^2/2, h^3/3!, -\sqrt{3}h^3/2, 3h^3/2, -\sqrt{3}h^3/2, h^4/4!, -\sqrt{3}h^4/3!, 3h^4/4, -\sqrt{3}h^4/2, 9h^4/4!, h^5/5!, -\sqrt{3}h^5/4!, h^5/4, -\sqrt{3}h^5/4, 9h^5/4!, -9\sqrt{3}h^5/(5!), h^6/6!, -\sqrt{3}h^6/5!, 3h^6/(4!2!), -\sqrt{3}h^6/(2!3!), 9h^6/(2!4!), -9\sqrt{3}h^6/5!, 27h^6/6!) \qquad (9)$$

Correspondingly, DA of biharmonic operator for the node # 14 looks as follows:

Taylor's vector for the node #20:

$$\vec{T}_{20} = (1, 2h, -h\sqrt{3}, 2h^2, -2\sqrt{3}h^2, 3h^2/2, 8h^3/3!, -2\sqrt{3}h^3, 3h^3, -\sqrt{3}h^3/2, 4h^4/3!, -8\sqrt{3}h^4/3!, h^4, -\sqrt{3}h^4, 9h^4/4!, 32h^5/5!, -4\sqrt{3}h^5/3!, 2h^5, -\sqrt{3}h^5, 18h^5/4!, -9\sqrt{3}h^5/(5!),$$

$$64h^6/6!,-32\sqrt{3}h^6/5!,2h^6/3!,-4\sqrt{3}h^6/3!,9h^6/12,$$
$$-18\sqrt{3}h^6/5!,27h^6/6!) \qquad (10)$$

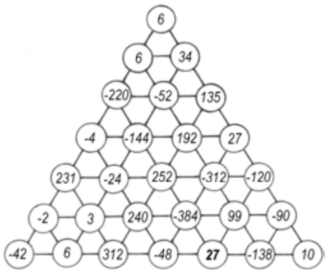

Fig. 5. Difference analogue of biharmonic operator $27/4h^4\nabla^4$ *for the 14-th node*

Correspondingly, DA of biharmonic operator for the node # 20 looks as follows:

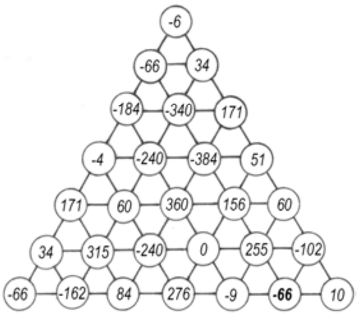

Fig. 6. Difference analogue of biharmonic operator $27/4h^4\nabla^4$ *for the 20-th node*

Taylor's vector for the node #26:

$$\vec{T}_{26} = (1,3h,-h\sqrt{3},9h^2/2,-3\sqrt{3}h^2,3h^2/2,9h^3/2,$$
$$-9\sqrt{3}h^3/2,9h^3/2,-\sqrt{3}h^3/2,81h^4/4!,-9\sqrt{3}h^4/2,$$
$$27h^4/(2!2!),-9\sqrt{3}h^4/2,27h^4/4!,243h^5/5!,-81\sqrt{3}h^5/4!,$$
$$27h^5/(2!2!),-8\sqrt{3}h^5/(2!2!),27h^5/4!,-9\sqrt{3}h^5/5!,$$
$$729h^6/6!,-243\sqrt{3}h^6/5!,243h^6/(4!2!),-9\sqrt{3}h^6/4,$$
$$27h^6/16,-27\sqrt{3}h^6/5!,27h^6/6!) \qquad (11)$$

Correspondingly, DA of biharmonic operator for the node # 26 looks as follows:

Fig. 7. Difference analogue of biharmonic operator $27/4h^4\nabla^4$ *for the 26-th node*

5. Conclusion

Application of invariant approximations methodology to the construction of difference and discrete analogues of differential operators gives us only one possible invariant analogue with required order of discretization error that ensures the objectivity of computational results and their independence from a chosen coordinate system.

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ПОБУДОВА ІНВАРІАНТНИХ РІЗНИЦЕВИХ ТА ДИСКРЕТНИХ АНАЛОГІВ ДИФЕРЕНЦІЙНИХ ОПЕРАТОРІВ З ВИСОКИМ ПОРЯДКОМ ПОХИБКИ ДИСКРЕТИЗАЦІЇ

М. Говикович

В статті пропонується алгоритм побудови інваріантних різницевих та дискретних аналогів диференційних операторів на підставі методології інваріантних наближень. Розраховано точні значення елементів матриці Тейлора шостого порядку, що дозволило сконструювати різницеві та дискретні аналоги високого порядку похибки дискретизації.

Mariya Howykowycz graduated form Lviv Polytechnic Institute (major "Electrical Systems") and Ivan Franko National University of Lviv (majors "English" and "German"). She defended Ph. D. dissertation thesis at Ivan Franko National University in Applied Mathematics. She has been working at Lviv Polytechnis National University since 1992 as Associate Professor in Electrical Engineering.