Improved Solution of Cramer-Rao Lower Bound for TOA/RSS Localization

Taras Holotyak, Svyatoslav Voloshynovskiy, Jose Rolim, Ivan Prudyus

Abstract – **In this paper the problem of evaluation of bound accuracy for the range based localization techniques in the wireless sensor networks is considered. The comprehensive analysis of the existing solution of the Cramer-Rao lower bound problem for the anchored localization based on the time of arrival or relative signal strength principles shows certain discrepancy between analytical solution and simulation results. Therefore, new bound for this problem using more accurate stochastic modeling of localization error is proposed.**

Keywords – **localization, Crame-Rao lower bound, wireless sensor networks.**

I. INTRODUCTION

The availability of the location information in the wireless network plays the vital role in such applications as geographical routing, target tracking and environmental monitoring and gives possibility to complete these problems in a more efficient (in terms of energy, latency, etc.) way. Another important aspect of localization methods design consists in the comparison of their performance with certain theoretical limits, a.k.a bounds, on the localization error. Among numerous existing bounds, the Cramer-Rao lower bound (*CRLB*) is probably the most popular tool for the benchmarking of performance of localization methods. Initially originated from the radar and remote sensing techniques [1-3], the existing solution of the *CRLB* problem was directly applied to the task of nodes localization in wireless sensor networks (WSN) [7,9-11]. However, obtained under certain assumptions, which are not valid anymore in the WSN, the sensor localization *CRLB* requires new and more detailed consideration.

The main contribution of this paper consists in the derivation of the *CRLB* for the range-based localization using more accurate ambiguity function that adequately represents the procedure of node position estimation.

II. GENERAL ASSUMPTIONS

Considering bounds on the sensor localization error, this paper will be concentrated only on the WSN, where time synchronization (TOA) or propagation of predefined and standardized signals (RSS) is available. By other words, the particularities of the *CRLB* for one-hope nodes localization based on the independent range estimations will be investigated. The problem consideration is restricted to pair-wise (anchor-node) measurements under static topology conditions. Also, the node localization under only ideal electromagnetic wave propagation conditions (absence of multipath, diffraction, etc) is analyzed. The localization problem consideration is restricted to a 2D case, whereas the extension to the higher dimensions cases is straightforward.

In wireless communications each anchor can be considered as a point source of the electromagnetic radiation, thus, the polar coordinate system is used as the most natural way to describe the pair-wise (anchor-node) measurements. Therefore, using measurements $\boldsymbol{\psi} = [\mathbf{r}, \boldsymbol{\varphi}]$, where $\boldsymbol{\psi}_i = [r_i, \phi_i]$ denotes values of the range r_i and angle φ_i that are obtained by means of the i^{th} anchor with coordinates $\xi_i = [\xi_{X i}, \xi_{Y i}]$, a vector of parameters $\boldsymbol{\omega} = [\omega_r, \omega_\varphi]$ should be estimated. This vector describes a position of node N in the polar coordinate system (Fig. 1). At the same time, the Cartesian coordinate system provides more convenient way for the WSN topology description. Here, the vector $\boldsymbol{\theta} = [\theta_X, \theta_Y]$ describes the position of node N. Therefore, the *CRLB* of the node N localization, which is evaluated based on the $\boldsymbol{\psi}$, is supposed to be introduced in the Cartesian coordinates as well. The problem analysis will be elaborated using the probabilistic description of the node N position. Then, the *CRLB* is represented as:

$$\operatorname{var}(\hat{\boldsymbol{\theta}}) \geq CRLB(\boldsymbol{\theta}) = -\left[\mathrm{E}\left[\left(\partial^2 \ln p_{\Xi|\boldsymbol{\theta}}(\xi|\boldsymbol{\theta}) / \partial \boldsymbol{\theta}^2\right)\right]\right]^{-1} = I^{-1}(\boldsymbol{\theta}), \tag{1}$$

where $I(\boldsymbol{\theta}) = -\mathrm{E}\left[\left(\partial^2 \ln p_{\Xi|\boldsymbol{\theta}}(\xi|\boldsymbol{\theta}) / \partial \boldsymbol{\theta}^2\right)\right]$ is the Fisher information matrix and $p_{\Xi|\boldsymbol{\theta}}(\xi|\boldsymbol{\theta}) = L(\boldsymbol{\theta})$ denotes a likelihood function that is associated with measurement data.

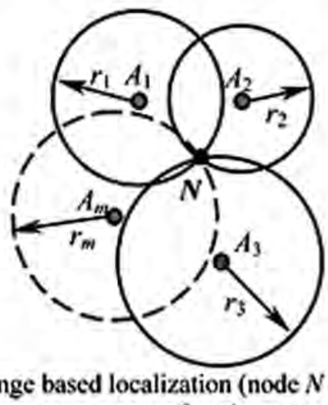

Fig. 1. Range based localization (node N is localized using measurements of anchors $A_1, \ldots, A_m$).

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III. PROPOSED SOLUTION TO *CRLB* PROBLEM

We start the analysis by an accurate stochastic modeling of the localization procedure. The problem linearization, which physically means assumption about plane wave propagation in far-field communications, is not valid in the description of localization problems in WSN.

The analysis of localization problem is based on the definition of joint pdf of measurements, which can be written based on the chain rule for probabilities as:

$$p(\xi_1,...,\xi_m) = p_\Xi(\xi_1)\, p_\Xi(\xi_2|\xi_1)..p_\Xi(\xi_m|\xi_1,...,\xi_{m-1}) = \prod_{i=1}^{m} p_\Xi(\xi_i), \quad (2)$$

where last step is possible due to the independence of the measurements. The localization procedure does not also depend on the coordinate system used for its description, therefore:

$$L(\boldsymbol{\omega}|\boldsymbol{\psi}) = \prod_{i=1}^{m} p_\Psi(\psi_i|\boldsymbol{\omega}) \doteq \prod_{i=1}^{m} p_\Xi(\chi_i|\boldsymbol{\theta}) = L(\boldsymbol{\theta}|\boldsymbol{\chi}). \quad (3)$$

According to (1) the $CRLB(\boldsymbol{\omega})$ in the polar coordinate system is equal to:

$$\mathrm{var}(\hat{\boldsymbol{\omega}}) \geq \left[-\mathrm{E}\left[\partial^2 \ln p(\boldsymbol{\psi}|\boldsymbol{\omega})/\partial\boldsymbol{\omega}^2\right]\right]^{-1}. \quad (4)$$

Then, in the Cartesian coordinates $CRLB(\boldsymbol{\theta})$ will be determined as:

$$\mathrm{var}(\hat{\boldsymbol{\theta}}) \geq \left[\frac{\partial f(\boldsymbol{\omega})}{\partial\boldsymbol{\omega}}\right]\left[-\mathrm{E}\left[\frac{\partial^2 \ln p(\boldsymbol{\psi}|\boldsymbol{\omega})}{\partial\boldsymbol{\omega}^2}\right]\right]^{-1}\left[\frac{\partial f(\boldsymbol{\omega})}{\partial\boldsymbol{\omega}}\right]^T, \quad (5)$$

where $\boldsymbol{\theta} = f(\boldsymbol{\omega})$ is the function that defines transformation of the coordinate system. $J = \partial f(\boldsymbol{\omega})/\partial\boldsymbol{\omega}$ is the Jacobian matrix, such that:

$$\boldsymbol{\omega} = \begin{bmatrix}\omega_r \\ \omega_\phi\end{bmatrix}; \quad \boldsymbol{\theta} = \begin{bmatrix}\theta_x \\ \theta_y\end{bmatrix} = \begin{bmatrix}\omega_r \cos\omega_\phi \\ \omega_r \sin\omega_\phi\end{bmatrix},$$

$$J = \frac{\partial f(\boldsymbol{\omega})}{\partial\boldsymbol{\omega}} = \begin{bmatrix}\cos\omega_\phi & -\omega_r \sin\omega_\phi \\ \sin\omega_\phi & \omega_r \cos\omega_\phi\end{bmatrix}. \quad (6)$$

It has to be highlighted, that the 2D likelihood function in the existing solution is defined by only range measurement projections causing singularity of the Fisher information matrix, while the proposed solution will operate with both the range and angle estimation models. The fact that angle measurements are not carried out is interpreted in the proposed solution as the angle measurement with uniformly distributed over support error, creating a random variable on the circle [12]. Taking into account the independence of polar coordinates, the error of node localization by each anchor can be separated onto independent components, which are related with range and angle estimations as:

$$p(\psi_i|\boldsymbol{\omega}) = p(\psi_i|\omega_r,\omega_\phi) = p(r_i|\omega_r)\,p(\phi_i|\omega_\phi). \quad (7)$$

The range estimation is described by the Gaussian distributed (3) ambiguity, whereas:

$$p(\phi_i|\omega_\varphi) = \frac{1}{2\pi}\left[\mathrm{H}(\phi_i+\pi) - \mathrm{H}(\phi_i-\pi)\right],\ \phi_i \in [-\pi,\pi), \quad (8)$$

where $\mathrm{H}(\cdot)$ is the Heaviside function. Then, the Fisher information matrix for the $CRLB(\boldsymbol{\omega})$ is equal to

$$I_i(\boldsymbol{\omega}) = -\begin{bmatrix}\mathrm{E}\left[\dfrac{\partial^2 \ln p(r_i|\omega_r)}{\partial\omega_r^2}\right] & 0 \\ 0 & \mathrm{E}\left[\dfrac{\partial^2 \ln p(\phi_i|\omega_\phi)}{\partial\omega_\phi^2}\right]\end{bmatrix}, \quad (9)$$

where $\mathrm{E}\left[\partial^2 \ln p(\boldsymbol{\psi}|\boldsymbol{\omega})/\partial\omega_r\partial\omega_\phi\right] = 0$ and $\mathrm{E}\left[\partial^2 \ln p(\boldsymbol{\psi}|\boldsymbol{\omega})/\partial\omega_\phi\partial\omega_r\right] = 0$ due to the independence of the coordinates of polar coordinate system, and $-\mathrm{E}\left[\partial^2 \ln p(r|\omega_r)/\partial\omega_r^2\right] = 1/\sigma_r^2$ [3,4]. The closed form solution for $\left\{-\mathrm{E}\left[\partial^2 \ln p(\phi|\omega_\phi)/\partial\omega_\phi^2\right]\right\}$ is still a challenging problem. However, inverting statement of (1): the *CRLB* is the lower bound of any unbiased estimator, one can conclude that the *CRLB* itself is upper bounded by the variance of any such an estimator. Therefore, in the Fisher information matrix this value will be approximated by the variance of the parameter estimation using, for instance, maximum likelihood approach that is known to be asymptotically efficient:

$$\ln p(\boldsymbol{\varphi}|\omega_\phi) = \ln\prod_{i=1}^{m} p(\phi_i|\omega_\phi) = \sum_{i=1}^{m} \ln p(\phi_i|\omega_\phi) = -m\ln(2\pi),\ \phi_i \in [-\pi,\pi). \quad (10)$$

Solving $\partial \ln p(\boldsymbol{\varphi}|\omega_\phi)/\partial\omega_\phi = 0$, the left-hand side of this equation is a constant being independent of ω_φ, therefore, $\hat{\omega}_\phi = \forall\omega_\phi \in [-\pi,\pi)$. This means, that for the measurement made by i^{th} anchor any value of the angle can be considered as the estimation of angle of the node N position. Due to the closed space of the support for real angles, which are random variables on the circle, the maximum likelihood estimation of the angle with uniform error distribution is asymptotically unbiased because of the symmetry of the error pdf, i.e., $\mathrm{E}\left[\hat{\omega}_\phi\right] = \omega_\phi$, where support symmetry is conditioned by the properties of distribution on the circle. The variance of this estimator is equal to:

$$\sigma_\varphi^2 = \mathrm{E}\left[(\phi - \hat{\omega}_\phi)^2\right] = \int_{-\infty}^{\infty}\phi^2 p(\varphi|\omega_\phi)\,d\phi = \int_{-\pi}^{\pi}\phi^2\frac{1}{2\pi}\left[\mathrm{H}(\phi+\pi) - \mathrm{H}(\phi-\pi)\right]d\phi = (2\pi)^2/12. \quad (11)$$

Therefore,

$\left[-\mathrm{E}\left[\partial^2 \ln p(\phi_i|\omega_\phi)/\partial\omega_\phi^2\right]\right]^{-1} \leq \mathrm{var}(\hat{\omega}_\phi) \leq (2\pi)^2/12$ and the Fisher information is defined as follows:

$$I_i(\boldsymbol{\omega}) \leq \begin{bmatrix}1/\sigma_r^2 & 0 \\ 0 & 1/\sigma_\varphi^2\end{bmatrix}. \quad (12)$$

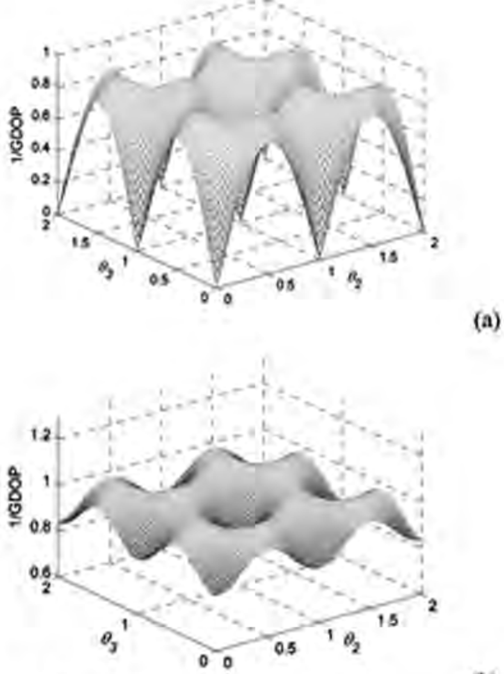

Fig. 2. Comparison of the *CRLB* for the range based localization: (a) existing solution, (b) proposed solution.

Using the additive property of the Fisher information the $I(\boldsymbol{\theta})$ based on the m independent pair-wise measurements can be written as:

$$I(\boldsymbol{\theta})=\sum_{i=1}^{m} I_i(\boldsymbol{\theta}) \le$$

$$\begin{bmatrix} \sum_{i=1}^{m}\frac{\sigma_{ri}^2\sin^2\phi_i+r_i^2\sigma_{\phi i}^2\cos^2\phi_i}{r_i^2\sigma_{ri}^2\sigma_{\phi i}^2} & \sum_{i=1}^{m}\frac{\left(r_i^2\sigma_{\phi i}^2-\sigma_{ri}^2\right)\cos\phi_i\sin\phi_i}{r_i^2\sigma_{ri}^2\sigma_{\phi i}^2} \\ \sum_{i=1}^{m}\frac{\left(r_i^2\sigma_{\phi i}^2-\sigma_{ri}^2\right)\cos\phi_i\sin\phi_i}{r_i^2\sigma_{ri}^2\sigma_{\phi i}^2} & \sum_{i=1}^{m}\frac{\sigma_{ri}^2\cos^2\phi_i+r_i^2\sigma_{\phi i}^2\sin^2\phi_i}{r_i^2\sigma_{ri}^2\sigma_{\phi i}^2} \end{bmatrix}, \quad (13)$$

where $I_i(\boldsymbol{\theta})$ denotes the Fisher information about vector $\boldsymbol{\theta}$ evaluated based in the i^{th} anchor measurement.

The comparison of the *CRLB* problem solutions is performed using geometrical dilution of precision (*GDOP*) [8]. This parameter is derived based on the *CRLB* and often used for the localization benchmarking providing an integral evaluation of the localization ambiguity. Using the setup from [5] the dependence of the *GDOP* from anchors locations based on existing and proposed solutions were obtained (Fig. 2). The presented results demonstrate same qualitative behavior, i.e., the positions of minima and maxima of the *GDOP*s: $GDOP=\max(GDOP)$ in the case, when $\phi_2\in\phi_1+\{0,\pi\}\cup\phi_3\in\{\phi_2-\pi,\phi_2,\phi_2+\pi\}$, and $GDOP=\min(GDOP)$ if $\phi_2\in\phi_1+\{\pi/3,2\pi/3,4\pi/3,5\pi/3\}\cup\ \phi_3\in\{\pi-\phi_2,2\pi-\phi_2,3\pi-\phi_2\}$. However, with the proposed solution of the *CRLB* problem, the *GDOP* achieves the finite non-zero values that correspond to the finite localization ambiguities defined in terms of the entropy of separate anchor localization.

IV. Conclusions

In this paper the problem of node localization bounds in the anchored wireless sensor networks was investigated. Focusing on the *CRLB* approach of the localization error evaluation, it was shown the lack of accuracy of the existing solution of this problem. Based on the more accurate measurement model an improved solution of the *CRLB* problem for the TOA/RSS localization in the WSN was derived and the impact of the network topology on the localization precision was studied. It was shown that the proposed solution overcomes the drawback of the existing one.

Future work in this area will consist in investigation of the conditions of *CRLB* applications as well as in development of information-theoretic criteria of the localization problem in WSN. Another extension will be focused on the application of the proposed solution to the error propagation model in multi-hop localization problems.

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