Sampling Theorem in Frequency-Time Domain and its Applications

Kalyuzhniy N.M.

Abstract: **In the paper, the results of the sampling theorem in frequency-time domain are given; its consequences are founded, and directions of its practical use are proposed for signals determination and restoration under conditions of the prior uncertainty of their kind and parameters.**

Key words: **prior uncertainty, sampling theorem in frequency-time domain, basis expansion functions, frequency-time samplings, determination-reproduction of signals, dispersion Fourier processor.**

INTRODUCTION

The efficiency of the complicated radio electronic condition control and radio electronic suppression of unauthorized radiations can be essentially increased by realization of quasi- and optimal methods of processing and reproduction of signals with the prior uncertainty of their structures and parameters. When solving these problems, of paramount importance is the development of the mathematical description of signals, such that could adequately simulate actual processing and allow for the aforementioned prior uncertainty.

MAIN PART

The mathematical description of signals is based on the proof of the generalized sampling theorem [1] whose essence is in the following.

Theorem. An arbitrary narrow-band signal *s(t)* with the limited spectrum enclosed in the frequency band $\pm \Delta f_s /2$ is fully specified by its frequency values taken with intervals $\Delta F_\ell = \Delta f_s/\ell$ and time values $\Delta T_\ell = 1/\Delta F_\ell$ at any integer $\ell = 1, 2, \ldots L+M$.

In [1], the theorem is proved basing on representation of spectrum-limited signals by a Kotelnikov series at the time domain with discretization $\Delta t=/2\Delta f_s$, uniform grouping of ℓ samplings with expansion functions at the time interval ΔT_ℓ and their Fourier transform into the frequency domain. Finally the frequency-time representation of the complex envelope of the arbitrary signal is obtained as a special double series:

$$\dot{S}(t) = \sum_{k=-\infty}^{\infty} \sum_{\ell=-(L-1/2)}^{L+1/2} \dot{S}_{k\ell}\dot{y}_{k\ell}(t) \qquad (1)$$

where $\dot{S}_{kl} = \dot{S}_{k\ell} e^{j\varphi_{k\ell}}$ are samplings of the complex envelope at time moments $k\Delta T\ell$ and frequencies $\ell\Delta F_\ell$,

$$\left.\begin{aligned} \dot{y}_{k\ell}(t) &= \frac{\sin \pi\Delta F_\ell (t-k\Delta T_\ell)}{\pi\Delta F (t-k\Delta T_\ell)} V_{k\ell}(t), \\ \dot{f}_{k\ell}(t) &= \Pi\left[\frac{(t-k\Delta T_\ell)}{\Delta T_\ell}\right] V_{k\ell}(t) \end{aligned}\right\} \text{ are basis expansion functions}$$

$V_{k\ell}(t) = e^{j2\pi\Delta F\ell (t-k\Delta T_\ell)}$ are the complex harmonics of the Fourier series.

Kalyuzhniy, N. M. – Kharkov National University of Radio Electronics, 14 prospekt Lenina, Kharkov, Ukraine, E-mail: rmorti@gmail.com

Expression (1) describes the complex envelope of the narrow-band signal as the time and frequency biaxial expansion. Therewith the frequency-time plane occupied by the signal is divided into frequency-time elements with the frequency band ΔF and duration ΔT. The basis expansion functions are of the *sin x/x* type with the frequency infill by harmonics of the Fourier series (except $\ell = 0$), that are biased in time by ΔT and in frequency by ΔF.

Let us define complex spectrum (1) using the inverse Fourier transform

$$\dot{C}(f) = \int_{-\infty}^{\infty} \dot{S}(t) e^{-j2\pi ft} dt$$

Introducing the integral over time under the symbol of summation in the right side of the Fourier transform, and using property of changeability of *F* and *t* for even functions [2], after calculation of the integral, we obtain

$$\dot{C}(f) = \sum_{\ell=-(L-1/2)}^{L+1/2} \sum_{k=-\infty}^{\infty} C_K(\ell\Delta F) e^{j\varphi_k(\ell\Delta F)} \Pi_k\left(\frac{f-\ell\Delta F}{\Delta F}\right) e^{-j2\pi(f-\ell\Delta F)k\Delta T}, \quad (2)$$

where $\dot{C}(\ell\Delta F, k\Delta T) = \Delta T \cdot \dot{S}(\ell\Delta F, k\Delta T)$,

$\Pi_k(f) = \begin{cases} 1, & npu\ |f| \le 1/2\,\Delta F \\ 0, & npu\ |f| > 1/2\,\Delta F \end{cases}$ is the rectangle strobing function with the frequency band ΔF and the center at the point $F_\ell = \ell\Delta F$, introduced by Woodworth.

The analogous proof of the generalized sampling theorem, based on representation of time-limited signals by the Kotelnikov series at the frequency domain, leads to the result

$$\dot{C}(f) = \sum_{\ell=-(L-1/2)}^{L+1/2} \sum_{k=1}^{K} C_k(\ell\Delta f) e^{j\varphi_k(\ell\Delta F)} \frac{\sin \pi\Delta T(f-\ell\Delta F)}{\pi\Delta T(f-\ell\Delta F)} e^{j2\pi(f-\ell\Delta F)k\Delta T} \quad \text{C} \quad (3)$$

$$\dot{S}(t) = \sum_{k=1}^{K} \sum_{k=-(L-1/2)}^{L+1/2} S_\ell(k\Delta T) e^{j\varphi_\ell(k\Delta T)} \prod\left(\frac{t-k\Delta T/2}{\Delta T}\right) e^{j2\pi\ell\Delta F(t-k\Delta T)}. \qquad (4)$$

omparing (3) and (4) with (2) and (1), respectively, we see that either the complex signal envelope or its spectrum can be presented by biaxial expansions Fourier conjugate by functions of sin x/x type or by Woodworth's rectangle strobing functions with the frequency infill by harmonics of the Fourier series. The presence of the frequency infill in basis expansion functions allows interpreting, e.g., expressions (1) and (4) as description of signals in the form of sets of radio-frequency pulses in the frequency-time plane with the corresponding envelopes, frequencies, amplitudes and initial phases at points with the *(kΔT, ℓΔF)* co-ordinates.

Analytical expressions (1) – (4) can be used for description of signals both in the complex form and in the real one.

Consider the consequences of this theorem and properties of the obtained frequency-time representation of signals (1) appropriate to be used at their practical application.

1. It is easy to make sure that the basis expansion *sin x/x* functions in (1) are orthogonal on the time axis. Given fixed ℓ

$$I_1 = \int_{-\infty}^{\infty} \Psi_{k\mathbf{l}}(t)\Psi_{m\mathbf{l}}^{*}(t)dt =$$

$$\equiv \frac{(-1)^{k+m}}{p}\int_{-\infty}^{\infty}\frac{\sin^2(p\Delta Ft)}{(\Delta Ft-k)(\Delta Ft-m)}dt. \qquad (5)$$

Replacing $t = \Delta F \cdot t$ in (5), obtain

$$I_1 = \frac{(-1)^{k+m}}{p\Delta \grave{A}}\int_{-\infty}^{\infty}\frac{\sin^2 pt}{(t-k)(t-m)}dt = \qquad (6)$$

$$\begin{cases} i\Delta T, & at\ k=m,\ \ \Delta F\cdot\Delta T = i,\ \ i=1,2,\ldots \\ 0, & at\ k \neq m. \end{cases}$$

Therefore, for any ℓ, the basis functions are orthogonal on the entire axis $-\infty < t < \infty$. The minimum shift of the argument, leading to orthogonalization on the time axis of *sin x/x*-type function, can be obtained under condition of $\Delta F \cdot \Delta T = 1$. In this case, a shift between the *sin x/x* functions along the frequency axis is $\Delta \mathrm{F} = 1/\Delta \mathrm{T}$.

Let us test the basis expansion function for a condition of orthogonality on the frequency axis. At $\Delta F\cdot\Delta T$=1, the expansion functions are

$$\Phi_{kl}(f) = \Pi_k\left(\frac{f-\mathbf{l}\Delta F}{\Delta F}\right)e^{-j2pfk\Delta T},$$

whereas the orthogonality condition on the frequency axis at the fixed *k* is

$$I_2 = \Delta F\int_{-\Delta f_c/2}^{\Delta f_c/2}\Phi_{k\mathbf{l}}(f)\Phi_{km}^{*}(f)df =$$

$$\equiv \int_{-\Delta f_c/2}^{\Delta f_c/2}\Pi_k\left[\frac{f-\mathbf{l}\Delta F}{\Delta F}\right]\Pi_k\left[\frac{f-m\Delta F}{\Delta F}\right]df. \qquad (7)$$

Replacing in (7) $f = S\cdot\Delta F$, obtain

$$I_2 = \Delta F\int_{-\Delta f_c/2}^{\Delta f_c/2}\Pi_k[s-\mathbf{l}]\Pi_k[s-m]ds = \begin{cases}\Delta F, & at\ \ \mathbf{l}=m \\ 0, & at\ \ \mathbf{l}\neq m.\end{cases} \qquad (8)$$

The Woodworth basis strobing functions are orthogonal, as they do not overlap in frequency, and consequently, the *sin x/x* functions in (1) are orthogonal on the frequency axis. Therewith frequency samplings taken at the $\ell\Delta F$ points and time samplings taken at the $k\Delta T$ points are uncorrelated by virtue of the orthogonality of the expansion functions.

The consequence of this result is in the possibility of using (1) – (4) for expansion of both deterministic and random signals into a double series of orthogonal by basis functions in time and frequency coordinates.

2. By virtue of the fact that, at $k\to\infty$, a norm of basis expansion functions, according to [2], is

$$\left\|y_{k\mathbf{l}\sin x/x}\right\|^2 = \left\|\Phi_{k\mathbf{l}\Pi}\right\|^2 = \Delta T, \qquad (9)$$

an accuracy of signal approximation by these functions is identical. This allows profitable using the approximating functions and the order of obtaining samplings in frequency and time coordinates at the signal processing.

3. In practice, models of signals limited both in spectrum width and in time are more often used. Such models describe actual signals observed with a sufficient accuracy [2]. Then the complex envelope and spectrum of the signal with duration τ_s and spectrum width Δf_s can be presented in the frequency-time plane by expressions (1) – (4), at fixed $k=K$ and $\ell=L$, respectively. For such signals, the total number of bins (sampling points) is

$$N = K\cdot L = \frac{t_s}{\Delta T}\cdot\frac{\Delta f_s}{\Delta F} = t_s\cdot\Delta f_s. \qquad (10)$$

Therewith for each element, two elements, being an amplitude and a phase, should be defined. Consequently, the total number of samplings corresponds to the number of samplings at signal representation by he Kotelnikov series and is equal to $2t_s\cdot\Delta f_s$. But in signal representations (1) – (4), samplings are taken both in frequency and in time.

Assuming either ℓ=0 in (1) and (4) at the known signal spectrum width $\Delta f_s=\Delta F$ or k=0 in (2) and (3) at the known signal duration $t_s=\Delta T$, obtain, respectively, the known signal expansions into the Kotelnikov series by the *sin x/x* functions or by Woodworth's rectangle strobing functions.

These results testify the correspondence of an accuracy of signal representation as (1) – (4) to an accuracy of their representation by the Kotelnikov series in accordance with the minimum root-mean-square error criterion.

4. The important consequence of the sampling theorem in frequency-time domain is in that the obtained representation of signals explicitly contains the frequency-time function $V_{k\mathbf{l}}(t) = e^{j2p\Delta F\mathbf{l}(t-k\Delta T_1)}$. This allows, in the process of taking samplings, defining evolution and parameters of functions of frequency-time signal modulation. Correspondingly, when determining signals, defined can be not only their parameters, but also their forms.

5. The remarkable consequence of the sampling theorem in frequency-time domain consists in its constructive character. The theorem not only proves a way of the signal expansion in the frequency-time plane into the corresponding double series, but also defines a way for restoration (reproduction) of signals by their frequency and time samplings.

For confirmation of practical importance of the obtained results, the experimental investigations were carried out for signal determination- restoration on the testing stand whose flow chart is given in Fig.1 where 1 is the simulator of radio signals; 2, 9 are the direct dispersion Fourier processors; 3, 10 are the analog-to-digital converters of frequency-time samplings; 4, 11 are RAM blocks; 5, 12 are the bit-mapped indicators of TV type; 6 is the inverse dispersion Fourier processor; 7 is the digital-to-analog converter of frequency-time samplings; 8 is the industrial spectrum analyzer C4-27.

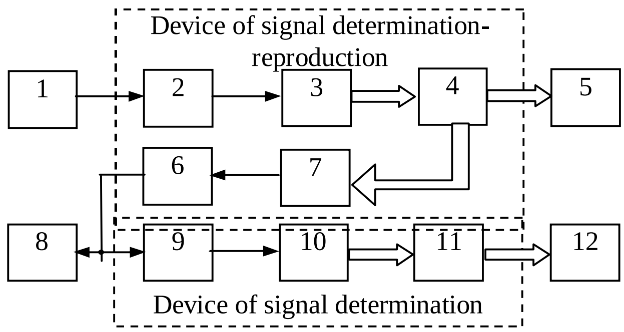


Fig. 1

The a priori domain of the spectrum width and duration of signals under determination and restoration constituted $\Pi \times T$ = 100 MHz×1000 μs with the central frequency f_0 = 750 MHz. The experimental investigations were carried out by definition and restoration continuous, simple pulse and LFM signals.

The simulated signals with the controllable parameters were carried to the input of the device for determination-reproduction. After the discretization, i.e. formation in block 2 of the biaxial signal expansion as (3), and the measurement-digitization of frequency-time samplings in block 3, they were recorded in memory of block 4 with the simultaneous multiple display on the frequency-time panorama of the bitmapped indicator 5 [3]. As memory was filled up, mode of block 4 was automatically changed into that of multiple reading of frequency-time samplings for their conversion in block 7 into the analog form as (3) and formation of the restored signals as (4). For controlling the results of processing, the reproduced signals were carried to the spectrum analyzer 8, and then to blocks 9, 10, 11 for redetermination in order to be displayed on the frequency-time panorama of the second bitmapped indicator 12.

The final results of experimental investigations on determination and restoration of two LFM signals ($\Delta f_s \times \Delta \tau_s$ = 10MHz×10μs and 30 MHz ×30μs) are visualized by photos presented in Figs. 2 and 3.

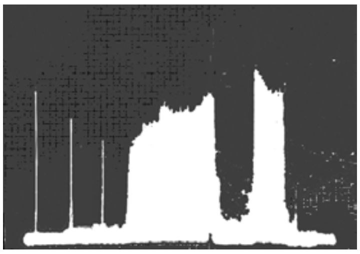
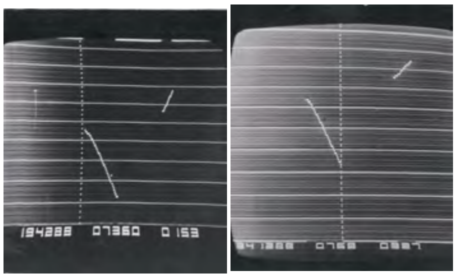


FIG. 2

Fig.2 shows spectra of reproduced signals taken from the screen of spectrum analyzer 8, whereas Fig.3 shows the frequency-time signal panoramas of the primary and secondary determinations after restoration [3], taken from screens of bitmapped indicators 1 and 12, respectively.

The interval between horizontal light-colored lines on the panoramas is 10 MHz. The frequency-time signal parameters within the panorama were measured by superimposing the dashdot adjusting mark with bright readings and displaying the results of measurements on the numeric display at the bottom of the screen.

The results of experimental investigations confirm that practical realization of the signal processing based on the generalized sampling theorem allows, when determining signals (Fig.3a), assessing a shape of the signal, the modulation function and its frequency-time parameters, whereas, when reproducing the signal, restoring them (Fig. 3b). Yet the presented results indicate the possibility to overcome the a priori uncertainty by the signal shape and parameters while processing them in the specified frequency-time domain $\Pi \times T$.



a) b)

FIG. 3

CONCLUSION

In the paper, the sampling theorem in frequency-time domain is formulated. As a result of its proof, the effective model of the frequency-time signal representation as a double series is obtained. The model unlike the known ones allows carrying out the biaxial expansion of an arbitrary signal by the system of orthogonal functions of the sin x/x type or by Woodworth's rectangle strobing functions with the frequency infill by harmonics of the Fourier series, singling out, in the explicit form, the function of frequency-time modulation, to represent it by $N = 2K \cdot L = 2t_s \cdot \Delta f_s$ samplings and implementing not only its determination, but also reproduction.

The practical application of the obtained results allow increasing the efficiency of the processing of a large ensemble of signals under conditions of their a priori parametric and structure uncertainty.

REFERENCES

1. Kalyuzhniy N. M. Generalized sampling theorem for realization of the effective signal processing under conditions of th a priori uncertainty. – "Applied Radioelectronics. State-of-the-art and outlook for the development", V.8, 2009, № 4, pp. 542–548. (in Rus.)

.2. Gonorovskiy I.S. Radiotechnical circuits and signals: remade and completed issue. – M.: Sovetskoe Radio, 1977. – 607 p.

3. N.M. Kalyuzhniy, M.V. Belash, S.V. Pshenichnyh, I.V. Puziy, V.M. Orlenko. Experience of time-frequency signal analysis pulse compression as an approach to design of "ANTI-LPI" radar. Applied Radio Electronics, 2005. – Vol. 4, No. 1, pp. 69-73.