ВІСНИК НАЦІОНАЛЬНОГО УНІВЕРСИТЕТУ
"ЛЬВІВСЬКА ПОЛІТЕХНІКА"
"Фізико-математичні науки"
Вип.718 № 718, (2011) с. 35–39

JOURNAL OF NATIONAL UNIVERSITY
"LVIVSKA POLITECHNIKA"
"Physical & mathematical sciences"
Vol.718 No 718, (2011) 35–39

ELLIPTIC BOUNDARY VALUE PROBLEMS IN WEIGHT SPACES OF GENERALIZED FUNCTIONS

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(Received 9 2011 .)

The uniqueness solvability of linear normal boundary value problems for elliptic at Petrovskyy systems of differential equations in weight spaces of generalized functions, which as a separate case include the spaces with strong power singularities on the whole boundary of the domain is established. The solutions' representation is obtained.

Key words: elliptic system of differential equations; normal boundary value problem; generalized function; weight functional spaces; Green function.

2000 MSC: 2000: 35K55

UDK: 517.95

Introduction

The linear elliptic boundary value problems in bounded domains were studied in Sobolev and Helder generalized functions' spaces [1–6], in full scales of Lizorkin-Tribel and Nikolskyy-Besov banah spaces [7, 8], in refined scales of banah spaces – Hermander-Volevich-Panejah spaces [9, 10] and bibl.

In [11, 12] the linear elliptic boundary value problems were studied under strong power singularities in right-hand sides, in the case of the strong power singularities by using the regularization of given distributions (so, with loss of the uniqueness). By Gorbachuk M.L., Gorbachuk V.I. ([13] and bibl.), Lions J.-L., Madjenes E. Saylor R., Gorodetskyy V.V. the study of the differential operators were carried in Shwarz spaces and the spaces of the analytic generalized functions.

We consider the actual problem of the extension of the existence and uniqueness theorems to solutions of boundary value problems for elliptic systems onto more general classes of right-hand sides, similar to [6] and [14] in scalar case.

I. Main definitions and lemmas

Let Ω be the domain in $\mathbb{R}^n$ $(n \geq 3)$ with the boundary S of class C^∞, $\nu = \nu(x,t)$– the unit vector of inner normal in the point x of the surface S,

$D(S) = C^\infty(S)$, $D(\overline{\Omega}) = C^\infty(\overline{\Omega})$.

We consider nicely elliptic at Petrovsky system of differential equations

$$A(x,D)u \equiv (A_{ij}(x,D))_{i,j=1,\dots,p}u = F, \qquad (1.1)$$

under boundary conditions

$$B(x,D)u \equiv (B_{hj}(x,D))_{h=1,\dots,m;j=1,\dots,p}u = P, \qquad (1.2)$$

where $A_{ij}(x,D) = \sum\limits_{|\alpha|\leq 2m} a_\alpha^{ij}(x)D^\alpha, \quad a_\alpha^{ij} \in D(\overline{\Omega})$,

$B_{hj}(x,D) = \sum\limits_{|\alpha|\leq m_h} b_\alpha^{hj}(x)D^\alpha, \quad b_\alpha^{hj} \in D(S)$, $h = \overline{1,m}$, $i,j = \overline{1,p}$, $\quad \alpha = (\alpha_1,\dots,\alpha_n), |\alpha| = \alpha_1 + \dots + \alpha_n$,

$$D_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}} = D_1^{\alpha_1}\cdot\dots\cdot D_n^{\alpha_n}.$$

The matrix B of the boundary differential expressions is called *normal*, and the problem (1.1)-(1.2) – the normal elliptic boundary value problem, if the matrix B may be complemented by new lines to Dirichlet matrix of $(2m,p)$-order [3].

Let the matrix B uniformly be cover A ([3]) and be normal.

If matrix $C(x,D)$ complements matrix $B(x,D)$ to Dirihlet matrix of $(2m,p)$-order then there exist [3] matrix $\hat{B}(x,D)$, $\hat{C}(x,D)$ such that Green formula

$$\int\limits_\Omega v^T(x)Au(x)dx + \sum_{h=1}^m \int\limits_S \hat{C}_h v(x)B_h u(x)dS =$$
$$= \int\limits_\Omega (A^*v(x))^T u(x)dx + \sum_{h=1}^m \int\limits_S \hat{B}_h v(x)C_h u(x)dS$$

holds for all vector-functions u,v with the elements from $D(\overline{\Omega})$. Here $A^* = A^*(x,D)$ – formally adjoint to A matrix differential expression (elliptic at Petrovsky), $B_h,\ \hat{B}_h,\ C_h,\ \hat{C}_h$ – the lines of the matrix $B,\ \hat{B},\ C,\ \hat{C}$ respectively with the coefficients from $D(S)$.

Let $\varrho(x)(x \in \overline{\Omega})$ be infinitely differentiable function, positive in Ω, having the order of the distance $d(x)$ from x to S near S ($\lim\limits_{d(x)\to 0+} \frac{\varrho(x)}{d(x)} = const$). We also regard $\varrho(x) \leq 1$, $x \in \overline{\Omega}$.

Let N be the set of natural numbers,

$[r]$ – the integer part of the number $r \geq 0$,

$C^r(\overline{\Omega})$ – the space of functions φ with continuous derivatives $D^\alpha\varphi$ for $|\alpha| \leq [r]$ and (for non-integer r) final $\sum\limits_{|\alpha|=[r]} \sup\limits_{x,y\in\overline{\Omega},x\neq y} \frac{\Delta_x^y D_x^\alpha \varphi(x)}{|x-y|^{r-[r]}}$, where $\Delta_x^y\psi(x) = \psi(y) - \psi(x)$.

We introduce next functional spaces:

$\mathcal{D}_r(\overline{\Omega}) = \{\varphi \in C^r(\overline{\Omega})$ $(D(\overline{\Omega})$ for integer $r)$: $\varrho^{|\alpha|-r}D^\alpha\varphi \in C(\overline{\Omega}), |\alpha| \leq [r]\}$,

$\mathcal{D}_r^*(\overline{\Omega}) = \{\varphi \in C^{r+2m}(\overline{\Omega})$ $(D(\overline{\Omega})$ for integer r): $A^*\varphi \in \mathcal{D}_r(\overline{\Omega})\}$,

$\mathcal{X}_r(\overline{\Omega}) = \{\varphi \in \mathcal{D}_r^*(\overline{\Omega}) : \hat{B}_j\varphi = 0,\ j = \overline{1,m}\}$.

We say that $\varphi_k \to 0$, $k \to \infty$ in $\mathcal{D}_r(\overline{\Omega})$ if

$\varrho^{|\alpha|-r}D^\alpha\varphi_k \to 0$, $k \to \infty$ for all $|\alpha| \leq [r]$ evenly in $\overline{\Omega}$ and (for non-integer r) $\sum\limits_{|\alpha|=[r]} \sup\limits_{x,y\in\overline{\Omega},x\neq y} \frac{\Delta_x^y D_x^\alpha\varphi_k(x)}{|x-y|^{r-[r]}} \to 0$, $k \to \infty$.

Let $S \subset \cup_{j=1}^M V_j$ with bounded V_j, $j = \overline{1,M}$. Then we get the covering of the part of Ω near S by $\cup_{j=1}^M U_j$ with bounded $U_j \subset \Omega$, called the boundary coordinate vicinity, conformable to coordinate vicinity V_j, $j = \overline{1,M}$.

Lemma 1. *For arbitrary coordinate vicinity $V_j \subset S$, vector-functions $\varphi_0, \varphi_1, \ldots, \varphi_{2m-1} \in C^\infty(V_j)$, number $r \in N \cup \{0\}$ there exist the vector-functions $\varphi_{2m}, \ldots, \varphi_{2m+r-1} \in C^\infty(V_j)$ and the infinitely differentiable and bounded in boundary coordinate vicinity U_j, conformable to V_j $(j = \overline{1,M})$, vector-function φ such that in local coordinate $(\xi_1, \ldots, \xi_n) = (\xi', \xi_n)$*

$$A\Big(\sum_{k=0}^{2m+r-1} \xi_n^k\varphi_k(\xi')\Big) = \xi_n^r\varphi(\xi', \xi_n).$$

Proof. Since

$$A_{ij}(x, D) = \tilde{Q}_0^{ij}(\xi)D_n^{2m} + \sum_{t=1}^{2m} \tilde{Q}_t^{ij}(\xi, \tfrac{\partial}{\partial\xi'})D_n^{2m-t},$$

where $\tilde{Q}_0^{ij}(\xi) \neq 0$, $\tilde{Q}_t^{ij}(\xi, \frac{\partial}{\partial\xi'})$ – tangent differential operators of the order t, $i, j = \overline{1,p}$, then

$$A(x, D)\Big(\sum_{k=0}^{2m+r-1} \xi_n^k\varphi_k(\xi')\Big) =$$
$$= \sum_{k\geq 0} \xi_n^k \Big[\tilde{Q}_0(\xi)(k + 2m)\ldots(k + 1)\varphi_{k+2m}(\xi') +$$
$$+ \sum_{t=1}^{2m} \tilde{Q}_t(\xi, \tfrac{\partial}{\partial\xi'})(k + 2m - t)\ldots(k + 1)\varphi_{k+2m-t}(\xi')\Big]$$

with non-degenerate matrix $\tilde{Q}_0(\xi)$ and $\tilde{Q}_t(\xi, \frac{\partial}{\partial\xi'})$ – tangent differential operators of the order t, $t = \overline{1,2m}$. Writing $\tilde{Q}_0(\xi)$ and $\tilde{Q}_t(\xi, \frac{\partial}{\partial\xi'})$ in a form of matrix decays on powers ξ_n and equating to zero the coefficients at $\xi_n^0, \xi_n^1, \ldots, \xi_n^{r-1}$, we find $\varphi_{2m}(\xi'), \ldots, \varphi_{2m+r-1}(\xi')$, the another factors give $\xi_n^r\varphi(\xi', \xi_n)$.

Remark that φ_{2m+j} linear depend on φ_k and its derivatives of the orders $2m + j - k, k = \overline{j, 2m - 1}$, the function $\varphi(\xi', \xi_n)$ linear depend on φ_j and its derivatives of the orders $2m - j + r - 1$, $j = \overline{0, 2m - 1}$.

Lemms 2. *Let $\tilde{B}(x, D) = (\tilde{B}_0^T, \ldots, \tilde{B}_{2m-1}^T)^T$ be Dirihlet matrix of $(2m, p)$-order. For arbitrary vector-functions $\varphi_l \in D(S)$, $l = \overline{0, 2m - 1}$ and the number $r \in N \cup \{0\}$ there exists the vector-function $\psi \in D_r(\overline{\Omega})$ such that*

$\tilde{B}_l(x, D)\psi \mid_S = \varphi_l(x)$, $x \in S$, $l = \overline{0, 2m - 1}$.

Proof. It is known [?] that

$$\tilde{D}_n^l = E_p(\tfrac{\partial}{\partial\nu})^l = \sum_{k=0}^{l} T_{lk}(x, D)\tilde{B}_k(x, D),\ x \in S,$$

$l = \overline{0, 2m - 1}$, where E_p – unit $p \times p$ matrix, $T_{lk}(x, D)$ – tangent $p \times p$-matrix differential operators of the order $\leq l - k$ with non-degenerating $T_{ll} = T_{ll}(x)$. Hence it is sufficiently to prove the existence of the vector-function $\psi \in D(\overline{\Omega})$ such that $\varrho^{-r}A^*(\cdot, D)\psi \in C(\overline{\Omega})$ and

$$\tilde{D}_n^l\psi \mid_S = \tilde{\varphi}_l(x) = \sum_{k=0}^{l} T_{lk}(x, D)\varphi_k(x), \quad x \in S,$$

$l = \overline{0, 2m - 1}$.

We apply the operators $\tilde{D}_n^l$ to vector-function Φ, which in arbitrary boundary coordinate vicinity U in conformable local coordinate system ξ has the form

$$\Phi = \sum_{k=0}^{2m+r-1} \xi_n^k\tilde{\tilde{\varphi}}_k(\xi'),$$

where $\tilde{\tilde{\varphi}}_k(\xi')$ – for the present unknown infinitely differentiable vector-functions in $V = U \cap S$. We have

$\tilde{D}_n^l\Phi\mid_V = (\frac{\partial}{\partial\xi_n})^l\Phi\mid_{\xi_n=0} =$
$= \sum\limits_{k=0}^{2m+r-1} k(k - 1)\ldots(k - l + 1)\xi_n^{k-l}\tilde{\tilde{\varphi}}_k(\xi')\mid_{\xi_n=0} =$
$= (l)!\tilde{\tilde{\varphi}}_l(\xi')$. Hence, $\tilde{\tilde{\varphi}}_l(\xi') = \frac{\tilde{\varphi}_l(\xi')}{(l)!}$, $l = \overline{0, 2m - 1}$.

Choosing the vector-functions $\tilde{\varphi}_{2m}, \ldots, \tilde{\varphi}_{2m+r-1}$ by lemma 1 (using that $A^*(x, D)$ is also elliptic at Petrovskyy matrix differential expression), we get

$$A^*(x, D)\Phi(x) = \xi_n^r\varphi(\xi', \xi_n).$$

Using resolution of identity conformable to covering of the surface S by the system $\cup_{j=1}^M V_j$ we find the vector-function ψ, which in each boundary coordinate vicinity U_l is equal to Φ_l, $l = \overline{1,M}$.

II. The existence and uniqueness theorem.

Let $V'(\overline{\Omega})$ be the space of linear continuous functionals onto $V(\overline{\Omega})$ (generalized vector-functions from $D'(\mathbb{R}^n)$ with supports in $\overline{\Omega}$ [15], p. 27),

$(\varphi, F) = \sum\limits_{j=1}^{p}(\varphi_j, F_j)$ – the value of the generalized vector-function $F = (F_1, \ldots, F_p) \in V'(\overline{\Omega})$ on the basic vector-function $\varphi = (\varphi_1, \ldots, \varphi_p) \in V(\overline{\Omega})$,

$< \varphi, F >= \sum\limits_{j=1}^{p} < \varphi_j, F_j >$ – the value of the generalized vector-function $F \in V'(S)$ on the basic vector-function $\varphi \in V(S)$.

For the matrix Φ with the elements $\Phi_{ij} \in V(\overline{\Omega})$ we have

$$(\Phi, F) = \Big(\sum_{j=1}^{p}(\Phi_{1j}, F_j), \ldots, \sum_{j=1}^{p}(\Phi_{pj}, F_j)\Big).$$

We denote by $s(F)$ the order of the singularities of generalized function F [16, p. 46]shylov,

$s(F) = \max\limits_{1\leq j\leq p} s(F_j)$ for $F = (F_1, \ldots, F_p)$,

also $|g|_p = |g_1| + \cdots + |g_p|$ for $g = (g_1, \ldots, g_p)$.

Supposition (F):
$P=(P_1,\dots,P_m)\in D'(S)$, $s(P_j)\le s_j$, $j=\overline{1,m}$,
$r\ge r_0=\max\limits_{1\le j\le m}(s_j-m_j)-1$, $\quad F\in\mathcal{X}'_r(\overline{\Omega})$.

Definition. Under supposition (F) the vector-function $u\in\mathcal{D}'_r(\overline{\Omega})$ is called the solution of the problem (1.1),(1.2) if the equality

$$(A^*\psi,u)=(\psi,F)+<\hat{C}\psi,P> \quad \forall\psi\in\mathcal{X}_r(\overline{\Omega}) \tag{2.3}$$

holds.

Let $\mathcal{N}$ be the kernel of the problem (1.1),(1.2),

$\mathcal{N}^*$ – the kernel of the adjoint problem,

$\{\tilde{u}_k\}_{k=1}^q$ – full system of linear independent vector-functions $\tilde{u}_k\in D(\overline{\Omega})$ from $\mathcal{N}$ such that

$$\int\limits_\Omega \tilde{u}'_k\tilde{u}_j dx=\delta_{kj},\ k,j=\overline{1,q},$$

where δ_{kj}–Kroneker symbols,

$\{\psi_k\}_{k=1}^l$ – full system of linear undependent vector-functions $\psi_k\in D(\overline{\Omega})$ from $\mathcal{N}^*$ such that

$$\overline{\psi}_k\overline{\psi}_j=\int\limits_\Omega\psi_k\psi_j dx+\int\limits_S(\hat{C}\psi_k)'(\hat{C}\psi_j)dS=\delta_{kj}$$

$$(\overline{\psi}_k=(\psi_{k1},\dots,\psi_{kp},\hat{C}_1\psi_k,\dots,\hat{C}_p\psi_k)) \quad [3\text{–}5].$$

We get from (2.3) the necessary condition of the problems' solvability

$$(\psi_k,F)+<\hat{C}\psi_k,P>=0, \quad k=\overline{1,l}, \tag{2.4}$$

which may be written as

$$(\Psi,F)+(\hat{C}\Psi,P)=0,$$

where Ψ – the matrix of the vector-functions $\psi_1,\dots,\psi_l$.

Let $G^0(x,y)$ $(x,y\in\overline{\Omega})$, $G(x,y)$ $(x\in\overline{\Omega},y\in S)$ be such matrix that for all vector-functions $f\in D(\overline{\Omega})$, $g\in D(S)$ the problem's solution

$$Lu=\overline{f}-\overline{P}\,\overline{f}, \qquad Pu=0,$$

where $L=(A,B)$, $\quad\overline{f}=(f,g)$, $\quad\overline{P}\,\overline{f}=\sum\limits_{k=1}^{l}(\overline{f}\,\overline{\psi}_k)\overline{\psi}_k$, $Pu=\sum\limits_{k=1}^{r}\int\limits_\Omega\tilde{u}_k^T u dy\cdot\tilde{u}_k$, has next impression

$$u(x)=\int\limits_\Omega G^0(x,y)f(y)dy+\int\limits_S G(x,y)g(y)dS, \quad x\in\Omega. \tag{2.5}$$

Matrix (G^0,G) is called the Green matrix of the problem (1.1)-(1.2). It follows from [4], part IX,X and [17], §13 its existence. It follows from the definition of Green matrix that

$$A(x,D)G^0(x,y)=\delta(x-y)E_p-\Psi(x)\Psi(y), \quad x,y\in\overline{\Omega},$$

$$A(x,D)G(x,y)=-\Psi(x)(\hat{C}(y,D)\Psi(y))^T, \quad x\in\overline{\Omega},y\in S,$$

$$B(x,D)G^0(x,y)=-\hat{C}(x,D)\Psi(x)\Psi(y), \quad x\in S,y\in\overline{\Omega}, \tag{2.6}$$

$$B(x,D)G(x,y)=\hat{\delta}(x-y)E_m-$$
$$\hat{C}(x,D)\Psi(x)(\hat{C}(y,D)\Psi(y))^T, \quad x,y\in S,$$

$$\int\limits_\Omega\tilde{U}^T(x)G^0(x,y)dx=0, y\in\overline{\Omega},$$

$$\int\limits_\Omega\tilde{U}^T(x)G(x,y)dx=0, y\in S,$$

where $(\varphi(x),\delta(x-y))=\varphi(y),\ \varphi\in D(\overline{\Omega})$,

$$<\varphi(x),\hat{\delta}(x-y)>=\varphi(y),\ \varphi\in D(S),$$

$\tilde{U}$ – the matrix of the vector-functions $\tilde{u}_1,\dots,\tilde{u}_q$.

Theorem 1. *Under supposition (F) and the conditions* (2.4) *the vector-function* $u\in\mathcal{D}'_r(\overline{\Omega})$*, defined by the formula*

$$(\varphi,u)=(\int\limits_\Omega G^{0T}(x,\cdot)\varphi(x)dx,F)+ \tag{2.7}$$

$$+<\int\limits_\Omega G^T(x,\cdot)\varphi(x)dx,P> \quad \forall\varphi\in\mathcal{D}_r(\overline{\Omega}),$$

is the solution of the problem (1.1)-(1.2), *unique in* $\mathcal{D}'_r(\overline{\Omega})/\mathcal{N}$.

Proof. We use the formulas from [18], p.100 (and [19] for $r=0$):

$$\int\limits_\Omega G^{0T}(x,y)A^*\psi(x)dx=\psi(y)-$$

$$-\Psi(y)\Big[\int\limits_\Omega\Psi(x)\psi(x)dx+\int\limits_S(\hat{C}\Psi(x))^T\hat{C}\psi(x)dS\Big], y\in\overline{\Omega},$$

$$\int\limits_\Omega G^T(x,y)A^*\psi(x)^T dx=\hat{C}\psi(y)- \tag{2.8}$$

$$-\hat{C}\Psi(y)\Big[\int\limits_\Omega\Psi(x)\psi(x)dx+\int\limits_S(\hat{C}\Psi(x))^T\hat{C}\psi(x)dS\Big],$$

$$y\in S,\ \forall\psi\in\mathcal{X}_r(\overline{\Omega}).$$

Since for all $\varphi_0\in\mathcal{D}_r(\overline{\Omega})$, satisfying the condition

$$\int\limits_\Omega\varphi_0^T(x)\tilde{u}_k(x)dx=0, \quad k=\overline{1,r}, \tag{2.9}$$

there exists $\psi\in\mathcal{X}_r(\overline{\Omega})$ such that $A^*\psi=\varphi_0$, it follows from the formulas above that for such φ_0

$$\int\limits_\Omega G^{0T}(x,\cdot)\varphi_0(x)dx=\psi-$$

$$-\Psi\Big[\int\limits_\Omega\Psi(x)\psi(x)dx+\int\limits_S(\hat{C}\Psi(x))^T\hat{C}\psi(x)dS\Big]\in\mathcal{X}_r(\overline{\Omega}),$$

$$\int\limits_\Omega G^T(x,\cdot)\varphi_0(x)dx=\hat{C}\psi-$$

$$-\hat{C}\Psi\Big[\int\limits_\Omega\Psi(x)\psi(x)dx+\int\limits_S(\hat{C}\Psi(x))^T\hat{C}\psi(x)dS\Big] \text{ on } S.$$

For arbitrary $\varphi\in\mathcal{D}_r(\overline{\Omega})$ the function

$$\varphi-\sum_{k=1}^{r}\int\limits_\Omega\varphi^T(y)\tilde{u}_k(y)dy\cdot\tilde{u}_k$$

satisfies the condition (2.9):

$\int\limits_\Omega\varphi_0^T\tilde{u}_k dx=\int\limits_\Omega[\varphi^T(x)-\sum\limits_{i=1}^{q}\int\limits_\Omega\varphi^T\tilde{u}_i dy\cdot\tilde{u}_i^T(x)]\tilde{u}_k(x)dx=\int\limits_\Omega\varphi^T\tilde{u}_k dx-\sum\limits_{i=1}^{q}\int\limits_\Omega\varphi^T\tilde{u}_i dy\int\limits_\Omega\tilde{u}_i^T\tilde{u}_k dx=\int\limits_\Omega\varphi^T\tilde{u}_k dx-\int\limits_\Omega\varphi^T\tilde{u}_k dx=0,\ k=\overline{1,q}.$ Also

$$= \int_\Omega G^{0T}(x,y)\varphi(x)dx -$$

$$- \int_\Omega \varphi^T(y) \sum_{k=1}^{q} \tilde{u}_k(y)dy \cdot \int_\Omega G^{0T}(x,y)\tilde{u}_k(x)dx =$$

$$= \int_\Omega G^{0T}(x,y)\varphi(y)dx, \qquad y \in \overline{\Omega}$$

$(\int_\Omega \varphi^T(y) \sum_{k=1}^{q} \tilde{u}_k(y)(\int_\Omega G^{0T}(x,y)\tilde{u}_k(x)dx)dy = 0$ according to (2.6)). Therefore

$$\int_\Omega G^{0T}(x,\cdot)\varphi(x)dx \in \mathcal{X}_r(\overline{\Omega}) \quad \forall \varphi \in \mathcal{D}_r(\overline{\Omega}).$$

Similar, for arbitrary $\varphi = (\varphi_1, \ldots, \varphi_p)^T \in \mathcal{D}_r(\overline{\Omega})$ we have

$$\int_\Omega G^T(x,\cdot)[\varphi(x) - \sum_{k=1}^{r} \int_\Omega \varphi^T(y)\tilde{u}_k(y)dy \cdot \tilde{u}_k(x)]dx =$$
$$= \int_\Omega G^T(x,\cdot)\varphi_0(x)dx \in D(S) \text{ if } r \in N \cup \{0\},$$

$\int_\Omega \sum_{s=1}^{p} G_{sj}(x,\cdot)\varphi_s(x)dx \in C^{[r]+m_j+1}(S)$ if $r > 0$ – non-integer and G_{sj} $(s,j = \overline{1,p})$ – the elements of G.

Since the generalized function P_j with $s(P_j) \le s_j$ belongs to $(C^{s_j}(S))'$, $j = \overline{1,m}$, under supposition (F) the formula (2.7) uniquely defines $u \in \mathcal{D}'_r(\overline{\Omega})$.

Now we shall show that the vector-function (2.7) satisfies the identity (2.3), that is

$$(\int_\Omega G^{0T}(x,\cdot)A^*\psi(x)dx, F) + < \int_\Omega G^T(x,\cdot)A^*\psi(x)dx, P >=$$
$$= (\psi, F) + < \hat{C}\psi, P > \quad \forall \psi \in \mathcal{X}_r(\overline{\Omega}).$$

Using the formulas (2.8), we write the left part of this equality as next one

$$(\psi(y) - \Psi(y)\Big[\int_\Omega \Psi(x)\psi(x)dx +$$

$$+ \int_S (\hat{C}\Psi(x))^T \hat{C}\psi(x)dS\Big], F(y)) +$$

$$+ < \hat{C}\psi(y) - \hat{C}\Psi(y)\Big[\int_\Omega \Psi(x)\psi(x)dx +$$

$$+ \int_S (\hat{C}\Psi(x))^T \hat{C}\psi(x)dS\Big], P(y) >=$$

$$= (\psi, F) + < \hat{C}\psi, P > -$$

$$- \Big[\int_\Omega \Psi(x)\psi(x)dx + \int_S (\hat{C}\Psi(x))^T \hat{C}\psi(x)dS\Big] \times$$

$$\times \Big[(\Psi, F) + < \hat{C}\Psi, P > \Big] = (\psi, F) + < \hat{C}\psi, P >.$$

We get the last equality from the condition (2.4).

Let u_1, u_2 be two problem's solutions, $v = u_1 - u_2$. It follows from (2.3) that

$$(A^*\psi, v) = 0 \quad \forall \psi \in \mathcal{X}_r(\overline{\Omega}). \qquad (2.10)$$

Then for arbitrary $\varphi_0 \in \mathcal{D}_r(\overline{\Omega})$, satisfying (2.9) (orthogonal to $\mathcal{N}$), and such that $A^*\psi = \varphi_0$, we get $(\varphi_0, v) = 0$. Since for arbitrary $\varphi \in \mathcal{D}_r(\overline{\Omega})$ the vector-function $\int_\Omega G^{0T}(x,\cdot)\varphi(x)dx \in \mathcal{X}_r(\overline{\Omega})$, then from (2.10) we get

$$(\varphi, v) = 0 \quad \forall \varphi \in \mathcal{D}_r(\overline{\Omega}).$$

It follows from Green formula that

$(A^*\psi, u^0) = 0$ for all $u^0 \in \mathcal{N}$ and $\psi \in \mathcal{X}_r(\overline{\Omega})$.

Then $(\varphi, u^0) = 0$ for all $\varphi \in \mathcal{D}_r(\overline{\Omega})$. There for $v = u^0$.

Remark, that the formula (2.7) generalizes the according formulas from [4] and also from [19] to case of right-hand sides from wider spaces of generalized functions.

Resume

The solvability and uniqueness theorem of the solution to normal boundary value problem for elliptic at Petrovskyy system of differential equations in new spaces of generalized functions and the character of strong power singularities of the solutions are established.

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ЭЛЛИПТИЧЕСКИЕ ГРАНИЧНЫЕ ЗАДАЧИ В ВЕСОВЫХ ПРОСТРАНСТВАХ ОБОБЩЕННЫХ ФУНКЦИЙ

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Установлено существование и единственность решений линейных нормальных граничных задач для эллиптических по Петровскому систем дифференциальных уравнений в весовых пространствах обобщенных функций, содержащих функции с сильными степенными особенностями на всей границе области или в ее отдельных точках. Получено представление решений.

Ключевые слова: эллиптическая система дифференцыальных уравнений, нормальная граничная задача, обобщенная функция, весовые функциональные пространства, функция Грина.

2000 MSC: 35K55

УДК: 517.95

ЕЛІПТИЧНІ КРАЙОВІ ЗАДАЧІ В ВАГОВИХ ПРОСТОРАХ УЗАГАЛЬНЕНИХ ФУНКЦІЙ

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Встановлена однозначна розв'язність лінійних нормальних крайових задач для еліптичних за Петровським систем диференціальних рівнянь у вагових просторах узагальнених функцій, які як окремий випадок містять функції з сильними степеневими особливостями на всій межі області. Одержано зображення розв'язків.

Ключові слова: еліптична система диференціальних рівнянь, нормальна крайова задача, узагальнена функція, ваговий функційний простір, функція Гріна.

2000 MSC: 2000: 35K55

UDK: 517.95