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COMPUTATION OF INITIAL CONDITIONS AND INPUTS FOR GIVEN OUTPUTS OF STANDARD AND POSITIVE DISCRETE-TIME LINEAR SYSTEMS

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Abstract. The problem of computation of initial conditions and inputs for given outputs of standard and positive discrete-time linear systems has been formulated and solved. Necessary and sufficient conditions for existence of solution to the problem have been established. It has been shown that there exist the unique solutions to the problem only if the pair (*A*,*C*) of the system is observable.

Keywords: computation, initial condition, observability, positive, discrete-time, linear, system

1. Introduction

Inputs, state variables, and outputs in positive systems take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc. An overview of state of the art in positive linear theory is given in the monographs [2, 5].

The notions of controllability and observability and the decomposition of linear systems have been introduced by Kalman [8, 9]. Those notions are the basic concepts of the modern control theory [1, 7, 10, 11, 4]. They were also extended to positive linear systems [2, 5].

The decomposition of the pair (*A*,*C*) and (*A*,*B*) of the positive discrete-time linear systems was addressed in [3].

In this paper the problem of computation of initial conditions and inputs for given outputs of standard and positive discrete-time linear systems will be formulated and solved. Necessary and sufficient conditions for existence of solutions to the problem will be established.

The paper is organized as follows. In section 2 the problem is formulated. The main results of the paper are given in section 3, where the necessary and sufficient conditions for existence of solutions to the problem for standard and positive systems are established. Concluding remarks are given in section 4.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ - the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, I_n - the $n\times n$ identity matrix.

2. Preliminaries

Consider the linear discrete-time systems

$$x_{i+1} = Ax_i + Bu_i, \quad i \in Z_+ = \{0,1,...\}, \qquad (2.1a)$$

$$y_i = Cx_i + Du_i, \qquad (2.1b)$$

where $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$. Without decrease of generality it is assumed that

$$\text{rank } B = m \text{ and rank } C = p. \qquad (2.2)$$

Definition 2.1. [2, 5] The system (2.1) is called (internally) positive if and only if $x_i \in \Re_+^n$, and $y_i \in \Re_+^p$, $i \in Z_+$ for every $x_0 \in \Re_+^n$, , and any input sequence $u_i \in \Re_+^m$, $i \in Z_+$.

Theorem 2.1. [2, 3] The system (2.1) is (internally) positive if and only if

$$A \in \Re_+^{n\times n}, \ B \in \Re_+^{n\times m}, \ C \in \Re_+^{p\times n}, \ D \in \Re_+^{p\times m}. \qquad (2.3)$$

The problem under considerations can be stated as follows.

Given the sequence of inputs $y_0, y_1, ..., y_n$ compute the initial condition x_0 and input sequence $u_0, u_1, ..., u_n$ for the standard and positive system (2.1).

The problem can be considered as a generalization of the observability problem of standard and positive discrete-time linear systems [1, 4, 7, 11].

The following two cases will be considered separately for standard and positive systems:

Case 1. The matrix *D=0*.

Case 2. The matrix *D≠0*.

3. Problem solution

3.1. Standard systems

Substituting the solution of the equation (2.1a)

$$x_i = A^i x_0 + \sum_{k=0}^{i-1} A^{i-k-1} Bu_k, \quad i \in Z_+ \qquad (3.1)$$

into (2.1b) we obtain

$$y_i = CA^i x_0 + \sum_{k=0}^{i-1} CA^{i-k-1} Bu_k + Du_i, \quad i \in Z_+ . \quad (3.2)$$

If $D = 0$ then using (3.2) for $i = 0,1,...,n-1$ we obtain

$$Hz = y \quad (3.3a)$$

where

$$H = \begin{bmatrix} C & 0 & 0 & ... & 0 \\ CA & CB & 0 & ... & 0 \\ CA^2 & CAB & CB & ... & 0 \\ \vdots & \vdots & \vdots & ... & \vdots \\ CA^{n-1} & CA^{n-2}B & CA^{n-3}B & ... & CB \end{bmatrix} \in \Re^{pn\times(n+(n-1)m)},$$

$$z = \begin{bmatrix} x_0 \\ u_0 \\ u_1 \\ \vdots \\ u_{n-2} \end{bmatrix} \in \Re^{n+(n-1)m}, \quad y = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} \in \Re^{pn}. \quad (3.3b)$$

If $D \neq 0$ then using (3.2) for $i = 0,1,...,n-1$ we obtain

$$\overline{H}\overline{z} = y \quad (3.4a)$$

where

$$\overline{H} = \begin{bmatrix} C & D & 0 & ... & 0 & 0 \\ CA & CB & D & ... & 0 & 0 \\ CA^2 & CAB & CB & ... & 0 & 0 \\ \vdots & \vdots & \vdots & ... & \vdots & \vdots \\ CA^{n-1} & CA^{n-2}B & CA^{n-3}B & ... & CB & D \end{bmatrix} \in \Re^{pn\times(m+1)n},$$

$$\overline{z} = \begin{bmatrix} x_0 \\ u_0 \\ u_1 \\ \vdots \\ u_{n-2} \end{bmatrix} \in \Re^{(m+1)n}. \quad (3.4b)$$

In the proof of the main result of this paper the following well-known Kronecker-Cappely Theorem will be used [6].

Theorem 3.1. The equation (3.3a) has a solution *z* for given *H* and *y* if and only if

$$\text{rank}\,[H \quad y] = \text{rank}\, H . \quad (3.5)$$

Case 1. $D = 0$.

Theorem 3.2. Let $D = 0$ and $n + (n-1)m \geq np$. Then the equation (3.3a) has a solution $x_0, u_0, u_1, ..., u_{n-2}$ for any given sequence $y_0, y_1, ..., y_{n-1}$ if and only if the matrix *H* has full row rank, i.e.

$$\text{rank}\, H = pn . \quad (3.6)$$

Moreover, the equation has the unique solution

$$z = H^{-1} y \quad (3.7)$$

if $n + (n-1)m = np$ and many solutions if $n + (n-1)m > np$.

Proof. If (3.6) holds then the condition (3.5) is satisfied for any vector *y*. If additionally $n + (n-1)m = np$ the matrix *H* is square and invertible. In this case the unique solution of (3.3a) is given by (3.7). If $n + (n-1)m > np$ the equation (3.3a) has many solutions. □

Theorem 3.3. Let $D = 0$ and $np > n + (n-1)m$. Then the equation (3.3a) has a solution $x_0, u_0, u_1, ..., u_{n-2}$ for a given sequence $y_0, y_1, ..., y_{n-1}$ if and only if the condition (3.5) is met. Moreover, the equation has the unique solution if

$$\text{rank}\, H = n + (n-1)m \quad (3.8)$$

and it has many solutions if

$$\text{rank}\, H < n + (n-1)m \quad (3.9)$$

Proof. By Theorem 3.1 the equation (3.3a) has a solution $x_0, u_0, u_1, ..., u_{n-2}$ for a given output sequence $y_0, y_1, ..., y_{n-1}$ if and only if the condition (3.5) is satisfied. The solution is unique if (3.8) is met since the matrix *H* has full column rank. Presume that the equation (3.3a) has two different solutions z_1 and z_2 satisfying $Hz_1 = y$ and $Hz_2 = y$. Then $H(z_1 - z_2) = 0$ and $z_1 - z_2 = 0$ since *H* has full column rank. If (3.5) and (3.9) hold then the equation (3.3a) has many solutions.

Remark 3.1. Note that the first matrix column

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \quad (3.10)$$

of *H* is the observability matrix of the system (2.1). The matrix (3.10) has full column rank if and only if the pair (A, C) is observable. Therefore, the equation (3.3a) has unique solution only if the pair (A, C) is observable.

Example 3.1. Consider the system (2.1) with the matrices

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = [1 \quad 0], \quad D = [0] . (3.11)$$

Compute the initial condition $x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$ and the input u_0 of the system for the given output sequence y_0, y_1 .

In this case we have $n = 2$, $m = p = 1$, $n + (n-1)m = 3 > np = 2$ and the matrix

$$H = \begin{bmatrix} C & 0 \\ CA & CB \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

has the full row rank.

The equation (3.3a) has the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_{10} \\ x_{20} \\ u_0 \end{bmatrix}=\begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \tag{3.12}$$

and it has many solutions for any sequence y_0, y_1. From (3.12) we have

$$\begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}=\begin{bmatrix} y_0 \\ y_1 - u_0 \end{bmatrix} \text{ for arbitrary } u_0$$

or

$$\begin{bmatrix} x_{10} \\ u_0 \end{bmatrix}=\begin{bmatrix} y_0 \\ y_1 - x_{20} \end{bmatrix} \text{ for arbitrary } x_{20}.$$

Example 3.2. Consider the system (2.1) with the matrices

$$A=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},\; B=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\; C=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\; D=\begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.13}$$

In this case we have $n=3$, $m=1$, $p=2$, $np=6>n+(n-1)m=5$ and the matrix

$$H=\begin{bmatrix} C & 0 & 0 \\ CA & CB & 0 \\ CA^2 & CAB & CB \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \tag{3.14}$$

has full column rank. Note that the second and the fifth rows of (3.14) are identical. Therefore, by Theorem 3.3 the equation (3.3a) for (3.14) has a unique solution if and only if $y_{02}=y_{21}$, where y_{ik} is the k-th component of the vector y_i, $i=0,1,2;\; k=1,2$.

Omitting the second row of (3.14) we obtain the equation

$$\hat{H}\begin{bmatrix} x_0 \\ u_0 \\ u_1 \end{bmatrix}=\begin{bmatrix} y_{01} \\ y_1 \\ y_2 \end{bmatrix} \tag{3.15}$$

where

$$\hat{H}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \tag{3.16}$$

is nonsingular, $\det\hat{H}=1$. The equation (3.15) and also (3.14) has the unique solution

$$\begin{bmatrix} x_0 \\ u_0 \\ u_1 \end{bmatrix}=\hat{H}^{-1}\begin{bmatrix} y_{01} \\ y_1 \\ y_2 \end{bmatrix}=\begin{bmatrix} y_{01} \\ y_{11} \\ y_{21} \\ y_{11}-y_{01} \\ y_{22}-y_{11} \end{bmatrix} \tag{3.17}$$

Case 2. $D \neq 0$.

Theorem 3.4. Let $D \neq 0$ and $m+1 \geq p$. Then the equation (3.4a) has a solution $x_0, u_0, u_1, ..., u_{n-1}$ for any given output sequence $y_0, y_1, ..., y_{n-1}$ if and only if the matrix $\overline{H}$ has full row rank, i.e.

$$\operatorname{rank}\overline{H}=pn. \tag{3.18}$$

Moreover, the equation has the unique solution

$$\bar{z}=\overline{H}^{-1}y \tag{3.19}$$

if $m+1=p$ and many solutions if $m+1>p$.

Proof is similar to the proof of Theorem 3.2.

Theorem 3.5. Let $D=0$ and $p>m+1$. Then the equation (3.4a) has a solution $x_0, u_0, u_1, ..., u_{n-1}$ for a given output sequence $y_0, y_1, ..., y_{n-1}$ if and only if the condition

$$\operatorname{rank} H=n+(n-1)m \tag{3.20}$$

is met. Moreover, the equation has the unique solution if

$$\operatorname{rank} H<n+(n-1)m \tag{3.21}$$

Proof is similar to the proof of Theorem 3.3.

Remark 3.2. The equation (3.4a) has unique solution only if the pair (A, C) is observable.

Example 3.3. Consider the system (2.1) with the matrices *A*, *B*, *C* given by (3.13) and $D=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

In this case the matrix

$$\overline{H}=\begin{bmatrix} C & 0 & 0 \\ CA & CB & 0 \\ CA^2 & CAB & CB \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \tag{3.22}$$

is nonsingular. Using (3.19) we obtain

$$\bar{z}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}^{-1} \times$$

$$\times\begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}=\begin{bmatrix} y_{01} \\ y_{11} \\ y_{21} \\ y_{02}-y_{21} \\ y_{12}+y_{21}-y_{01}-y_{02} \\ y_{01}+y_{02}+y_{22}-y_{11}-y_{12}-y_{21} \end{bmatrix} \tag{3.23}$$

3.2. Positive systems

Case 1. $D=0$.

From Definition 2.1 and Theorem 2.1 that for positive systems

$$H \in \Re_+^{pn\times(n+(n-1)m)},\ \ \overline{H} \in \Re_+^{pn\times(m+1)n},\ \ y \in \Re_+^{pn},\ \ z \in \Re_+^{n+(n-1)m},\ \ \overline{z} \in \Re_+^{(m+1)n}. \quad (3.24)$$

it follows.

Definition 3.1. [5] A square matrix *A* (a vector) is called the monomial matrix (vector) if its every row and its every column contains only one positive entry (one positive component) and the remaining entries (components) are zero.

Lemma 3.1. [5] The inverse matrix A^{-1} of a matrix $A \in \Re_+^{n\times n}$ is the positive matrix $A^{-1} \in \Re_+^{n\times n}$ if and only if *A* is monomial matrix.

Theorem 3.6. Let $D = 0$ and $n + (n-1)m \geq np$. Then the equation (3.3a) has a solution $x_0 \in \Re_+^n$, $u_i \in \Re_+^m$, $i = 0,1,...,n-2$ for any given output sequence $y_i \in \Re_+^p$, $i = 0,1,...,n-1$ if and only if the matrix $H \in \Re_+^{pn\times(n+(n-1)m)}$ contains *np* linearly independent monomial columns.

Moreover, the equation has the unique solution

$$z = H_m^{-1} y \quad (3.25)$$

if the matrix *H* contains only one monomial matrix H_m and many solutions if it contains many such monomial matrices.

Proof. The equation (3.3a) has a solution for any given sequence $y_0, y_1, ..., y_{n-1}$ if and only if the condition (3.6) is met. By Lemma 3.1 the solution is nonnegative $x_0 \in \Re_+^n$, $u_i \in \Re_+^m$, $i = 0,1,...,n-2$ for a nonnegative sequence $y_i \in \Re_+^p$, $i = 0,1,...,n-1$ if and only if the matrix *H* contains at least one monomial matrix $H_m \in \Re_+^{pn\times pn}$. The solution (3.25) is unique if the matrix *H* contains only one monomial matrix and many solutions if it contains more then on such monomial matrices. □

Theorem 3.7. Let $D = 0$ and $np > n + (n-1)m$. Then the equation (3.3a) has a solution $x_0 \in \Re_+^n$, $u_i \in \Re_+^m$, $i = 0,1,...,n-2$ for any given output sequence $y_i \in \Re_+^p$, $i = 0,1,...,n-1$ if and only if the following conditions are satisfied:

1) the condition (3.5) is met,
2) the matrix $H \in \Re_+^{pn\times(n+(n-1)m)}$ contains $n + (n-1)m$ linearly independent monomial rows.

Moreover, the equation has the unique solution

$$z = \widetilde{H}_m^{-1} y \quad (3.26)$$

where $\widetilde{H}_m$ is the monomial matrix consisting of linearly independent monomial rows of the matrix *H*.

Proof. For $np > n + (n-1)m$ the equation (3.3a) has a solution if and only if the condition (3.5) is met. By Lemma 3.1 the solution is nonnegative $x_0 \in \Re_+^n$, $u_i \in \Re_+^m$, $i = 0,1,...,n-2$ for a nonnegative sequence $y_i \in \Re_+^p$, $i = 0,1,...,n-1$ if and only if the matrix *H* contains one monomial matrix $\widetilde{H}_m \in \Re_+^{(n+(n-1)m)\times(n+(n-1)m)}$. The solution (3.26) is unique since the matrix *H* contains only one monomial matrix. ϒ

Remark 3.3. The equation (3.3a) has the unique solution (3.26) only if the positive pair (A, C) is observable. In this case the observability matrix (3.10) contains *n* linearly independent monomial rows.

Example 3.4. Consider the positive system (2.1) with the matrices (3.11). In this case the matrix

$$H = \begin{bmatrix} C & 0 \\ CA & CB \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \in \Re_+^{2\times 3} \quad (3.27)$$

contains two linearly independent monomial columns and the equation (3.3a) has the form (3.12). The equation (3.12) has the following two nonnegative solutions for any nonnegative sequence $y_i \geq 0$, $i = 0,1$.

$$\begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \text{ for } u_0 = 0 \quad (3.28a)$$

and

$$\begin{bmatrix} x_{10} \\ u_0 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \text{ for } x_{20} = 0. \quad (3.28b)$$

Example 3.5. Consider the positive system with the matrices (3.13). The matrix (3.14) has full column rank and the condition (3.5) is satisfied if ad only if $y_{02} = y_{21}$ (see Example 3.2). The matrix (3.14) contains only three linearly independent monomial rows. Therefore, by Theorem 3.7 the equation (3.3a) has not a nonnegative solution $x_0 \in \Re_+^2$, $u_i \in \Re_+$, $i = 0,1$ for a nonnegative sequence $y_i \in \Re_+^2$, $i = 0,1,2$ satisfying the condition $y_{02} = y_{21}$. Note that the in this case the positive pair (A, C) is observable since the matrix

$$\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad (3.29)$$

contains three linearly independent monomial rows.

Example 3.6. Consider the positive system (2.1) with the matrices

$$A=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix},\quad B=\begin{bmatrix}0\\0\\1\end{bmatrix},\quad C=\begin{bmatrix}1&0&0\\0&0&1\end{bmatrix},\quad D=\begin{bmatrix}0\\0\end{bmatrix}. \tag{3.30}$$

In this case the matrix (3.3b) has the form

$$H=\begin{bmatrix}C&0&0\\CA&CB&0\\CA^2&CAB&CB\end{bmatrix}=\begin{bmatrix}1&0&0&0&0\\0&0&1&0&0\\0&1&0&0&0\\0&0&0&1&0\\0&0&1&0&0\\0&0&0&0&1\end{bmatrix} \tag{3.31}$$

and it has five linearly independent monomial rows. The second and the fifth its rows are identical. Therefore the condition (3.5) is met if and only if $y_{02}=y_{21}$. Omitting the fifth row in (3.31) from (3.26) and (3.31) we obtain

$$z=\begin{bmatrix}1&0&0&0&0\\0&0&1&0&0\\0&1&0&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}^{-1}\begin{bmatrix}y_0\\y_1\\y_{22}\end{bmatrix}=\begin{bmatrix}y_{01}\\y_{11}\\y_{02}\\y_{12}\\y_{22}\end{bmatrix} \tag{3.32}$$

Generalizing Example 3.6 we obtain the following theorem.

Theorem 3.8. It is possible to compute $x_0\in\Re_+^n$, $u_i\in\Re_+^m$, $i=0,1,...,n-2$ of the positive system (2.1) with the matrices

$$A=\begin{bmatrix}0&1&0&...&0\\0&0&1&...&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&...&1\\0&0&0&...&0\end{bmatrix}\in\Re_+^{n\times n},\quad B=\begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}\in\Re_+^n, \tag{3.32}$$

$$C=\begin{bmatrix}1&0&...&0&0\\0&0&...&0&1\end{bmatrix}\in\Re_+^{2\times n},\quad D=\begin{bmatrix}0\\0\end{bmatrix}\in\Re_+^2$$

for a given output sequence $y_i\in\Re_+^2$, $i=0,1,...,n-1$ if the condition

$$y_{02}=y_{n-1,1} \tag{3.33}$$

is met.

Proof. Using (3.3b) and (3.32) we obtain

$$H=\begin{bmatrix}C&0&0&...&0\\CA&CB&0&...&0\\CA^2&CAB&CB&...&0\\\vdots&\vdots&\vdots&...&\vdots\\CA^{n-1}&CA^{n-2}B&CA^{n-3}B&...&CB\end{bmatrix}=$$

$$=\begin{bmatrix}1&0&0&...&0&0&0&0&...&0\\0&0&0&...&0&1&0&0&...&0\\0&1&0&...&0&0&0&0&...&0\\0&0&0&...&0&0&1&0&...&0\\0&0&1&...&0&0&0&0&...&0\\0&0&0&...&0&0&0&1&...&0\\\vdots&\vdots&\vdots&...&\vdots&\vdots&\vdots&\vdots&...&\vdots\\0&0&0&...&0&1&0&0&...&0\\0&0&0&...&0&0&0&0&...&1\end{bmatrix}\in\Re^{2n\times(2n-1)} \tag{3.37}$$

The matrix (3.34) has the second and the $2(n-1)$-th rows identical. Therefore, the condition (3.5) is met if and only if (3.33) holds. Omitting the $2(n-1)$-th row in (3.34) we obtain monomial matrix $\tilde{H}_m$ and from (3.26) we can compute the desired vector z consisting of $x_0\in\Re_+^n$ and $u_i\in\Re_+^m$, $i=0,1,...,n-2$. ϒ

Case 2. $D\neq 0$.

Theorem 3.9. Let $D\neq 0$ and $m+1\geq p$. Then the equation (3.4a) has a nonnegative solution $x_0\in\Re_+^n$, $u_i\in\Re_+^m$, $i=0,1,...,n-1$ for any given output sequence $y_i\in\Re_+^p$, $i=0,1,...,n-1$ if and only if the matrix $\overline{H}\in\Re_+^{pn\times(m+1)n}$ contains np linearly independent monomial columns.

Proof is similar to the proof of Theorem 3.6.

Theorem 3.10. Let $D\neq 0$ and $p>m+1$. Then the equation (3.4a) has a nonnegative solution $x_0\in\Re_+^n$, $u_i\in\Re_+^m$, $i=0,1,...,n-1$ for any given output sequence $y_i\in\Re_+^p$, $i=0,1,...,n-1$ if and only if the following conditions are satisfied:

1) the condition (3.17) is met,
2) the matrix $\overline{H}\in\Re_+^{pn\times(m+1)n}$ contains $(m+1)n$ linearly independent monomial rows.

Proof is similar to the proof of Theorem 3.7.

Remark 3.7. The equation (3.4a) has the unique nonnegative solution only if the positive pair (A,C) is observable.

Example 3.4. Consider the positive system (2.1) with the matrices *A*, *B*, *C* given by (3.11) and $D=[1]$. In this case the matrix (3.4b) has the form

$$\overline{H}=\begin{bmatrix}C&0&0\\CA&CB&D\end{bmatrix}=\begin{bmatrix}1&0&1&0\\0&1&1&1\end{bmatrix} \tag{3.35}$$

and it contains two linearly independent monomial columns. The equation

$$\begin{bmatrix}1&0&1&0\\0&1&1&1\end{bmatrix}\begin{bmatrix}x_{10}\\x_{20}\\u_0\\u_1\end{bmatrix}=\begin{bmatrix}y_0\\y_1\end{bmatrix} \tag{3.36}$$

has the following two nonnegative solutions for any nonnegative sequence $y_i \geq 0$, $i = 0,1$

$$\begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \text{ for } u_0 = u_1 = 0 \qquad (3.37a)$$

and

$$\begin{bmatrix} x_{20} \\ u_0 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \text{ for } x_{10} = u_0 = 0 \,. \qquad (3.37b)$$

4. Concluding remarks

The problem of computation of initial conditions and inputs for given outputs of standard and positive discrete-time linear systems has been formulated and solved. Two cases $D = 0$ and $D \neq 0$ have been considered for standard and positive systems. Necessary and sufficient conditions have been established for existence of solution to the problem. It has been shown that there exist the unique solutions to the problem only if the pair (*A*, *C*) of the system is observable. Therefore, the computation of initial conditions and inputs for given outputs can be considered as a generalized observability problem for standard and positive linear systems. The considerations have been illustrated by numerical examples.

The considerations can be extended to the fractional standard and positive discrete-time linear systems. An extension of these considerations for standard and positive continuous-time linear systems is an open problem.

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РОЗРАХУНОК ПОЧАТКОВИХ УМОВ ТА ВХІДНИХ ДАНИХ ЗА ЗАДАНИМИ ВИХІДНИМИ ДЛЯ КЛАСИЧНИХ ТА ДОДАТНІХ ДИСКРЕТНИХ СИСТЕМ

Т. Качорек

У статті сформульовано та розв’язано задачу обчислення початкових умов та вхідних даних за заданими вихідними для класичних та додатних дискретних в часі лінійних систем. Встановлено необхідні та достатні умови існування розв’язку поставленої задачі. Показано, що єдиний розв’язок даної задачі існує лише за умови спостережуваності пари (*А*, *С*) досліджуваної системи.

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