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TAYLOR EXPANSION OF SPECIAL ATEB-FUNCTIONS

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Peculiarities of special *Ateb*-functions Taylor series are investigated. Taylor series of functions $sa(\overline{n},\overline{m},w)$ and $ca(\overline{m},\overline{n},w)$ for some particular parameters $\overline{n},\overline{m}$ are constructed.

Key words: Taylor series, uncomplete *Beta*-function, special *Ateb*-functions.

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Special *Ateb*-functions $sa(\overline{n},\overline{m},w)$ and $ca(\overline{m},\overline{n},w)$ $\left(\overline{m}=\dfrac{2\nu_1+1}{2\nu_2+1},\overline{n}=\dfrac{2\mu_1+1}{2\mu_2+1},\nu_1,\nu_2,\mu_1,\mu_2=0,1,2,\dots\right)$ are continuous in the variable w, periodic with a period 2Π functions $\left(\Pi=\mathrm{B}\left(\dfrac{1}{\overline{n}+1};\dfrac{1}{\overline{m}+1}\right)\text{ "— Beta-function}\right)$, obtained by inversion of uncomplete *Beta*-function [1, 2].

Taking into account the rules for differentiation of *Ateb*-functions

$$\frac{d\,sa(\overline{n},\overline{m},w)}{dw}=\frac{2}{\overline{n}+1}\big(ca(\overline{m},\overline{n},w)\big)^{\overline{m}},$$

$$\frac{d\,ca(\overline{m},\overline{n},w)}{dw}=-\frac{2}{\overline{m}+1}\big(sa(\overline{n},\overline{m},w)\big)^{\overline{n}}$$

and their values at $w=0$ $\Big(sa(\overline{n},\overline{m},0)=0,\ ca(\overline{m},\overline{n},0)=1\Big)$ obviously, the functions $sa(\overline{n},\overline{m},w)$ and $ca(\overline{m},\overline{n},w)$ have the derivatives of all orders in the neighborhood of $w=0$ when $\overline{n}=2k-1$ $\big(k\in\mathrm{N}\big)$ and arbitrary possible values $\overline{m}$. Thus, the functions can be expanded in the Taylor series [3].

Theorem 1. *Let* $sa(\overline{n},\overline{m},w)=\sum\limits_{k=0}^{\infty}a_kw^k$ *be the Taylor expansion of* $sa(\overline{n},\overline{m},w)$ *in the neighborhood of the point* $w=0$ *and*

$$\overline{n}=2k-1, \tag{1}$$

$k\in\mathrm{N}$. *Then* $a_0=0$, $a_1=\dfrac{2}{\overline{n}+1}$ *and* a_s $\big(s\geq 2\big)$ *can be different from zero only when* $s-1$ *is a multiple of* $\overline{n}+1$ *while* $\overline{m}$ *takes arbitrary possible values.*

The proof. Let find the coefficients directly:

$$s=0,\quad a_0=sa(\overline{n},\overline{m},0)=0;$$

$$s=1,\quad a_1=\left.\frac{d\,sa(\overline{n},\overline{m},w)}{dw}\right|_{w=0}=\frac{2}{\overline{n}+1}\left(ca(\overline{m},\overline{n},0)\right)^{\overline{m}}=\frac{2}{\overline{n}+1};$$

$$s=2,\quad a_2=\left.\frac{d^2\,sa(\overline{n},\overline{m},w)}{dw^2}\right|_{w=0}=\frac{-4\overline{m}}{(\overline{n}+1)(\overline{m}+1)}\times\left.\Big(\left(ca(\overline{m},\overline{n},w)\right)^{\overline{m}-1}\left(sa(\overline{n},\overline{m},w)\right)^{\overline{n}}\Big)\right|_{w=0}=0;$$

$$s=3,\quad a_3=\left.\frac{d^3\,sa(\overline{n},\overline{m},w)}{dw^3}\right|_{w=0}=\frac{-4\overline{m}}{(\overline{n}+1)(\overline{m}+1)}\left(\frac{-2(\overline{m}-1)}{\overline{m}+1}\left(ca(\overline{m},\overline{n},w)\right)^{\overline{m}-2}\times\right.$$
$$\left.\left.\times\left(sa(\overline{n},\overline{m},w)\right)^{2\overline{n}}+\frac{2}{\overline{n}+1}\left(ca(\overline{m},\overline{n},w)\right)^{2\overline{m}-1}\left(sa(\overline{n},\overline{m},w)\right)^{\overline{n}-1}\right)\right|_{w=0}.$$

The coefficient a_3 is different from zero if and only if the exponent of the $sa(\overline{n},\overline{m},w)$ function equals to zero $\big($when $\overline{n}$ is of the form (1)$\big)$. It happens only if $\overline{n}=1$ for which $s-1=2$ is a multiple of $\overline{n}+1$. Thus $a_3=\begin{cases}-\dfrac{2\overline{m}}{\overline{m}+1}, & \overline{n}=1,\\ 0, & \overline{n}\neq 1.\end{cases}$

Similarly, when $s = 4$ and $s = 5$:

$$a_4 = \left.\frac{d^4\, sa(\overline{n},\overline{m},w)}{dw^4}\right|_{w=0} = \left(\frac{-16\overline{m}(\overline{m}-1)(\overline{m}-2)}{(\overline{n}+1)(\overline{m}+1)^3}\left(ca(\overline{m},\overline{n},w)\right)^{\overline{m}-3}\left(sa(\overline{n},\overline{m},w)\right)^{3\overline{n}}+\right.$$

$$+\frac{16\overline{m}\overline{n}(4\overline{m}-3)}{(\overline{n}+1)^2(\overline{m}+1)^2}\left(ca(\overline{m},\overline{n},w)\right)^{2\overline{m}-2}\left(sa(\overline{n},\overline{m},w)\right)^{2\overline{n}-1}-$$

$$\left.\left.-\frac{16\overline{m}\overline{n}(\overline{n}-1)}{(\overline{n}+1)^3(\overline{m}+1)}\left(ca(\overline{m},\overline{n},w)\right)^{3\overline{m}-1}\left(sa(\overline{n},\overline{m},w)\right)^{\overline{n}-2}\right)\right|_{w=0} = 0;$$

$$a_5 = \left.\frac{d^5\, sa(\overline{n},\overline{m},w)}{dw^5}\right|_{w=0} = \left(\frac{-16\overline{m}(\overline{m}-1)(\overline{m}-2)}{(\overline{n}+1)(\overline{m}+1)^3}\left(\frac{-2(\overline{m}-3)}{\overline{m}+1}\left(ca(\overline{m},\overline{n},w)\right)^{\overline{m}-4}\times\right.\right.$$

$$\left.\times\left(sa(\overline{n},\overline{m},w)\right)^{4\overline{n}}+\frac{6\overline{n}}{\overline{n}+1}\left(ca(\overline{m},\overline{n},w)\right)^{2\overline{m}-3}\left(sa(\overline{n},\overline{m},w)\right)^{3\overline{n}-1}\right)+$$

$$+\frac{16\overline{m}\overline{n}(4\overline{m}-3)}{(\overline{n}+1)^2(\overline{m}+1)^2}\left(\frac{-4(\overline{m}-1)}{\overline{m}+1}\left(ca(\overline{m},\overline{n},w)\right)^{2\overline{m}-3}\left(sa(\overline{n},\overline{m},w)\right)^{3\overline{n}-1}+\right.$$

$$\left.+\frac{2(2\overline{n}-1)}{\overline{n}+1}\left(ca(\overline{m},\overline{n},w)\right)^{3\overline{m}-2}\left(sa(\overline{n},\overline{m},w)\right)^{2\overline{n}-2}\right)-$$

$$-\frac{16\overline{m}\overline{n}(\overline{n}-1)}{(\overline{n}+1)^3(\overline{m}+1)}\left(\frac{-2(3\overline{m}-1)}{\overline{m}+1}\left(ca(\overline{m},\overline{n},w)\right)^{3\overline{m}-2}\left(sa(\overline{n},\overline{m},w)\right)^{2\overline{n}-2}+\right.$$

$$\left.\left.\left.+\frac{2(\overline{n}-2)}{\overline{n}+1}\left(ca(\overline{m},\overline{n},w)\right)^{4\overline{m}-1}\left(sa(\overline{n},\overline{m},w)\right)^{\overline{n}-3}\right)\right)\right|_{w=0}.$$

The value a_5 is different from zero only when $\overline{n} = 1$ or $\overline{n} = 3$ for which $s - 1 = 4$ is a multiple of $\overline{n} + 1$. Thus,

$$a_5 = \begin{cases} \dfrac{4\overline{m}(4\overline{m}-3)}{(\overline{m}+1)^2}, & \overline{n} = 1, \\ \dfrac{-3\overline{m}}{4(\overline{m}+1)}, & \overline{n} = 3, \\ 0, & \overline{n} \neq 1;\ \overline{n} \neq 3. \end{cases}$$

Taking into account the rules for differentiation of *Ateb*-functions and the above found derivatives let assume that the expression for a_s can be written

$$a_s = \left.\frac{d^s\, sa(\overline{n},\overline{m},w)}{dw^s}\right|_{w=0} = \left(\left(b_1\left(ca(\overline{m},\overline{n},w)\right)^{(\overline{m}+1)-s}\left(sa(\overline{n},\overline{m},w)\right)^{(s-1)\overline{n}}+\right.\right.$$

$$+b_2\left(ca(\overline{m},\overline{n},w)\right)^{2(\overline{m}+1)-s}\left(sa(\overline{n},\overline{m},w)\right)^{(s-2)\overline{n}-1}+b_3\left(ca(\overline{m},\overline{n},w)\right)^{3(\overline{m}+1)-s}\left(sa(\overline{n},\overline{m},w)\right)^{(s-3)\overline{n}-2}+\ldots+$$

$$\left.\left.+b_{s-1}\left(ca(\overline{m},\overline{n},w)\right)^{(s-1)(\overline{m}+1)-s}\left(sa(\overline{n},\overline{m},w)\right)^{\overline{n}-(s-2)}\right)\right|_{w=0},$$

where b_i are some constants, represented in terms of $\overline{n},\overline{m}$.

When $w = 0$ the expression for a_s is different from zero only for odd s and $\overline{n}$ of the form (1) when $\overline{n} = \frac{1}{s-2}$, $\overline{n} = \frac{3}{s-4}$, …, $\overline{n} = \frac{s-2}{1}$. For them $\overline{n}+1 = \frac{s-1}{s-2}$, $\overline{n}+1 = \frac{s-1}{s-4}$, $\overline{n}+1 = \frac{s-1}{1}$, thus $s-1$ is a multiple of $\overline{n}+1$.

Lets show that if the theorem 1 is fulfilled for a_s then a_{s+1} has the same structure as a_s and for a_{s+1} the theorem is also true

$$a_{s+1} = \left.\frac{d^{s+1}\, sa(\overline{n},\overline{m},w)}{dw^{s+1}}\right|_{w=0} =$$

$$= \Bigg(b_1 \Bigg(\frac{-2(\overline{m}+1-s)}{m+1} \left(ca(\overline{m},\overline{n},w)\right)^{\overline{m}-s} \left(sa(\overline{n},\overline{m},w)\right)^{(s-1)\overline{n}+\overline{n}} +$$

$$+ \frac{2\overline{n}(s-1)}{\overline{n}+1} \left(ca(\overline{m},\overline{n},w)\right)^{\overline{m}+1-s+\overline{m}} \left(sa(\overline{n},\overline{m},w)\right)^{(s-1)\overline{n}-1} \Bigg) +$$

$$+ b_2 \Bigg(\frac{-2(2(\overline{m}+1)-s)}{m+1} \left(ca(\overline{m},\overline{n},w)\right)^{2(\overline{m}+1)-s-1} \left(sa(\overline{n},\overline{m},w)\right)^{(s-2)\overline{n}-1+\overline{n}} +$$

$$+ \frac{2((s-2)\overline{n}-1)}{\overline{n}+1} \left(ca(\overline{m},\overline{n},w)\right)^{2(\overline{m}+1)-s+\overline{m}} \left(sa(\overline{n},\overline{m},w)\right)^{(s-2)\overline{n}-2} \Bigg) +$$

$$+ b_3 \Bigg(\frac{-2(3(\overline{m}+1)-s)}{m+1} \left(ca(\overline{m},\overline{n},w)\right)^{3(\overline{m}+1)-s-1} \left(sa(\overline{n},\overline{m},w)\right)^{(s-3)\overline{n}-2+\overline{n}} +$$

$$+ \frac{2((s-3)\overline{n}-2)}{\overline{n}+1} \left(ca(\overline{m},\overline{n},w)\right)^{3(\overline{m}+1)-s+\overline{m}} \left(sa(\overline{n},\overline{m},w)\right)^{(s-3)\overline{n}-3} \Bigg) + \ldots +$$

$$+ b_{s-1} \Bigg(\frac{-2((s-1)(\overline{m}+1)-s)}{m+1} \left(ca(\overline{m},\overline{n},w)\right)^{(s-1)(\overline{m}+1)-s-1} \left(sa(\overline{n},\overline{m},w)\right)^{\overline{n}+2-s+\overline{n}} +$$

$$+ \left.\frac{2(\overline{n}+2-s)}{\overline{n}+1} \left(ca(\overline{m},\overline{n},w)\right)^{(s-1)(\overline{m}+1)-s+\overline{m}} \left(sa(\overline{n},\overline{m},w)\right)^{\overline{n}+2-s-1} \Bigg) \Bigg)\right|_{w=0} =$$

$$= \Big(\widetilde{b}_1 \left(ca(\overline{m},\overline{n},w)\right)^{(\overline{m}+1)-(s+1)} \left(sa(\overline{n},\overline{m},w)\right)^{((s+1)-1)\overline{n}} +$$

$$+ \widetilde{b}_2 \left(ca(\overline{m},\overline{n},w)\right)^{2(\overline{m}+1)-(s+1)} \left(sa(\overline{n},\overline{m},w)\right)^{((s+1)-2)\overline{n}-1} +$$

$$+ \widetilde{b}_3 \left(ca(\overline{m},\overline{n},w)\right)^{3(\overline{m}+1)-(s+1)} \left(sa(\overline{n},\overline{m},w)\right)^{((s+1)-3)\overline{n}-2} + \ldots +$$

$$+ \left.\widetilde{b}_s \left(ca(\overline{m},\overline{n},w)\right)^{((s+1)-1)(\overline{m}+1)-(s+1)} \left(sa(\overline{n},\overline{m},w)\right)^{(\overline{n}+2)-(s+1)} \Big)\right|_{w=0},$$

where $\widetilde{b}_1 = \dfrac{-2(\overline{m}+1-s)}{\overline{m}+1} b_1$; $\widetilde{b}_2 = \dfrac{2\overline{n}(s-1)}{\overline{n}+1} b_1 - \dfrac{2(2\overline{m}+2-s)}{\overline{m}+1} b_2$;
$\widetilde{b}_3 = \dfrac{-2(3\overline{m}+3-s)}{\overline{m}+1} b_3 + \dfrac{2((s-2)\overline{n}-1)}{\overline{n}+1} b_2;\ \ldots;\ \widetilde{b}_s = \dfrac{2(\overline{n}+2-s)}{\overline{n}+1} b_{s-1}$.

The values a_{s+1} may be different from zero only for odd s when $\overline{n} = \dfrac{1}{(s+1)-2}$; $\overline{n} = \dfrac{3}{(s+1)-4};\ldots;\overline{n} = \dfrac{s-1}{1}$, for which $\overline{n}+1 = \dfrac{(s+1)-1}{s-1}; \overline{n}+1 = \dfrac{(s+1)-1}{s-3}$; $\overline{n}+1 = \dfrac{(s+1)-1}{1}$. Thus $(s+1)-1$ is a multiple of $\overline{n}+1$.

According to the theorem, the expansion of $sa(\overline{n},\overline{m},w)$ into the Taylor series about w in the neighborhood of $w=0$ is

$$\begin{aligned}sa(\overline{n},\overline{m},w) = \frac{2}{\overline{n}+1}w + \frac{a_{(\overline{n}+1)+1}}{((\overline{n}+1)+1)!}w^{\overline{n}+2} + \frac{a_{2(\overline{n}+1)+1}}{((2(\overline{n}+1)+1)!}w^{2(\overline{n}+1)+1} + \ldots + \\ + \frac{a_{k(\overline{n}+1)+1}}{(k(\overline{n}+1)+1)!}w^{k(\overline{n}+1)+1} + \ldots,\end{aligned} \tag{2}$$

where $k \in \mathrm{N}$; $a_{(\overline{n}+1)+1}, a_{2(\overline{n}+1)+1}, \ldots, a_{k(\overline{n}+1)+1}$ can be obtained directly.

Theorem 2. *Let* $ca(\overline{m},\overline{n},w) = \sum_{k=0}^{\infty} c_k w^k$ *be the Taylor expansion of* $ca(\overline{m},\overline{n},w)$ *in the neighborhood of the point* $w=0$ *and* $\overline{n}$ *is of the form* (1). *Then* $c_0 = 1$ *and* c_s $(s \geq 1)$ *is different from zero when* s *is not a multiple of* $\overline{n}+1$ *while* $\overline{m}$ *takes arbitrary possible values.* (The proof of the theorem is analogues to that of the theorem 1).

According to the theorem 2 the Taylor expansion of $ca(\overline{m},\overline{n},w)$ in the neighborhood of $w=0$ can be written

$$\begin{aligned}ca(\overline{m},\overline{n},w) = 1 + \frac{d_{\overline{n}+1}}{(\overline{n}+1)!}w^{\overline{n}+1} + \\ + \frac{d_{2(\overline{n}+1)}}{(2(\overline{n}+1))!}w^{2(\overline{n}+1)} + \ldots + \frac{d_{k(\overline{n}+1)}}{(k(\overline{n}+1))!}w^{k(\overline{n}+1)} + \ldots,\end{aligned} \tag{3}$$

where $k \in \mathrm{N}$; $d_{\overline{n}+1}, d_{2(\overline{n}+1)}, \ldots, d_{k(\overline{n}+1)}$ can be obtained directly.

As appears from the above the parameters $\overline{m}$ and $\overline{n}$ play unequal role, specified by the differentiation rules. The parameter $\overline{n}$ (in the form (1)) defines the power of w in the Taylor expansion; both parameters $\overline{m}$ and $\overline{n}$ define the values of coefficients a_s and d_s.

For some values of the parameter $\overline{n}$ the expansion can be written

$$sa(1,\overline{m},w) = w - \frac{2\overline{m}}{\overline{m}+1}\frac{w^3}{3!} + \frac{4\overline{m}(4\overline{m}-3)}{(\overline{m}+1)^2}\frac{w^5}{5!} + \ldots,$$

$$sa(3,\overline{m},w) = \frac{1}{2}w - \frac{3\overline{m}}{4(\overline{m}+1)}\frac{w^5}{5!} + \ldots,$$

$$\begin{aligned}ca(\overline{m},1,w) = 1 - \frac{2}{\overline{m}+1}\frac{w^2}{2!} + \frac{4\overline{m}}{(\overline{m}+1)^2}\frac{w^4}{4!} - \\ - \frac{8\overline{m}(2\overline{m}-1)}{(\overline{m}+1)^3}\frac{w^6}{6!} + \ldots,\end{aligned}$$

$$ca(\overline{m},3,w) = 1 - \frac{3}{2(\overline{m}+1)}\frac{w^4}{4!} + \ldots,$$

$$ca(\overline{m},5,w) = 1 - \frac{160}{27(\overline{m}+1)}\frac{w^6}{6!} + \ldots,$$

$$sa(5,\overline{m},w) = \frac{1}{3}w - \frac{80\overline{m}}{243(\overline{m}+1)}\frac{w^7}{7!} + \ldots.$$

It should be observed that when $\overline{m} = \overline{n} = 1$ special *Ateb*-functions $sa(\overline{n},\overline{m},w)$ and $ca(\overline{m},\overline{n},w)$ reduce to trigonometric functions $\sin w$ and $\cos w$ and from (2) and (3) we obtain the Taylor series of $\sin w$ and $\cos w$.

Література

[1] Сеник П.М. Обернення неповної *Beta*-функції // УМЖ. – 1969. №1. – 21. – №3. – С.325–333.

[2] Сеник П.М. Про *Ateb*-функції // ДАН УРСР, сер.А. – 1968. – №1. – С.23–26.

[3] Фихтенгольц Г. М. Курс дифференциального и интегрального исчисления. Т.2. – М.: ОГИЗ Гостехиздат, 1948. – 860 с.

РАЗЛОЖЕНИЕ В РЯД ТЕЙЛОРА СПЕЦИАЛЬНЫХ ATEB-ФУНКЦИЙ

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Исследуются особенности рядов Тейлора специальных *Ateb*-функций. Построены разложения в ряды Тейлора функций $sa(\overline{n}, \overline{m}, w)$ и $ca(\overline{m}, \overline{n}, w)$ для отдельных значений параметров $\overline{n}, \overline{m}$.

Ключевые слова: ряд Тейлора, неполная *Beta*-функция, специальные *Ateb*-функции.

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РОЗКЛАД В РЯД ТЕЙЛОРА СПЕЦІАЛЬНИХ ATEB-ФУНКЦІЙ

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Досліджено особливості рядів Тейлора спеціальних *Ateb*-функцій. Побудовано розклади у ряди Тейлора функцій $sa(\overline{n}, \overline{m}, w)$ та $ca(\overline{m}, \overline{n}, w)$ для окремих значень параметрів $\overline{n}, \overline{m}$.

Ключові слова: ряд Тейлора, неповна *Beta*-функція, спеціальні *Ateb*-функцiЇ.

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